Lecture 3 : Cauchy Criterion, Bolzano-Weierstrass Theorem

We have seen one criterion, called monotone criterion, for proving that a sequence converges without knowing its limit. We will now present another criterion.

Suppose that a sequence (x_n) converges to x. Then for $\epsilon > 0$, there exists an N such that $|x_n - x| < \epsilon/2$ for all $n \geq N$. Hence for $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon.$$

Thus we arrive at the following conclusion:

If a sequence (x_n) converges then it satisfies the **Cauchy's criterion:** *for* $\epsilon > 0$, *there exists* N *such that* $|x_n - x_m| < \epsilon$ *for all* $n, m \geq N$.

If a sequence converges then the elements of the sequence get close to the limit as n increases. In case of a sequence satisfying Cauchy criterion the elements get close to each other as m, n increases.

We note that a sequence satisfying Cauchy criterion is a bounded sequence (verify!) with some additional property. Moreover, intuitively it seems as if it converges. We will show that a sequence satisfying Cauchy criterion does converge. We need some results to prove this.

Theorem 3.1 : ***(Nested interval Theorem)*** *For each* n, *let* $I_n = [a_n, b_n]$ *be a (nonempty) bounded interval of real numbers such that*

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. *Then* $\bigcap\limits_{n=1}^{\infty} I_n$ *contains only one point.*

Proof (*): Note that the sequences (a_n) and (b_n) are respectively increasing and decreasing sequences; moreover both are bounded. Hence both converge, say $a_n \to a$ and $b_n \to b$. Then $a_n \leq a$ and $b \leq b_n$ for all $n \in N$. Since $b - a = \lim(b_n - a_n) = 0$, $a = b$. Since $a_n \leq b_n$ for all n we have $a \in \bigcap\limits_{n=1}^{\infty} I_n$. Clearly if $x \neq a$ then x does not belong to $\bigcap\limits_{n=1}^{\infty} I_n$. $\square$

Subsequences : Let (x_n) be a sequence and let (n_k) be any sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$. *The sequence* (x_{n_k}) *is called a subsequence.* Note that here k varies from 1 to ∞.

A subsequence is formed by deleting some of the elements of the sequence and retaining the remaining in the same order. For example, $(\frac{1}{k^2})$ and $(\frac{1}{2^k})$ (k varies from 1 to ∞) are subsequences of $(\frac{1}{n})$, where $n_k = k^2$ and $n_k = 2^k$.

Sequences $(1, 1, 1, \ldots)$ and $(0, 0, 0, \ldots)$ are both subsequences of $(1, 0, 1, 0, \ldots)$. From this we see that a given sequence may have convergent subsequences though the sequence itself is not convergent. We note that every sequence is a subsequence of itself and if $x_n \to x$ then every subsequence of (x_n) also converges to x.

The following theorem which is an important result in calculus, is a consequence of the nested interval theorem.

Theorem 3.2 ***(Bolzano-Weierstrass theorem)*****:** *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof (*):*(Sketch).* Let (x_n) be a bounded sequence such that the set $\{x_1, x_2, \cdots\} \subset [a, b]$. Divide this interval into two equal parts. Let I_1 be that interval which contains an infinite number of elements (or say terms) of (x_n). Let x_{n_1} be one of the elements belonging to the interval I_1. Divide I_1 into two equal parts and let I_2 be that interval which contains an infinite number of elements. Choose a point x_{n_2} in I_2 such that $n_2 > n_1$. Keep dividing the intervals I_k, to generate I_k's and x_{n_k}'s. By nested interval theorem $\bigcap_{k=1}^{\infty} I_k = \{x\}$, for some $x \in [a, b]$. It is easy to see that the subsequence (x_{n_k}) converges to x. □

Theorem 3.3: *If a sequence (x_n) satisfies the Cauchy criterion then (x_n) converges.*

Proof (*): Let (x_n) satisfy the Cauchy criterion. Since (x_n) is bounded, by the previous theorem there exists a subsequence (x_{n_k}) convergent to some x_0. We now show that $x_n \to x_0$. Let $\epsilon > 0$. Since (x_n) satisfies the Cauchy criterion,

$$\text{there exists } N_1 \; s.t. \; \mid x_n - x_m \mid \leq \epsilon/2 \quad \text{for all } n, m \geq N_1 \; \text{......} \; (1)$$

Since $x_{n_k} \longrightarrow x_0$,

$$\text{there exists } N_2 \; s.t. \; \mid x_{n_k} - x_0 \mid \leq \epsilon/2 \; \text{ for all } \; n_k \geq N_2 \; \text{.......} \; (2)$$

Let $N = max\{N_1, N_2\}$. For $n \geq N$, choose some $n_k \geq N$, then by (1) and (2) we have

$$\mid x_n - x_0 \mid \leq \mid x_n - x_{n_k} \mid + \mid x_{n_k} - x_0 \mid \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves that $x_n \to 0$. □

Checking the Cauchy criterion directly from the definition is very difficult. The following result will help us to check the Cauchy criterion.

Problem 3.4: *Suppose $0 < \alpha < 1$ and (x_n) is a sequence satisfying the* ***contractive condition:***

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n| \qquad n = 1, 2, 3, \ldots.$$

Then show that (x_n) satisfies the Cauchy criterion.

Solution : Note that $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n| \leq \alpha^2|x_n - x_{n-1}| \leq \cdots \leq \alpha^n|x_2 - x_1|$.

For $n > m$, $|x_n - x_m| \leq (\alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{m-1})|x_2 - x_1| \leq \frac{\alpha^m}{1-\alpha}|x_2 - x_1| \to 0$ as $m \to \infty$.

Thus (x_n) satisfies the Cauchy criterion. □

Examples 3.5: 1. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$. Then

$$\mid x_{n+2} - x_{n+1} \mid = \frac{1}{(2 + x_{n+1})(2 + x_n)} \mid x_n - x_{n+1} \mid < \frac{1}{4} \mid x_n - x_{n+1} \mid .$$

Therefore (x_n) satisfies the contractive condition with $\alpha = 1/4$ and hence it satisfies the Cauchy criterion. Therefore it converges. Suppose $x_n \to l$. Then $l = \frac{1}{2+l}$. Find l !.

Remark : *Whenever we use the result given in the above exercise, we have to show that the number α that we get, satisfies $0 < \alpha < 1$.*

2. If $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ then $\mid x_{n+2} - x_{n+1} \mid < \frac{1}{4} \mid x_n - x_{n+1} \mid$ (verify !). Therefore the sequence (x_n) converges.