Lecture 22: Areas of surfaces of revolution, Pappus's Theorems

Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(x) \geq 0$. Consider the curve C given by the graph of the function f. Let S be the surface generated by revolving this curve about the x-axis. We will define the surface area of S in terms of an integral expression.

Consider a partition $P : a = x_0 < x_1 < x_2 < ... < x_n = b$ and consider the points $P_i = (x_i, f(x_i)), i = 0, 1, 2, .., n$. Join these points by straight lines as shown in Figure 1. Consider the segment $P_{i-1}P_i$. The area A of the surface generated by revolving this segment about the x-axis is $\pi(f(x_{i-1}) + f(x_i))\ell_i$ where ℓ_i is the length of the segment $P_{i-1}P_i$. This can be verified as follows. Note that the area $A = \pi f(x_i)(\ell + \ell_i) - \pi f(x_{i-1})\ell$ (see Figure 2). Since

$$\frac{\ell}{f(x_{i-1})} = \frac{\ell + \ell_i}{f(x_i)} = \frac{\ell_i}{f(x_i) - f(x_{i-1})} = \alpha$$

for some α, the area

$$A = \pi f(x_i)\alpha f(x_i) - \pi f(x_{i-1})\alpha f(x_{i-1}) = \pi\alpha(f(x_i) + f(x_{i-1}))(f(x_i) - f(x_{i-1})) = \pi\ell_i(f(x_{i-1}) + f(x_i)).$$

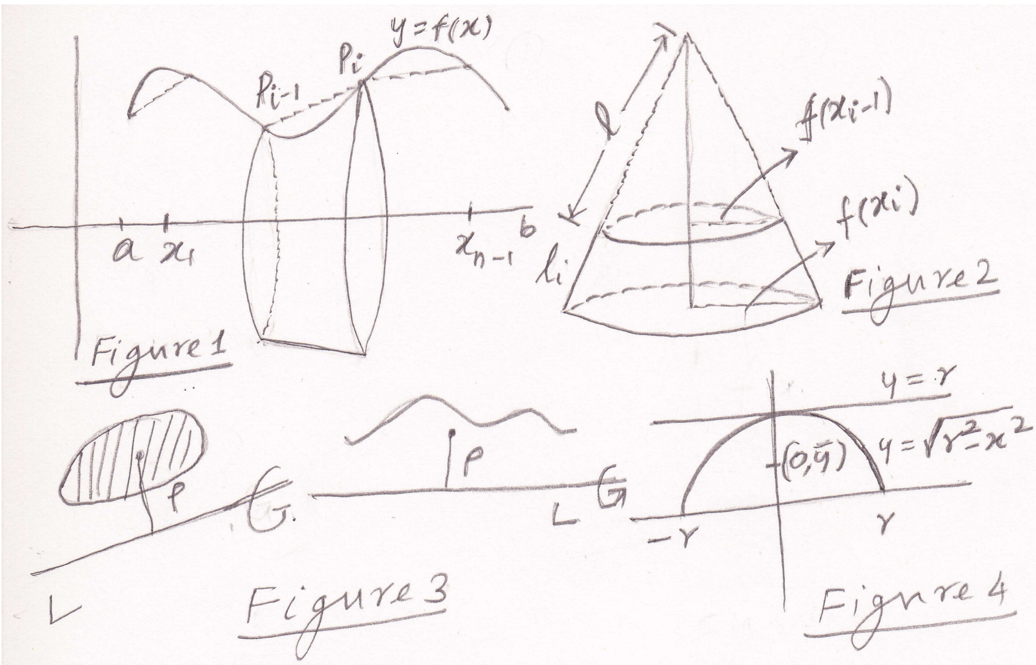
The sum of the areas of the surfaces generated by the line segments is

$$\sum_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i))\ell_i = \sum_{i=1}^{n} \pi f(x_{i-1})\sqrt{(\triangle x_i)^2 + (\triangle y_i)^2} + \sum_{i=1}^{n} \pi f(x_i)\sqrt{(\triangle x_i)^2 + (\triangle y_i)^2}$$

where $\triangle y_i = f(x_i) - f(x_{i-1})$. If f' is continuous, one can show that each of the sum given in the RHS of the above equation converges to $\int_a^b \pi f(x)\sqrt{1 + (f'(x))^2}\, dx$ as $\| P \| \to 0$. In view of this we define the surface area generated by revolving the curve about the x-axis to be

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

In case $f(x) \leq 0$, the formula for the area is $\int_a^b 2\pi \mid f(x) \mid \sqrt{1 + (f'(x))^2}\, dx$.



Example: Let us find the area of the surface generated by revolving the curve $y = \frac{1}{2}(x^2 + 1), 0 \leq x \leq 1$ about the y-axis. Here the function y is increasing hence it is one-one and onto. Hence we can write x in terms of y : $x = g(x) = \sqrt{2y-1}$. In this case the formula is $\int_a^b 2\pi \mid g(y) \mid \sqrt{1+(g'(y))^2}\, dy$ where $a = 1/2$ and $b = 1$.

Parametric case: If the curve is given in the parametric form $\{(x(t), y(t)) : t \in [a,b]\}$, and x' and y' are continuous, then the surface area generated is

$$\int_a^b 2\pi\rho(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

where $\rho(t)$ is the distance between the axis of revolution and the curve.

Example : The curve $x = t+1,\ y = \frac{t^2}{2} + t,\ 0 \le t \le 4$ is rotated about the y-axis. Let us find the surface area generated. The surface area is $\int_0^4 2\pi \mid t+1 \mid \sqrt{1+(1+t)^2} dt$.

Polar case: If the curve is given in the polar form, the surface area generated by revolving the curve about the x-axis is

$$\int_a^b 2\pi y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_a^b 2\pi\ r(\theta)\sin\theta\ \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example : The lemniscate $r^2 = 2a^2\cos 2\theta$ is rotated about the x-axis. Let us find the area of the surface generated. A simple calculation shows that $\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \frac{2a^2}{r}$. The curve is given in the notes of the previous lecture. The surface area is $2\int_0^{\frac{\pi}{4}} 2\pi r \sin\theta \frac{2a^2}{r} d\theta = 8\pi a^2(1 - \frac{1}{\sqrt{2}})$.

Pappus's Theorems: There are two results of Pappus which relate the centroids to surfaces and solids of revolutions. The first result relates the centroid of a plane region with the volume of the solid of revolution generated by it.

Theorem: *Let R be a plane region. Suppose R is revolved about the line L which does not cut through the interior of R, then the volume of the solid generated is*

$$V = 2\pi\rho A$$

where ρ is the distance from the axis of revolution to the centroid and A is the area of the region R(see Figure 3).

Note that in the above formula $2\pi\rho$ is the distance traveled by the centroid during the revolution. The second result relates the centroid of a plane curve with the area of the surface of revolution generated by the curve.

Theorem: *Let C be a plane curve. Suppose C is revolved about the line L which does not cut through the interior of C, then the area of the surface generated is*

$$S = 2\pi\rho L$$

where ρ is the distance from the axis of revolution to the centroid and L is the length of the curve C (see Figure 3).

Example: Use a theorem of Pappus to find the centroid of the semi circular arc $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$. If the arc is revolved about the line $y = r$, find the volume of the surface area generate.

Solution: We know the surface area generated by the curve $4\pi r^2$ (see Figure 4). Let the centroid of the curve be $(0, \overline{y})$. By Pappus theorem $4\pi r^2 = 2\pi\overline{y}\pi r$ which implies that $\overline{y} = \frac{2r}{\pi}$. Again by Pappus theorem, the area of the surface generated by revolving the curve around $y = r$ is $2\pi(r - \overline{y})\pi r = 2\pi r^2(\pi - 2)$.