Assignment 6: Integration

1. **(T)** If f is a bounded function such that $f(x) = 0$ except at a point $c \in [a, b]$, then show that f is integrable on $[a, b]$ and that $\int_a^b f = 0$.

2. **(D)** Let $f : [0,\ 1] \longrightarrow \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that f is integrable on $[0,\ 1]$ and $\int\limits_0^1 f(x)dx = 0$.

3. **(T)** If f and g are continuous functions on $[a,\ b]$ and if $g(x) \geq 0$ for $a \leq x \leq b$, then show that there exists $c \in [a,\ b]$ such that $\int\limits_a^b f(x)g(x)dx = f(c)\int\limits_a^b g(x)dx$.

   (This result is sometimes called the second mean value theorem for integrals. The special case $g = 1$ yields the first mean value theorem for integrals.)

4. **(T)** Does there exist a continuous function f on $[0, 1]$ such that $\int\limits_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$.

5. **(D)** Let $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \leq y < 1 \\ 0 & \text{if } y = 1 \end{cases}$. Then prove that $\lim\limits_{n \longrightarrow \infty} \int\limits_0^1 g_n(y)dy = \frac{1}{2}$ whereas
   $\int\limits_0^1 \lim\limits_{n \longrightarrow \infty} g_n(y)dy = 0$.

6. **(T)** Show that $\int\limits_0^x (\int\limits_0^u f(t)dt)du = \int\limits_0^x f(u)(x-u)du$, assuming f to be contiuous.