Lecture 38: Stokes' Theorem

As mentioned in the previous lecture Stokes' theorem is an extension of Green's theorem to surfaces. Green's theorem which relates a double integral to a line integral states that

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C M dx + N dy$$

where D is a plane region enclosed by a simple closed curve C. Stokes' theorem relates a surface integral to a line integral. We first rewrite Green's theorem in a different form as mentioned in the previous lecture. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = M(x, y)i + N(x, y)j$. Then $curlF = \nabla \times F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k$. Therefore Green's theorem is stated as follows:

$$\iint_D (curlF) \cdot k \; dxdy = \oint_C F \cdot dR. \tag{1}$$

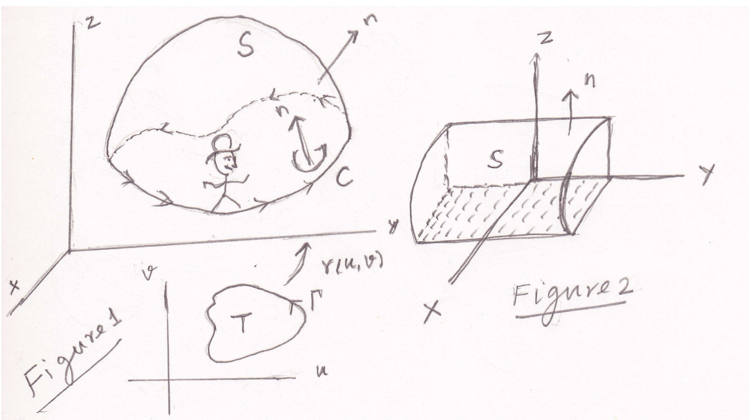
We will extend this form of Green's theorem to a vector field F defined on a surface S having the boundary curve C. We need the following definitions.

Smooth Surface: Let $S = r(u, v)$ be a parametric surface defined on a parameter domain T. We say that S is smooth if r_u and r_v are continuous on T and $r_u \times r_v$ is never zero on T. A level surface S defined by $f(x, y, z) = c$ is said to be smooth if ∇f is continuous and never zero on S

We have already seen that the vectors $r_u \times r_v$ and ∇f are normals to the parametric surface and the level surface respectively.

Orientable Surface: A smooth surface is said to be orientable if there exists a continuous unit normal vector function defined at each point of the surface.

Basically a surface is oriented by orienting its normals in a continuous manner. In practice, we consider an orientable surface as a smooth surface with two sides. For example spheres, planes and paraboloids are orientable surfaces. The Möbious strip is not an orientable surface and is not one sided.



We will be dealing with only orientable surfaces. Let S be an orientable surface with the boundary curve C (see Figure 1). Since S has two sides, consider a side of the surface, that is, consider a normal $\mathbf{n}$ to the surface S. With respect to this normal (that is corresponding to a side of the surface), we define an orientation on C. We need this orientation to evaluate the line integral involved in Stokes' theorem.

Orientation on the boundary w.r.to a normal: Orientation is formally defined as follows. Suppose, for example, $S := r(u, v)$ is a parametric (orientable) surface defined by a one-one map r on the parameter domain T. Consider the unit normal $\mathbf{n} = \frac{r_u \times r_v}{\|r_u \times r_v\|}$ of the surface S. Let Γ be the boundary of T and $C = r(\Gamma)$ (see Figure 1). If we assume that Γ is oriented in the counterclockwise direction then we get an orientation for C inherited from Γ through the mapping r. This orientation for C

is considered to be the orientation w.r.to **n**.

In practice, we get the orientation of the boundary curve (w.r.to a given **n**) using the right-hand rule: If we wrap around the surface with the right hand by pointing the thumb in the direction of the normal then roughly the fingers of the right hand are pointing towards the orientation of the curve. One can also use the following method. Consider a person walking on C by keeping his or her head towards the direction of the normal and the surface to the left. The orientation of the curve is the direction in which the person is walking on C (see Figure 1). For example, if S is a plane region and $\mathbf{n} = k$ then the orientation of C w.r.to **n** is the counterclockwise direction. Let us state Stokes' theorem.

Theorem: Let S be a (piecewise) smooth orientable surface and a piecewise simple closed curve C be its boundary. Let $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$ be a vector field such that P, Q and R are continuous and have continuous first partial derivatives in an open set containing S. If **n** is a unit normal to S, then

$$\iint_S (curlF) \cdot n \; d\sigma = \oint_C F \cdot dR \qquad (2)$$

where the line integral is taken around C in the direction of the orientation of C w.r.to **n**.

Remarks: 1. The value of the surface integral in (2) depends only on the boundary C. This means that the shape of the surface is irrelevant. So Stokes theorem is an analog of the 2nd FTC.

2. If S is a plane region, then the identity given in (2) reduces to the identity given in (1). Therefore Stokes' theorem is consider to be a direct extension of Green's theorem.

3. For a closed oriented surface such as sphere or donut, there is no boundary and in this case $\iint_S (curlF) \cdot n \; d\sigma = 0$. For example for a sphere, this can be seen by cutting the sphere into two hemispheres. If we apply Stokes' theorem to each and add the resulted identities, the two boundary integrals cancel and we get what we claimed.

4. Stokes' theorem can also be extended to a smooth surface which has more than one simple closed curve forming the boundary of the surface.

Problem: Let S be the part of the cylinder $z = 1 - x^2,\ 0 \le x \le 1,\ -2 \le y \le 2$. Let C be the boundary curve of the surface S. Let $F(x, y, z) = yi + yj + zk$. Find the unit outer normal to S, *curl F* and evaluate $\oint_C F \cdot dR$ where C is oriented counterclockwise as viewed from above the surface.

Solution: The surface is given by $z = 1 - x^2$ (see Figure 2). This surface can be considered as a graph of the function $f(x, y) = 1 - x^2$ or a parametric surface $r(x, y) = (x, y, f(x, y))$. An unit normal is $\frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{-f_x - f_y + k}{\sqrt{1+f_x^2+f_y^2}} = \frac{2xi+k}{\sqrt{1+4x^2}}$. This has to be the outer normal, because if we calculate at $(0, 1)$, it coincides with the outer normal k. Simple calculation shows that $curlF = -k$. We will use Stokes' theorem to solve this problem. Note that $curlf \cdot \mathbf{n} = \frac{-1}{\sqrt{1+4x^2}}$. By Stokes' theorem

$$\oint_C F \cdot dR = \iint_S \frac{-1}{\sqrt{1+4x^2}} \; d\sigma = \iint_R \frac{-1}{\sqrt{1+4x^2}} \sqrt{1 + f_x^2 + f_y^2} dxdy = \iint_R -1 dxdy = \int_0^1 \int_{-2}^2 -1 dydx. \; \square$$

Remarks: 1. The above problem can be done directly by calculating the line integral. In that case we have to parametrize the boundary which consists of four smooth curves. One has to be careful with the direction in each piece. The calculation also becomes lengthy.

2. We can also consider the surface given in the previous problem as a level surface defined by $f(x, y, z) = z - 1 + x^2 = 0$. In this case an unit normal is $\mathbf{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{2xi+k}{\sqrt{1+4x^2}}$.