# Lecture 6 : Rolle's Theorem, Mean Value Theorem

The reader must be familiar with the classical maxima and minima problems from calculus. For example, the graph of a differentiable function has a horizontal tangent at a maximum or minimum point. This is not quite accurate as we will see.

**Definition :** Let $f : I \to \mathbb{R}$, $I$ an interval. A point $x_0 \in I$ is a local maximum of $f$ if there is a $\delta > 0$ such that $f(x) \leq f(x_0)$ whenever $x \in I \cap (x_0 - \delta, x_0 + \delta)$. Similarly, we can define local minimum.

**Theorem 6.1 :** *Suppose $f : [a, b] \to \mathbb{R}$ and suppose $f$ has either a local maximum or a local minimum at $x_0 \in (a, b)$. If $f$ is differentiable at $x_0$ then $f'(x_0) = 0$.*

**Proof**: Suppose $f$ has a local maximum at $x_0 \in (a, b)$. For small (enough) $h$, $f(x_0 + h) \leq f(x_0)$. If $h > 0$ then

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Similarly, if $h < 0$, then

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

By elementary properties of the limit, it follows that $f'(x_0) = 0$. □

We remark that the previous theorem is not valid if $x_0$ is $a$ or $b$. For example, if we consider the function $f : [0, 1] \to \mathbb{R}$ such that $f(x) = x$, then $f$ has maximum at 1 but $f'(x) = 1$ for all $x \in [0, 1]$.

The following theorem is known as *Rolle's theorem* which is an application of the previous theorem.

**Theorem 6.2 :** *Let $f$ be continuous on $[a, b]$, $a < b$, and differentiable on $(a, b)$. Suppose $f(a) = f(b)$. Then there exists $c$ such that $c \in (a, b)$ and $f'(c) = 0$.*

**Proof**: If $f$ is constant on $[a, b]$ then $f'(c) = 0$ for all $c \in [a, b]$. Suppose there exists $x \in (a, b)$ such that $f(x) > f(a)$. (A similar argument can be given if $f(x) < f(a)$). Then there exists $c \in (a, b)$ such that $f(c)$ is a maximum. Hence by the previous theorem, we have $f'(c) = 0$. □

**Problem 1 :** *Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one (real) root.*

*Solution :* Let $f(x) = x^{13} + 7x^3 - 5$. Then $f(0) < 0$ and $f(1) > 0$. By the IVP there is at least one positive root of $f(x) = 0$. If there are two distinct positive roots, then by Rolle's theorem there is some $x_0 > 0$ such that $f'(x_0) = 0$ which is not true. Moreover, observe that $f(x) < 0$ for $x < 0$.

**Problem 2 :** *Let $f$ and $g$ be functions, continuous on $[a, b]$, differentiable on $(a, b)$ and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a, b)$ such that $g'(c)f(c) + f'(c) = 0$.*

*Solution :* Define $h(x) = f(x)e^{g(x)}$. Here, $h(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Since $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$.

Since $h'(x) = [f'(x) + g'(x)f(x)]e^{g(x)}$ and $e^\alpha \neq 0$ for any $\alpha \in \mathbb{R}$, we see that $f'(c) + g'(c)f(c) = 0$.

A geometric interpretation of the above theorem can be given as follows. If the values of a differentiable function $f$ at the end points $a$ and $b$ are equal then somewhere between $a$ and $b$ there is a horizontal tangent. It is natural to ask the following question. If the value of $f$ at the end points $a$ and $b$ are not the same, is it true that there is some $c \in [a, b]$ such that the tangent line at $c$ is parallel to the line connecting the endpoints of the curve? The answer is yes and this is essentially the Mean Value Theorem.

**Theorem 6.3 : (Mean Value Theorem)** *Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

**Proof**: Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a).$$

Then $g(a) = g(b) = f(a)$. The result follows by applying Rolle's Theorem to $g$. □

The mean value theorem is an important result in calculus and has some important applications relating the behaviour of $f$ and $f'$. For example, if we have a property of $f'$ and we want to see the effect of this property on $f$, we usually try to apply the mean value theorem. Let us see some examples.

**Example 1 :** *Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then $f$ is constant if and only if $f'(x) = 0$ for every $x \in [a, b]$.*

**Proof :** Suppose that $f$ is constant, then from the definition of $f'(x)$ it is immediate that $f'(x) = 0$ for every $x \in [a, b]$.

To prove the converse, let $a < x \leq b$. By the mean value theorem there exists $c \in (a, x)$ such that $f(x) - f(a) = f'(c)(x - a)$. Since $f'(c) = 0$, we conclude that $f(x) = f(a)$, that is $f$ is constant. (*If we try to prove the converse directly from the definition of $f'(x)$ we will be in trouble.)* □

**Example 2 :** *Suppose $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.*

*(i) If $f'(x) \neq 0$ for all $x \in (a, b)$, then $f$ is one-one (i.e, $f(x) \neq f(y)$ whenever $x \neq y$).*

*(ii) If $f'(x) \geq 0$ (resp. $f'(x) > 0$) for all $x \in (a, b)$ then $f$ is increasing (resp. strictly increasing) on $[a, b]$.* (We have a similar result for decreasing functions.)

**Proof :** Apply the mean value theorem as we did in the previous example. (Note that $f$ can be one-one but $f'$ can be 0 at some point, for example take $f(x) = x^3$ and $x = 0$.)

**Problem 3 :** *Use the mean value theorem to prove that $\mid sinx - siny \mid \leq \mid x - y \mid$ for all $x, y \in \mathbb{R}$.*

*Solution :* Let $x, y \in \mathbb{R}$. By the mean value theorem $sinx - siny = cosc\ (x - y)$ for some $c$ between $x$ and $y$. Hence $\mid sinx - siny \mid \leq \mid x - y \mid$.

**Problem 4 :** *Let $f$ be twice differentiable on $[0, 2]$. Show that if $f(0) = 0, f(1) = 2$ and $f(2) = 4$, then there is $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.*

*Solution :* By the mean value theorem there exist $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that

$$f'(x_1) = f(1) - f(0) = 2 \quad \text{and} \quad f'(x_2) = f(2) - f(1) = 2.$$

Apply Rolle's theorem to $f'$ on $[x_1, x_2]$.

**Problem 5 :** *Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.*

*Solution :* Let $g(x) = f(x) - x$ on $[-a, a]$. Note that $g'(x) \leq 0$ on $(-a, a)$. Therefore, $g$ is decreasing. Since $g(a) = g(-a) = 0$, we have $g = 0$.

This problem can also be solved by applying the MVT for $g$ on $[-a, x]$ and $[x, a]$.