Lecture 34 : Change of Variable in a Triple Integral; Area of a Parametric Surface

The change of variable formula for a double integral can be extended to triple integrals. We will straightaway present the formula.

Formula: $\iiint\limits_S f(x,y,z)dxdydz = \iiint\limits_T f[X(u,v,w),Y(u,v,w),Z(u,v,w)] \mid J(u,v,w) \mid dudvdw$

where the Jacobian determinant $J(u,v,w)$ is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \\ \frac{\partial X}{\partial w} & \frac{\partial Y}{\partial w} & \frac{\partial Z}{\partial w} \end{vmatrix}.$$

The above formula is valid under some assumptions which are similar to the assumptions we had for the two dimensional case.

Special cases : 1. Cylindrical coordinates. In this case the variables x, y and z are changed to r, θ and z by the following three equations:

$$x = X(r,\theta) = r\cos\theta,\ y = Y(r,\theta) = r\sin\theta \quad \text{and} \quad z = z.$$

We assume that $r > 0$ and θ lies in $[0, 2\pi)$ or $\theta_0 \le \theta < \theta_0 + 2\pi$ for some θ_0 as in the double integral case. We have basically replaced x and y by their polar coordinates in the xy plane and left z unchanged. The Jacobian is

$$J(u,v,z) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Therefore the change of variable formula is $\iiint\limits_S f(x,y,z)dxdydz = \iiint\limits_T f(r\cos\theta, r\sin\theta, z)\ rdrd\theta dz.$

Example 1: Let us evaluate $\iiint_D (z^2x^2+z^2y^2)dxdydz$ where D is the region determined by $x^2+y^2 \le 1, -1 \le z \le 1$. Note that we can describe D in cylindrical coordinates: $0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ -1 \le z \le 1$. Therefore,

$$\iiint\limits_D (z^2x^2 + z^2y^2)dxdydz = \int\limits_{-1}^{1}\int\limits_{0}^{2\pi}\int\limits_{0}^{1}(z^2r^2)rdrd\theta dz = \int\limits_{-1}^{1}\int\limits_{0}^{2\pi} z^2\frac{r^4}{4}\Big|_{r=0}^{1} d\theta dz = \int\limits_{-1}^{1}\frac{2\pi}{4}z^2dz = \frac{\pi}{3}.$$

2 Spherical Coordinates: Suppose (x,y,z) be a point $\mathbb{R}^3$. We will represent this point in terms of spherical coordinates (ρ,θ,ϕ). The coordinates ρ, θ and ϕ are defined below.

Given a point (x,y,z), let $\rho = \sqrt{x^2+y^2+z^2}$ and ϕ is the angle that the position vector $xi + yj + zk$ makes with the (positive side of the) z-axis. The coordinate of z is given by $z = \rho\cos\phi$. To represent x and y in terms of spherical coordinates, represent x and y by polar coordinates in the xy-plane: $x = r\cos\theta$ and $y = r\sin\theta$. Since $r = \rho\sin\phi$, the point (x,y,z) is represented in terms of the spherical coordinates (ρ,θ,ϕ) as follows:

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

We keep $\rho > 0,\ 0 \le \theta < 2\pi$ and $0 \le \phi < \pi$ to get a one-one mapping. The Jacobian determinant is $J(\rho,\theta,\phi) = -\rho^2\sin\phi$. Since $\sin\phi \ge 0$, we have $\mid J(\rho,\theta,\phi) \mid = \rho^2\sin\phi$ and the change of variable formula is

$$\iiint\limits_S f(x,y,z)dxdydz = \iiint\limits_T f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho d\theta d\phi.$$

Example 2: Let $D = \{(x,y,z) : x^2+y^2+z^2 \le 4a^2,\ z \ge a\}$. Let us evaluate $\iiint\limits_D \frac{z}{(x^2+y^2+z^2)^{3/2}}dV$. We will use the spherical coordinates to solve this problem. If we allow ϕ to vary independently, then ϕ varies from 0 to $\frac{\pi}{3}$ (see Figure 2). If we fix ϕ and allow θ to vary from 0 to 2π then we obtain a surface of a cone (see Figure 1). Since only a part of the cone is lying in the given region, for a fixed ϕ and θ, ρ varies from $a\sec\phi$ to $2a$ (see Figure 1). Therefore the integral is

$$\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_{a\sec\phi}^{2a} \frac{\cos\phi}{\rho^2} \mid J(\rho,\theta,\phi) \mid d\rho d\theta d\phi = 2\pi\int_0^{\frac{\pi}{3}}(2a\sin\phi\cos\phi - a\sin\phi)d\phi = \frac{\pi a}{2}.$$

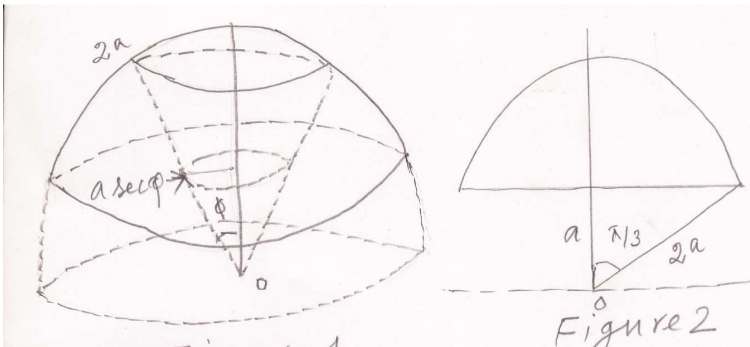

Figure 1

Figure 2

Parametric Surfaces: We defined a parametric curve in terms of a continuous vector valued function of one variable. We will see that a continuous vector valued function of two variables is associated with a surface, called parametric surface.

Let T be a region in $\mathbb{R}^2$ and $r(u,v) = X(u,v)i + Y(u,v)j + Z(u,v)k$ be a continuous function on T. The range of r, $\{r(u,v) : (u,v) \in T\}$ is called a parametric surface (with the parameter domain T and the parameters u and v). We assume that the map r is one-one in the interior of T so that the surface does not cross itself. Sometimes the surface defined by $r(u,v)$ is also expressed as

$$x = X(u,v),\ y = Y(u,v),\ z = Z(u,v) \quad \text{where} \quad (u,v) \in T$$

and the above equations are called parametric equations of the surface.

Examples: 1. For a constant $a > 0$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$ the equations $x = a\sin\phi\cos\theta,\ y = a\sin\phi\sin\theta,\ \ z = a\cos\phi$ represent a sphere. Here the parameters are θ and ϕ.

2. For a fixed a, $\infty < t < \infty$, $0 \le \theta \le 2\pi$, the equations $x = a\cos\theta,\ y = a\sin\theta,\ z = t$ represent a cylinder. Here the parameters are t and θ.

3. A cone is represented by $r(u,v) = \rho\sin\alpha\cos\theta i + \rho\sin\alpha\sin\theta j + \rho\cos\alpha k$ where $\rho \ge 0$, $0 \le \theta \le 2\pi$ and α is fixed. Here the parameters are ρ and θ.

Area of a Parametric Surface: Let $S = r(u,v)$ be a parametric surface defined on a parameter domain T. Suppose r_u and r_v are continuous on T and $r_u \times r_v$ is never zero on T. Then the area of S, denoted by $a(S)$, is defined by the double integral

$$a(S) = \iint_T \parallel \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \parallel dudv.$$

The formula can be justified as follows. Consider a small rectangle $\triangle A$ in T with the sides on the lines $u = u_0, u = u_0 + \triangle u, v = v_0$ and $v = v_0 + \triangle v$. Consider the corresponding patch in S, that is $r(\triangle A)$. Note that the sides of $\triangle A$ are mapped to the boundary curves of the patch $r(\triangle A)$ by the map r. The vectors $r_u(u_0,v_0)$ and $r_v(u_0,v_0)$ are tangents to the boundary curves of $r(\triangle A)$ meeting at $r(u_0,v_0)$. We now approximate the surface patch $r(\triangle A)$ by the parallelogram whose sites are determined by the vectors $\triangle u r_u$ and $\triangle v r_v$. The area of this parallelogram is $\mid r_u \times r_v \mid \triangle u \triangle v$. This will lead to the Riemann sum corresponding to the double integral $\iint_T \parallel \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \parallel dudv$.