Assignment 2 : Continuity, Intermediate Value Property

1. **(D)** Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $\mid f(x) - f(y) \mid \leq \mid x - y \mid$. Show that f is continuous.

2. **(T)** Determine the points of continuity for the function $f : [0,1] \longrightarrow [0,1]$ defined by
$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

3. **(D)** Let $f : (-1,1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

4. **(T)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

5. **(D)** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.

6. **(T)** Show that the polynomial $x^4 + 6x^3 - 8$ has at least two real roots.

7. **(T)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which takes only rational values. Show that f is a constant function.

8. **(D)** Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Show that the range $\{f(x) : x \in [a,b]\}$ is a closed and bounded interval.

9. **(T)** Let $f : [0,2] \to \mathbb{R}$ be a continuous function and $f(0) = f(2)$. Prove that there exist real numbers $x_1, x_2 \in [0,2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

10. **(D)** Show that a polynomial of odd degree has at least one real root.