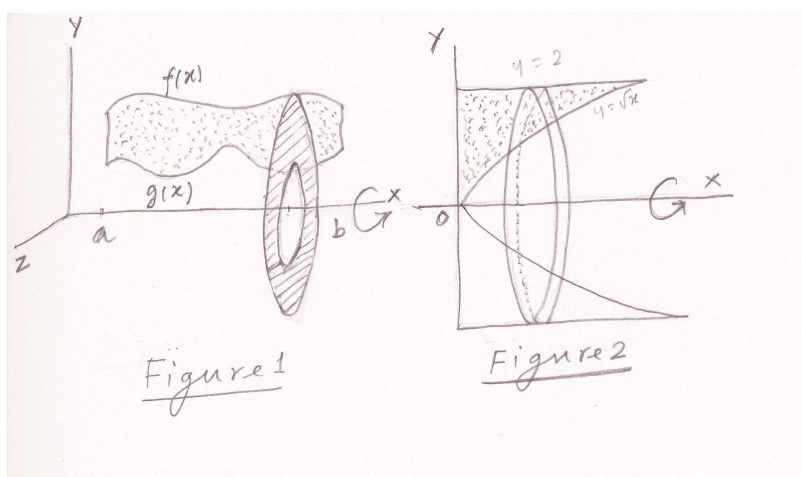


## Lecture 21: Washer and Shell Methods; Length of a plane curve

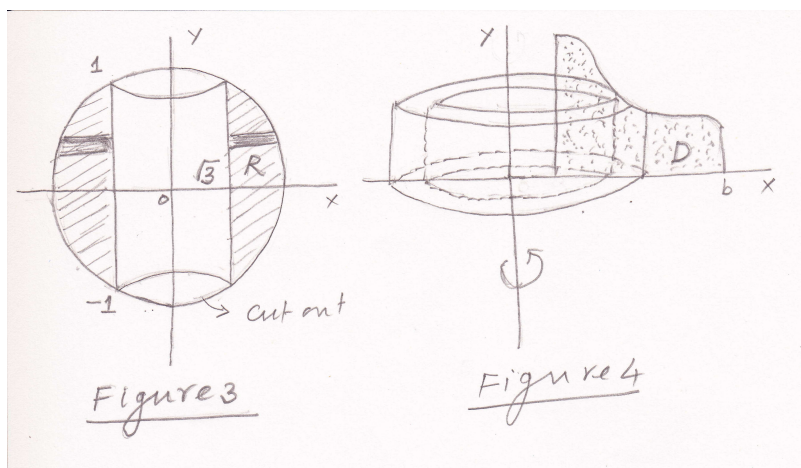
In the last lecture we considered the region between the graph of a continuous function  $f(x)$ ,  $a \leq x \leq b$  where  $f(x) \geq 0$  and the x-axis, and defined the volume of the solid generated by revolving this region about the x-axis. Instead of the x-axis we can take the graph of another function  $g(x)$  such that  $0 \leq g(x) \leq f(x)$ ,  $a \leq x \leq b$  and consider the region between the two graphs (see Figure 1). The volume of the solid generated by this region is

$$V = \int_a^b \pi(f(x)^2 - g(x)^2) dx.$$

In this case, the cross sections of the solid (i.e, the slice) perpendicular to the x-axis are washers instead of disks. Therefore this method of finding volume is called **washer method**.



**Examples:** 1. Let us find the volume generated by revolving the region bounded by  $y = \sqrt{x}$ ,  $y = 2$  and  $x = 0$  about the x-axis (see Figure 2). By washer method the volume is  $V = \int_0^4 \pi(f(x)^2 - g(x)^2) dx$  where  $f(x) = 2$  and  $g(x) = \sqrt{x}$ .



2. A round hole of radius  $\sqrt{3}$  cms is bored through the center of a solid sphere of radius 2 cms. Let us find the volume cut out (see Figure 3). We know that the volume of the whole sphere is  $\frac{4}{3}\pi r^3 = \frac{32}{3}\pi$ . We will find the volume of the shaded region given in Figure 3 using the washer method. Note that the region  $R$  is revolved about the y-axis. We get the range for  $y$  by solving the equations  $x^2 + y^2 = 4$  and  $x = \sqrt{3}$ . By washer method the volume is  $\int_{-1}^1 \pi(f(y)^2 - 3) dy = \int_{-1}^1 \pi(4 - y^2 - 3) dy = \frac{4\pi}{3}$ . The volume cut out is  $\frac{32\pi}{3} - \frac{4\pi}{3} = \frac{28\pi}{3}$ .

**Shell Method:** Recall that in the washer method we consider the slices perpendicular to the axis of revolution which look like washers. We now describe another method, called shell method, in which we will consider the slices parallel to the axis of revolution which will look like shells.

Let  $D$  be a plane region between the graph of the function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a > 0$ , and the x-axis as shown in Figure 4. We define the volume of the solid generated by revolving  $D$  about the y-axis to be

$$V = \int_a^b 2\pi x f(x) dx.$$

This formula is motivated by the following fact. For  $x \in [a, b]$ , consider the small interval  $I = [x - \Delta x/2, x + \Delta x/2]$  and the rectangle with the base  $I$  and the height  $f(x)$  (see Figure 4). If this rectangle is revolved about the y-axis, a cylindrical shell is generated and the shell's volume is  $2\pi x f(x) \Delta x$ .

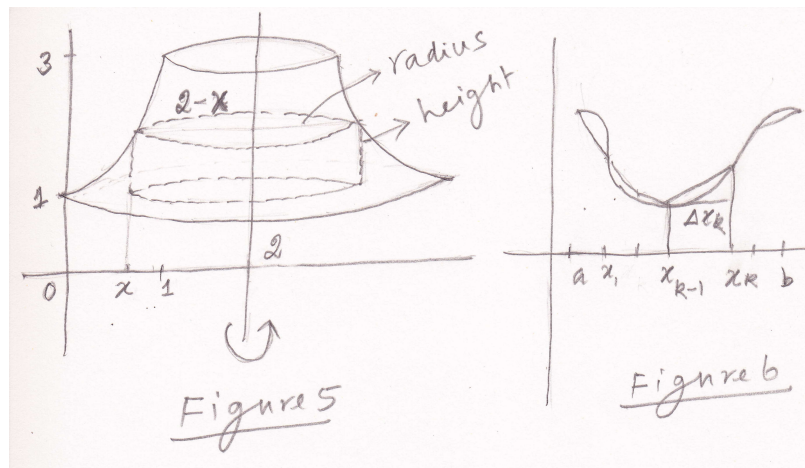
**Example :** Consider the solid obtained by revolving the region bounded by the functions

$$y = x^2 + x + 1, \quad y = 1, \quad \text{and} \quad x = 1$$

about the line  $x = 2$  (see Figure 5). Let us find the volume of the solid by the shell method. By the shell method, the volume is

$$V = \int_a^b 2\pi (\text{shell radius}) (\text{shell height}) dx.$$

For each  $x$  from 0 to 1, we consider a shell (see Figure 5). The shell radius at  $x$  is  $2 - x$  and the shell height is  $x^2 + x + 1 - 1$ . Therefore the volume is  $\int_0^1 2\pi (2 - x) (x^2 + x) dx$ .



**Remark:** In the last two lectures we defined area of some region and volume of some solid in terms of some integral expressions. Note that we have not derived these integral expressions or the formulae. We have given some justifications for the definitions. For example, in the shell method, we approximated a shell by a cylindrical shell and justified the definition or the formula. Instead of the cylindrical approximation, if we take some other approximation we may end up with some other formula which may not give the same result. So in order to find the area of some region or volume of some solid one has to use the definition (or say the formula). One should not try to approximate the given region by some other region and derive the area or the volume in terms of some integral expression.

**Length of a plane curve:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable function such that  $f'$  is continuous. Such a function is said to be smooth and its graph is said to be a smooth curve. We will describe the length of such a curve in terms of an integral expression.

Let  $P : a = x_0 < x_1 < x_2 \dots < x_n = b$  be a partition of  $[a, b]$ . Join  $(x_{k-1}, f(x_{k-1}))$  and  $(x_k, f(x_k))$  by a straight line (see Figure 6). Then we define the length  $L$  of the curve to be

$$L = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}.$$

By MVT, there exists  $c_k \in (x_{k-1}, x_k)$  such that  $f(x_k) - f(x_{k-1}) = f'(c_k)\Delta x_k$ . Therefore

$$L = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + f'(c_k)(\Delta x_k)^2} = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{1 + f'(c_k)^2} \Delta x_k = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

*Length of a plane curve given in parametric form:*

Suppose a plane curve is given by  $\{(x(t), y(t)) : t \in [a, b]\}$  where  $x$  and  $y$  are continuous function. Such a curve is called a parametric curve. For example unit circle is given by  $\{(\cos t, \sin t) : t \in [0, 2\pi]\}$ . We assume that  $x'$  and  $y'$  are continuous. Such a curve is said to be smooth. For a smooth curve the length of the curve  $L$  is defined to be

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

**Example:** Consider the curve is given by  $\{(a \cos^3 t, a \sin^3 t) : t \in [0, 2\pi]\}$  for some  $a > 0$ . The length of this curve is

$$L = \int_0^{2\pi} \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2} dt = \int_0^{2\pi} 3a |\cos t \sin t| dt = 4 \frac{3a}{2} \int_0^{\frac{\pi}{2}} \sin 2t dt.$$

*Curve given in polar form:*

If a curve is given in the polar form  $r = f(\theta)$ ,  $f(\theta) \geq 0$ ,  $\alpha \leq \theta \leq \beta$  then the curve can be represented in parametric form by taking  $x(\theta) = r(\theta) \sin \theta$  and  $y(\theta) = r(\theta) \cos \theta$ . In this case

$$L = \int_{\alpha}^{\beta} \sqrt{(d(r \cos \theta))^2 + (d(r \sin \theta))^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

**Example:** Consider the cardioid  $r = a(1 + \cos \theta)$ , where  $a > 0$ ,  $0 \leq \theta \leq 2\pi$ . The length of the curve is

$$L = \int_0^{2\pi} \sqrt{2a^2(1 + \cos \theta)} d\theta = 2a \int_0^{2\pi} |\cos(\theta/2)| d\theta = 4a \int_0^{\pi} \cos(\theta/2) d\theta = 8a.$$

**Problem 1:** The region bounded by the curve  $y = x^2 + 1$  and the line  $y = -x + 3$  is revolved about the  $X$ -axis to generate a solid. Find the volume of the solid.

**Solution:** We use the washer method. The outer radius  $r_2(x) = -x + 3$  and the inner radius  $r_1(x) = x^2 + 1$ . Therefore, the volume is

$$V = \int_{-2}^1 \pi((-x + 3)^2 - (x^2 + 1)^2) dx = 117 \frac{\pi}{3}.$$

**Problem 2:** The region in the first quadrant bounded by the parabola  $y = x^2$ , the  $Y$ -axis and the line  $y = 1$  is revolved about the line  $x = 2$  to generate a solid. Find the volume of the solid.

**Solution:** We use the shell method. Observe that shell radius is  $2 - x$  and the shell height is  $1 - x^2$ . The volume  $V = \int_0^1 2\pi(2 - x)(1 - x^2) dx = \frac{13\pi}{6}$ .

The volume can also be computed by the washer method.