Assignment 5 : Series, Power Series, Taylor Series

1. **(D)** Let $a_n \geq 0$. Then show that both the series $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

2. **(T)** Prove that $\sum (a_n - a_{n+1})$ converges if and only if the sequence a_n converges. Use this to decide the convergence/divergence of the following series:

   (1) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ (2) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

3. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$ where a_n equals:

   (a)**(D)** $1-n\sin\frac{1}{n}$ (b)**(D)** $\frac{1}{n}\log(1+\frac{1}{n})$ (c)**(T)** $1-\cos\frac{1}{n}$

   (d)**(T)** $2^{-n-(-1)^n}$ (e)**(T)** $\left(1+\frac{1}{n}\right)^{n(n+1)}$ (f)**(T)** $\frac{n\ln n}{2^n}$

4. **(T)** Let $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq N$. Show that if $\sum\limits_{n\geq 1} b_n$ converges then $\sum\limits_{n\geq 1} a_n$ also converges. Test the series $\sum\limits_{n\geq 1} \frac{n^{n-2}}{e^n n!}$ for convergence.

5. **(D)** Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim\limits_{n\longrightarrow\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that $\sum\limits_{n\geq 1}(-1)^n b_n$ converges.

6. **(T)** Determine the values of x for which the series $\sum\limits_{n\geq 1} \frac{(x-1)^{2n}}{n^2 3^n}$ converges.

7. **(T)** Show that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n},\ x \in \mathbb{R}$ and $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,\ x \in \mathbb{R}$.