Assignment 6: Integration

2. **(D)** Let $f : [0,\ 1] \longrightarrow \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that f is integrable on $[0,\ 1]$ and $\int_0^1 f(x)dx = 0$.

5. **(D)** Let $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \leq y < 1 \\ 0 & \text{if } y = 1 \end{cases}$. Then prove that $\lim_{n \longrightarrow \infty} \int_0^1 g_n(y)dy = \frac{1}{2}$ whereas
$\int_0^1 \lim_{n \longrightarrow \infty} g_n(y)dy = 0.$

Assignment 6 - Solutions

2. The solution is given in the lecture notes.

5. From the ratio test for the sequence we can show that $\lim\limits_{n\longrightarrow\infty} \frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\int_0^1 \lim\limits_{n\longrightarrow\infty} g_n(y)dy = 0$.

   For the other part, use integration by parts to see that $\int\limits_0^1 \frac{ny^{n-1}}{1+y}dy = \frac{1}{2} + \int\limits_0^1 \frac{y^n}{(1+y)^2}dy$. Note that $\int\limits_0^1 \frac{y^n}{(1+y)^2}dy \leq \int\limits_0^1 y^n = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \int\limits_0^1 g_n(y)dy = \frac{1}{2}$.