Lecture 18 : Improper integrals

We defined $\int_a^b f(t)dt$ under the conditions that f is defined and bounded on the bounded interval $[a, b]$. In this lecture, we will extend the theory of integration to bounded functions defined on unbounded intervals and also to unbounded functions defined on bounded or unbounded intervals.

Improper integral of the first kind: Suppose f is (Riemann) integrable on $[a, x]$ for all $x > a$, i.e., $\int_a^x f(t)dt$ exists for all $x > a$. If $\lim\limits_{x\to\infty} \int_a^x f(t)dt = L$ for some $L \in \mathbb{R}$, then we say that the improper integral (of the first kind) $\int_a^\infty f(t)dt$ converges to L and we write $\int_a^\infty f(t)dt = L$. Otherwise, we say that the improper integral $\int_a^\infty f(t)dt$ diverges.

Observe that the definition of convergence of improper integrals is similar to the one given for series. For example, $\int_a^x f(t)dt,\ x > a$ is analogous to the partial sum of a series.

Examples : 1. The improper integral $\int_1^\infty \frac{1}{t^2}dt$ converges, because, $\int_1^x \frac{1}{t^2}dt = 1 - \frac{1}{x} \to 1$ as $x \to \infty$. On the other hand, $\int_1^\infty \frac{1}{t}dt$ diverges because $\lim_{x\to\infty} \int_1^x \frac{1}{t}dt = \lim_{x\to\infty} \log x$. In fact, one can show that $\int_1^\infty \frac{1}{t^p}dt$ converges to $\frac{1}{p-1}$ for $p > 1$ and diverges for $p \le 1$.

2. Consider $\int_0^\infty te^{-t^2}dt$. We will use substitution in this example. Note that

$$\int_0^x te^{-t^2}dt = \frac{1}{2}\int_0^{x^2} e^{-s}ds = \frac{1}{2}(1 - e^{-x^2}) \to \frac{1}{2} \ as\ x \to \infty.$$

3. The integral $\int_0^\infty sint dt$ diverges, because, $\int_0^x sint dt = 1 - cosx$.

We now derive some convergence tests for improper integrals. These tests are similar to those used for series. We do not present the proofs of the following three results as they are similar to the proofs of the corresponding results for series.

Theorem 17.1 : *Suppose f is integrable on $[a, x]$ for all $x > a$ where $f(t) \ge 0$ for all $t > a$. If there exists $M > 0$ such that $\int_a^x f(t)dt \le M$ for all $x \ge a$ then $\int_a^\infty f(t)dt$ converges.*

This result is similar to the result: *If $a_n \ge 0$ for all n and the partial sum $S_n \le M$ for all n, then $\sum a_n$ converges.* The proofs are also similar. One uses the above theorem to prove the following theorem which is analogous to the comparison test of series.

In the following two results we assume that f and g are integrable on $[a, x]$ for all $x > a$.

Theorem 17.2 : (Comparison test) *Suppose $0 \le f(t) \le g(t)$ for all $t > a$. If $\int_a^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ converges.*

Examples : 1. The improper integral $\int_1^\infty \frac{cos^2 t}{t^2}dt$ converges, because $0 \le \frac{cos^2 t}{t^2} \le \frac{1}{t^2}$.

2. The improper integral $\int_1^\infty \frac{2+sint}{t}dt$ diverges, because $\frac{2+sint}{t} \ge \frac{1}{t} > 0$ for all $t > 1$.

Theorem 17.3 : (Limit Comparison Test(LCT)) *Suppose $f(t) \ge 0$ and $g(t) > 0$ for all $x > a$. If $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = c$ where $c \ne 0$, then both the integrals $\int_a^\infty f(t)dt$ and $\int_a^\infty g(t)dt$ converge or both diverge. In case $c = 0$, then convergence of $\int_a^\infty g(t)dt$ implies convergence of $\int_a^\infty f(t)dt$.*

Examples : 1. The integral $\int_1^\infty sin\frac{1}{t}dt$ diverges by LCT, because $\frac{sin\frac{1}{t}}{\frac{1}{t}} \to 1$ as $t \to \infty$.

2. For $p \in \mathbb{R}$, $\int_1^\infty e^{-t}t^p dt$ converges by LCT because $\frac{e^{-t}t^p}{t^{-2}} \to 0$ as $x \to \infty$.

So far we considered the convergence of improper integrals of only non-negative functions. We will now consider any real valued functions. The following result is anticipated.

Theorem 17.4 : *If an improper integral* $\int_a^\infty \mid f(t) \mid dt$ *converges then* $\int_a^\infty f(t)dt$ *converges i.e., every absolutely convergent improper integral is convergent.*

Proof : Suppose $\int_a^\infty \mid f(t) \mid dt$ converges and $\int_a^x f(t)dt$ exists for all $x > a$. Since $0 \leq f(x) + \mid f(x) \mid \leq 2 \mid f(x) \mid$, by comparison test $\int_a^\infty (f(x) + \mid f(x) \mid)dx$ converges. This implies that $\int_a^\infty (f(x) + \mid f(x) \mid - \mid f(x) \mid)dx$ converges. □

The converse of the above theorem is not true (see Problem 2).

The following result, known as **Dirichlet test**, is very useful.

Theorem 17.5 : *Let* $f, g : [a, \infty) \to \mathbb{R}$ *be such that*

(i) f *is decreasing and* $f(t) \to 0$ *as* $t \to \infty$*,*

(ii) g *is continuous and there exists* M *such that* $\int_a^x g(t)dt \leq M$ *for all* $x > a$*.*

Then $\int_a^\infty f(t)g(t)dt$ *converges.*

We will not present the proof of the above theorem but we will use it.

Examples : Integrals $\int_\pi^\infty \frac{sint}{t}dt$ and $\int_\pi^\infty \frac{\cos t}{t}dt$ are convergent.

Improper integrals of the form $\int_{-\infty}^b f(t)dt$ are defined similarly. We say that $\int_{-\infty}^\infty f(t)dt$ is convergent if both $\int_{-\infty}^c f(t)dt$ and $\int_c^\infty f(t)dt$ are convergent for some element c in $\mathbb{R}$ and $\int_{-\infty}^\infty f(t)dt = \int_{-\infty}^c f(t)dt + \int_c^\infty f(t)dt$.

Improper integral of second kind : Suppose $\int_x^b f(t)dt$ exists for all x such that $a < x \leq b$ (the function f could be unbounded on $(a, b]$). If $\lim\limits_{x \to a^+} \int_x^b f(t)dt = M$ for some $M \in \mathbb{R}$, then we say that the improper integral (of the second kind) $\int_a^b f(t)dt$ converges to M and we write $\int_a^b f(t)dt = M$.

Example : The improper integral $\int_0^1 \frac{1}{t^p}dt$ converges for $p < 1$ and diverges for $p \geq 1$.

Comparison test and limit comparison test for improper integral of the second kind are analogous to those of the first kind. If an improper integral is a combination of both first and second kind then one defines the convergence similar to that of the improper integral of the kind $\int_{-\infty}^\infty f(t)dt$,

Problem 1: Determine the values of p for which $\int\limits_0^\infty f(x)dx$ converges where $f(x) = \frac{1-e^{-x}}{x^p}$.

Solution : Let $I_1 = \int\limits_0^1 f(x)dx$ and $I_2 = \int\limits_1^\infty f(x)dx$. We have to determine the values of p for which the integrals I_1 and I_2 converge. Now one has to see how the function $f(x)$ behaves in the respective intervals and apply the LCT. Since $\lim\limits_{x \to 0} \frac{1-e^{-x}}{x} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that I_1 is convergent iff $p - 1 < 1$, *i.e.*,$p < 2$. Similarly, I_2 is convergent (by applying LCT with $\frac{1}{x^p}$) iff $p > 1$. Therefore $\int\limits_0^\infty f(x)dx$ converges iff $1 < p < 2$.

Problem 2 : Prove that $\int_1^\infty \frac{\sin x}{x^p}dx$ converges but not absolutely for $0 < p \leq 1$.

Solution : Let $0 < p \leq 1$. By Dirichlet's Test, the integral converges. We claim that $\int\limits_1^\infty \frac{|\sin x|}{x^p}dx$ does not converge. Since, $|\sin x| \geq \sin^2 x$, we see that $|\frac{\sin x}{x^p}| \geq \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p}$. By Dirichlet's Test, $\int\limits_1^\infty \frac{\cos 2x}{2x^p}dx$ converges $\forall\ p > 0$. But $\int\limits_1^\infty \frac{1}{2x^p}$ diverges for $p \leq 1$. Hence, $\int\limits_1^\infty \mid \frac{\sin x}{x^p} \mid dx$ does not converge.