Assignment 3 : Derivatives, Maxima and Minima, Rolle's Theorem

1. **(T)** Show that the function $f(x) = x \mid x \mid$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

2. **(T)** Examine the function $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ for differentiability.

3. **(D)** Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

4. **(T)** Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function (i.e., $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an odd function (i.e.,$f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

5. **(D)** Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer$k$, show that

$$\lim_{x \to 0} \frac{1}{x} \left\{ f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + ... + f(\frac{x}{k}) \right\} = 1 + \frac{1}{2} + ... + \frac{1}{k}$$

6. (T) Show that among all triangles with given base and the corresponding vertex angle, the isosceles triangle has the maximum area.

7. **(D)** Prove that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.

8. **(T)** Show that exactly two real values of x satisfy the equation $x^2 = xsinx + cosx$.

9. **(T)** Suppose f is continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. The show that the equation $f'(x)f(x) = x$ has at least one root in $(a,\ b)$.

10. **(D)** Let f and g be functions, continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a,\ b)$ such that $g'(c)f(c) + f'(c) = 0$.