Lecture 20: Area in Polar coordinates; Volume of Solids

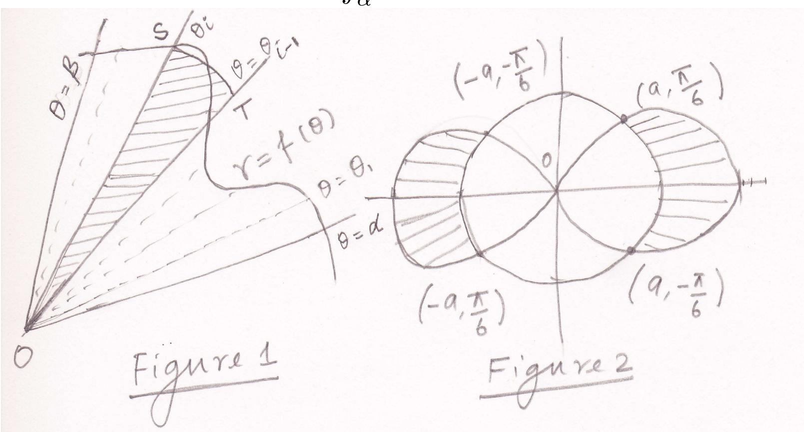
We will define the area of a plane region between two curves given by polar equations.

Suppose we are given a continuous function $r = f(\theta)$, defined in some interval $\alpha \leq \theta \leq \beta$. Let us also assume that $f(\theta) \geq 0$ and $\beta \leq \alpha + 2\pi$. We want to define the area of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ (see Figure 1). Consider a partition $P : \alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta$. Corresponding to P, we consider the circular sectors (as shown in Figure 1) with radii $f(\theta_i)'s$. Note that the area of the union of these circular sectors is approximately equal to the area of the desired region. The area of the sector OTS is

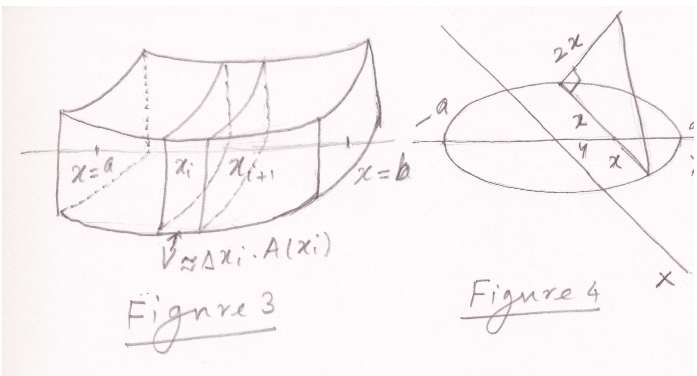
$$\left(\frac{\theta_i - \theta_{i-1}}{2\pi}\right)\pi f(\theta_i)^2 = \frac{1}{2}\triangle\theta_i f(\theta_i)^2.$$

So the sum of the areas of all the sectors is $\sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \triangle\theta_i$. Since f is continuous, this Riemann sum converges to $\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 d\theta$. In view of this we define the area bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ to be

$$\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 d\theta.$$



Example: Let us find the area inside the curve $r^2 = 2a^2 \cos 2\theta$ and not included in the circle $r = a$. The graphs of the curves are given in Figure 2. To find the points of intersection we solve the equations which imply that $2a^2 \cos 2\theta = a^2$. This implies that $\cos 2\theta = 1/2$ which in turn gives that $2\theta = \pi/3$. Therefore $\theta = \pi/6$.



Using the symmetry we can get the other points of intersection. Actually we do not need the other points of intersection to solve this problem. The area of the desired region is

$$4\int_0^{\pi/6} \frac{1}{2} r^2 d\theta = 4\int_0^{\pi/6} \frac{1}{2}(2a^2 \cos 2\theta - a^2) d\theta.$$

Volume of a solid by slicing: We will see that volumes of certain solid bodies can be defined as integral expressions.

Consider a solid which is bounded by two parallel planes perpendicular to x-axis at $x = a$ and $x = b$ (as shown in Figure 3). Let $P : x_0 = a < x_1 < x_2 < ... < x_n = b$ be a partition of $[a, b]$. For P consider the planes perpendicular to the x-axis at $x = x_i$'s. This would slice the given solid. Consider one slice of the solid that is bounded between two planes at $x = x_i$ and $x = x_{i+1}$ (see Figure 3). The volume of this slice is approximately $A(x_i)\triangle x_i$ where $A(x_i)$ is the area of the cross section of the solid made by the plane at $x = x_i$. Therefore it is natural to consider $\sum_{i=1}^{n} A(x_i)\triangle x_i$ as an approximation of the volume of the given solid. If $A(x)$ is continuous, then the above Riemann sum converges to $\int_a^b A(x)dx$. In view of the above, we define the volume of the solid to be

$$\int_a^b A(x)dx.$$

Example: The base of certain solid is the disk: $x^2 + y^2 \leq a^2$. Each section of the solid cut out by a plane perpendicular to the y-axis is an isosceles right triangle with one leg in the base of the solid (see Figure 4). Let us find the volume of the solid. By the above formula the volume is $V = \int_{-a}^{a} A(y)dy$ where $A(y)$ is the area of the cross section of the solid at y. Note that

$$A(y) = \frac{1}{2}(2x)^2 = 2(a^2 - y^2)dy.$$

Therefore, $V = \int_{-a}^{a} 2(a^2 - y^2)dy = \frac{8a^3}{3}$.

Volumes of solids of revolution: By revolving a planer region about an axis, we can generate a solid in $\mathbb{R}^3$. Such a solid is called a solid of revolution. We will use the method of finding volume by slicing and find the volume of a solid of revolution.

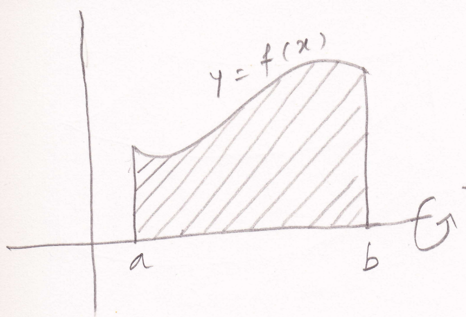


Figure 5

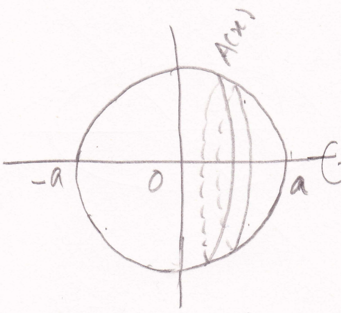


Figure 6

Consider a planer region which is bounded by the graph of a continuous function $f(x)$, $a \leq x \leq b$ where $f(x) \geq 0$ and the x-axis (see Figure 5). Suppose the region is revolved about the x-axis. The volume of the solid of revolution, by the slice method, is

$$V = \int_a^b A(x)dx = \int_a^b \pi\,(f(x))^2\,dx.$$

Example: For understanding, let us take a simple example of evaluating the volume of a sphere which is generated by the circular disk : $x^2 + y^2 \leq a^2$ by revolving it about the x-axis (see Figure 6). The volume is

$$V = \int_{-a}^{a} \pi\,(f(x))^2\,dx = \int_{-a}^{a} \pi(a^2 - x^2)dx = \frac{4}{3}\pi a^3.$$