Assignment 8: Applications of Integration, Pappus Theorem

1. (D) Find the area of the region in the first quadrant bounded on the left by the \( Y \)-axis, below by the curve \( x = 2\sqrt{y} \), above left by the curve \( x = (y - 1)^2 \), and above right by the line \( x = 3 - y \).

2. (D) Sketch the graph of \( r = 1 + \sin \theta \). Find the area of the region that is inside the circle \( r = 3 \sin \theta \) and also inside \( r = 1 + \sin \theta \).

5. (D) The region bounded by the curve \( y = x^2 + 1 \) and the line \( y = -x + 3 \) is revolved about the \( X \)-axis to generate a solid. Find the volume of the solid.

6. (D) The region in the first quadrant bounded by the parabola \( y = x^2 \), the \( Y \)-axis and the line \( y = 1 \) is revolved about the line \( x = 2 \) to generate a solid. Find the volume of the solid.

8. (D) Find the area of the surface generated by revolving the curve \( y = x^3 \), \( 0 \leq x \leq 1/2 \), about the \( X \)-axis.

10. (D) A regular hexagon is inscribed in the circle \( x^2 + (y - 2)^2 = 1 \) and is rotated about the \( X \)-axis. Find the volume and the surface area of the solid so formed.
1. Area= \[ \int_0^1 (1 + \sqrt{x} - \frac{x^2}{4})dx + \int_1^2 (3 - x - \frac{x^2}{4})dx = \frac{5}{2}. \]

2. Area= \[ 2 \cdot \frac{1}{2} \left[ \int_0^{\frac{\pi}{6}} (3 \sin \theta)^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin \theta)^2 d\theta \right] = \frac{5\pi}{4}. \]

3. Let \( r = \cos 2\theta \). Between \( \theta = 0 \) to \( \theta = \frac{\pi}{4} \), we plot \((r, \theta)\) (in polar coordinate) i.e., for each \( \theta \) we find \( r \). The graph lies in the first quadrant for these \( \theta \)'s.

Note that, since \( r \) is negative for \( \theta = \frac{\pi}{4} \) to \( \theta = \frac{\pi}{2} \), if we sketch the graph for these \( \theta \)'s, the graph appears in the third quadrant.

Whenever \((r, \theta) \in G\), the graph, we see that \((r, -\theta), (r, \pi - \theta), (r, \pi + \theta) \in G\).

Therefore, there is symmetry about the x-axis, y-axis and the origin.

Let \( r = \sin 2\theta \). Again, we see that there is symmetry about the x-axis, y-axis and the origin.

4. Each cross section is a rectangle of area \( A(x) = x^2 \sqrt{9 - x^2} \). Therefore the volume \( V = \int_0^2 x \sqrt{9 - x^2} dx = \frac{18}{7} \).

5. Use the washer method: The outer radius , \( r_2(x) = -x + 3 \) and the inner radius \( r_1(x) = x^2 + 1 \).

\[ V = \int_{-2}^1 \pi [((-x + 3)^2 - (x^2 + 1)^2)] dx = 117\pi \frac{7}{5}. \]

6. By the shell method: Observe that shell radius =2 − x and the shell height =1 − x^2.

The volume \( V = \int_0^1 2\pi(2 - x)(1 - x^2)dx = \frac{13\pi}{6} \).

The volume can also be computed by the washer method.

8. \( S = \int_0^{\frac{\pi}{2}} 2\pi y ds \).

Since \( y = x^3 \), \( ds = \sqrt{dx^2 + dy^2} \Rightarrow ds = \sqrt{1 + 9x^2}dx \). Therefore \( S = \frac{61\pi}{1728} \).
10. From the figure, the area of the hexagon is $\frac{3\sqrt{3}}{2}$ and the perimeter is 6. $\rho = 2$ since the centroid is (0, 2).

By Pappus theorem, $V = 6\sqrt{3}\pi$ and $S = 24\pi$. 

Distance of centroid from the line $y = -mx$ is $\rho = mr + 2r\pi\sqrt{1 + m^2}$.

Again, by Pappus Theorem, we see that $A = 2\pi\rho r$. 

$dA/dm = 0 \Rightarrow m = \frac{\pi}{2}$. Easy to see that $A$ has a maxima at $\frac{\pi}{2}$. 