Assignment 10: Functions of several variables (Continuity and Differentiabilty)

1. **(D)** Examine the following functions for continuity at the point $(0,0)$, where $f(0,0) = 0$ and $f(x,y)$ for $(x,y) \neq (0,0)$ is given by

   $i)\ \mid x \mid + \mid y \mid \qquad i)\ \frac{xy}{\sqrt{x^2+y^2}} \qquad ii)\ \frac{xy}{x^2+y^2} \qquad iii)\ \frac{x^4-y^2}{x^4+y^2} \qquad iv)\ \frac{x^2y}{x^4+y^2}.$

2. **(T)** Identify the points, if any, where the following functions fail to be continuous:

   $$(i)\ f(x,y) = \begin{cases} xy & \text{if } xy \geq 0 \\ -xy & \text{if } xy < 0 \end{cases} \qquad (ii)\ f(x,y) = \begin{cases} xy & \text{if } xy \text{ is rationnal} \\ -xy & \text{if } xy \text{ is irrational.} \end{cases}$$

3. **(T)** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

   $$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if} (x,y) = (0,0) \end{cases}$$

   Show that the function satisfy the following:

   (a) The iterated limits $\lim\limits_{x \longrightarrow 0} \left[\lim\limits_{y \longrightarrow 0} f(x,y) \right]$ and $\lim\limits_{y \longrightarrow 0} \left[\lim\limits_{x \longrightarrow 0} f(x,y) \right]$ exist and equals 0;

   (b) $\lim\limits_{(x,y) \longrightarrow (0,0)} f(x,y)$ does not exist;

   (c) $f(x,y)$ is not continuous at $(0,0)$;

   (d) the partial derivatives exist at $(0,0)$.

4. **(D)** Let $f(x,y)$ be defined in $S = \{(x,y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$. Suppose that the partial derivatives of f exist and are bounded in S. Then show that f is continuous in S.

5. **(D)** Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Prove that

   (a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y;

   (b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and $\qquad (c)$ $f(x,y)$ is differentiable at $(0,0)$.

6. **(T)** Let $f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

7. **(T)** Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) = c$, a constant.