Lecture 7 : Cauchy Mean Value Theorem, L'Hospital Rule

L'Hospital (pronounced Lopeetal) Rule is a useful method for finding limits of functions. There are several versions or forms of L'Hospital rule. Let us start with one form called $\frac{0}{0}$ form which deals with $\lim_{x\to x_0}\frac{f(x)}{g(x)}$, where $\lim_{x\to x_0} f(x) = 0 = \lim_{x\to x_0} g(x)$.

Theorem 1: (L'Hospital Rule) *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Suppose $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$. Then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.*

Proof : Note that $\frac{f'(x_0)}{g'(x_0)} = \frac{\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}} = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0}\frac{f(x)}{g(x)}$. □

Example : The condition $f(x_0) = g(x_0) = 0$ is essential in the previous result. For example, $\lim_{x\to 0}\frac{x+17}{2x+3} = \frac{17}{3}$ but $\frac{f'(0)}{g'(0)} = \frac{1}{2}$.

The following result is a stronger version of the previous result.

Theorem 2 : (L'Hospital Rule) *Let $f, g : [x_0, b) \to \mathbb{R}$. Suppose $f(x_0) = g(x_0) = 0$ and f, g are differentiable on (x_0, b). Let $g'(x) \neq 0$ for all $x \in (x_0, b)$. Then*

$$\lim_{x\to x_0^+}\frac{f(x)}{g(x)} = \lim_{x\to x_0^+}\frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to x_0^+}\frac{f'(x)}{g'(x)}$ exists.

We note that the rule given in Theorem 2 is also valid for left hand limits with similar assumptions on f and g.

The difference between Theorem 1 and Theorem 2 is that in Theorem 2 the function g may not be differentiable at x_0 but this is not the case in Theorem 1. So in order to prove Theorem 2, we have to modify the technique used in the proof of Theorem 1. Basically we have to handle the quotient $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}$ appearing in the proof of Theorem 1 in a different way. For this, we need the following theorem.

Theorem 3 : (Cauchy Mean Value Theorem) *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof (*): Consider the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Then F is continuous on $[a, b]$, differentiable on (a, b) and $F(a) = F(b) = 0$. By Rolle's Theorem there exists $c \in (a, b)$ such that $F'(c) = 0$. This proves the theorem. □

Remark : Cauchy mean value theorem (CMVT) is sometimes called generalized mean value theorem. Because, if we take $g(x) = x$ in CMVT we obtain the MVT. We will use CMVT to prove Theorem 2. We will now see an application of CMVT.

Problem 1: *Using Cauchy Mean Value Theorem, show that $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.*

Solution: Apply CMVT to $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$. We get $\frac{1-\cos x}{x^2/2} = \frac{\sin c}{c} < 1$ for some c between 0 and x.

Problem 2: *Let f be continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that $\frac{bf(a)-af(b)}{b-a} = f(c) - cf'(c)$.*

Solution: Apply CMVT to $\frac{f(x)}{x}$ and $\frac{1}{x}$.

Proof of Theorem 2(*): We will show that if $\lim_{x \to x_0^+} \frac{f'(x)}{g'(x)} = l$ then $\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = l$. Suppose $x_n \to x_0^+$. By CMVT (on $[x_0, x_n]$), there exists $c_n \in (x_0, x_n)$ such that

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f(x_n)}{g(x_n)}.$$

Therefore, $\lim_{x_n \to x_0^+} \frac{f(x_n)}{g(x_n)} = \lim_{c_n \to x_0^+} \frac{f'(c_n)}{g'(c_n)} = l$. $\square$

Remarks : 1. In Theorem 2, we can also take $l = \infty$ or $l = -\infty$ and in this case the proof is essentially the same.

2. We can replace the condition $f(x_0) = g(x_0) = 0$ by the condition $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ in Theorem 2. In this case the functions f and g may not be defined at x_0. For the proof, extend the functions to $[x_0, b)$ such that $f(x_0) = g(x_0) = 0$.

3. Theorem 2 is also true for $x_0 = \infty$ or $-\infty$ i.e., if $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = l$ then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$ with appropriate assumptions on f and g. To prove this fact, consider the functions $F(x) = f(\frac{1}{x})$ and $G(x) = g(\frac{1}{x}), x \neq 0$ and allow $x \to 0^-$. In this case also we can take $l = \infty$ or $l = -\infty$.

4. In case of the $\frac{\infty}{\infty}$ form we have

$$\{\lim_{x \to x_0^+} f(x) = \pm\infty = \lim_{x \to x_0^+} g(x)\} \quad \Rightarrow \quad \{\text{if } \lim_{x \to x_0^+} \frac{f'(x)}{g'(x)} = l \text{ then } \lim_{x \to x_0^+} \frac{f(x)}{g(x)} = l\}.$$

Of course we assume the standard necessary conditions as above on f and g. This result is also true when $x_0 = \infty$ or $-\infty$ and/or $l = \infty$ or $l = -\infty$. We will not give the proofs of these results but we will use them.

5. Other forms such as $\infty - \infty, 0(\infty), 1^\infty, ..$ are usually reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Examples : 1. $\lim_{x \to 0^+} \frac{log(sinx)}{logx} = \lim_{x \to 0^+} \frac{cosx/sinx}{1/x} = 1$ ($\frac{\infty}{\infty}$ form).

2. $\lim_{x \to 0^+}(xlogx) = \lim_{x \to 0^+} \frac{logx}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0$. ($0 \cdot \infty$ form).

3. $\lim_{x \to 0^+}(\frac{1}{x} - \frac{1}{sinx}) = \lim_{x \to 0^+} \frac{sinx-x}{xsinx} = \lim_{x \to 0^+} \frac{cosx-1}{sinx+xcosx} = \lim_{x \to 0^+} \frac{-sinx}{2cosx-xsinx} = 0$.

4. Let us evaluate $\lim_{x \to \infty}(1 + \frac{1}{x})^x = \lim_{x \to \infty} e^{xlog(1+\frac{1}{x})}$. Note that $xlog(1 + \frac{1}{x}) = \lim_{x \to \infty} \frac{log(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x \to \infty} \frac{1}{1+\frac{1}{x}} = 1$. Therefore, $\lim_{x \to \infty}(1 + \frac{1}{x})^x = e$.

5. Note that $\lim_{x \to \infty} \frac{x-sinx}{2x+sinx} = \frac{1}{2}$. However, one cannot apply L'Hospital Rule, because $\lim_{x \to \infty} \frac{1-cosx}{2+cosx}$ does not exist.

6. Note that $\lim_{x \to 0} \frac{x+1}{x}$ does not exist but if we apply L'Hospital rule, we get a wrong answer : $\lim_{x \to 0} \frac{x+1}{x} = \lim_{x \to 0} \frac{1}{1} = 1$. The reason is that in this case we cannot apply L'Hospital Rule because the given quotient is neither $\frac{0}{0}$ form nor $\frac{\infty}{\infty}$ form