# Assignment 1 : Real Numbers, Sequences

1. **(D)** Find the supremum and infimum of the sets $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$ and $\left\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$.

2. **(D)** Let $(x_n)$ be a sequence of strictly positive real numbers such that $\lim\limits_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n} = \ell$. Then prove the following:

   (a) if $\ell < 1$ then $\lim\limits_{n\longrightarrow\infty} x_n = 0$,

   (b) if $\ell > 1$ then $\lim\limits_{n\longrightarrow\infty} x_n = \infty$

   (c) if $\ell = 1$ then give example of sequences to show that both conclusions can hold.

3. Investigate the convergence of the following sequences:

   (a) **(T)** $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}$,

   (b) **(D)** $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$,

   (c) **(T)** $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0, 1)$,

   (d) **(D)** $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$,

   (e) **(D)** $x_n = \frac{2^n}{n!}$.

   (f) **(T)** $x_n = \frac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$,

4. **(T)** Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence $(x_n)$ converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.

5. **(D)** Suppose that $0 < \alpha < 1$ and that $(x_n)$ is a sequence which satisfies one of the following conditions

   (a) $|x_{n+1} - x_n| \leq \alpha^n$, $\quad n = 1, 2, 3, \ldots$

   (b) $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|$, $\quad n = 1, 2, 3, \ldots$.

   Then prove that $(x_n)$ satisfies the Cauchy criterion. *Whenever you use this result, you have to show that the number* $\alpha$ *that you get, satisfies* $0 < \alpha < 1$. *The condition* $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ *does not guarantee the convergence of* $(x_n)$. *Give examples.*

6. **(T)** Let $x_1 \in \mathbb{R}$ and let $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ for $n \in \mathbb{N}$. Show that $(x_n)$ converges for $0 < x_1 < 1$. Also conclude that it converges to a root of $x^3 - 7x + 2$ lying between 0 and 1. Does the sequence converge for any starting value of $x_1 > 1$?