Assignment 7: Improper Integrals

2. **(D)** In each case, determine the values of p for which the following improper integrals converge

$(a)\ \int\limits_0^\infty \frac{1-e^{-x}}{x^p}dx$ $\qquad$ $(b)\ \int\limits_0^\infty \frac{t^{p-1}}{1+t}dt.$

5. **(D)** Prove that improper integral $\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$.

Assignment 7 - Solutions

2. (a) $\int\limits_0^\infty \frac{1-e^{-x}}{x^p}dx = \int\limits_0^1 \frac{1-e^{-x}}{x^p}dx + \int\limits_1^\infty \frac{1-e^{-x}}{x^p}dx = I_1 + I_2.$

Now one has to see how the function $\frac{1-e^{-x}}{x^p}$ behaves in the respective intervals and apply the LCT.

Since $\lim\limits_{x\to 0} \frac{1-e^{-x}}{x} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that I_1 is convergent iff $p-1<1$, *i.e.*,$p < 2$. Similarly, I_2 is convergent (by applying LCT with $\frac{1}{x^p}$) iff $p > 1$. Therefore $\int\limits_0^\infty \frac{1-e^{-x}}{x^p}dx$ converges iff $1 < p < 2$.

(b) $\int_0^\infty \frac{t^{p-1}}{1+t}dt = \int_0^1 \frac{t^{p-1}}{1+t}dt + \int_1^\infty \frac{t^{p-1}}{1+t}dt = I_1 + I_2$. For I_1, use LCT with t^{p-1}. We see that the integral converges iff $p > 0$. Similarly, for I_2, Use LCT with t^{p-2}. The integral converges iff $p < 1$. Therefore, $\int_0^\infty \frac{t^{p-1}}{1+t}dt$ converges iff $0 < p < 1$.

5. By Dirichlet's Test, $\int\limits_1^\infty \frac{\sin x}{x^p}dx$ converges for all $p > 0$.

$\int\limits_1^\infty \frac{|\sin x|}{x^p}dx \leq \int\limits_1^\infty \frac{dx}{x^p}$. Therefore, the function converges absolutely for $p > 1$.

Now, let $0 < p \leq 1$.

Since, $|\sin x| \geq \sin^2 x$, we see that $|\frac{\sin x}{x^p}| \geq \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p}$.

By Dirichlet's Test, $\int\limits_1^\infty \frac{\cos 2x}{2x^p}dx$ converges $\forall\ p > 0$. But $\int\limits_1^\infty \frac{1}{2x^p}$ diverges for $p \leq 1$.

Hence, $\int\limits_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$.