Assignment 4 : Mean Value Theorem, Taylor's Theorem, Curve Sketching

1. **(T)** Using Mean Value Theorem show that

   (i) $\frac{x-1}{x} \; < \; \log x \; < \; x-1$ for $x > 1$.

   (ii) $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

2. **(D)** Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.

3. **(T)** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

4. Using Cauchy Mean Value Theorem, show that

   (a) **(D)** $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.

   (b) **(T)** $x - \frac{x^3}{3!} < \sin x$ for $x > 0$.

   (c) **(T)** $\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $x \neq 0$.

   (d) **(T)** $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x > 0$.

5. **(D)** Let f be continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

6. **(T)** Find $\lim\limits_{x \longrightarrow 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \longrightarrow 0^+} (1 + \frac{1}{x})^x$.

7. **(T)** Sketch the graphs of $f(x) = x^3 - 6x^2 + 9x + 1$ and $f(x) = \frac{x^2}{x^2-1}$.

8. **(T)** Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem prove that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

9. **(D)** Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k}x^{2k} \; < \; \log(1 + x) \; < \; x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$