Lecture 10 : Taylor's Theorem

In the last few lectures we discussed the mean value theorem (which basically relates a function and its derivative) and its applications. We will now discuss a result called Taylor's Theorem which relates a function, its derivative and its higher derivatives. We will see that Taylor's Theorem is an extension of the mean value theorem. Though Taylor's Theorem has applications in numerical methods, inequalities and local maxima and minima, it basically deals with approximation of functions by polynomials. To understand this type of approximation let us start with the linear approximation or tangent line approximation.

Linear Approximation : Let f be a function, differentiable at $x_0 \in \mathbb{R}$. Then the linear polynomial

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

is the natural linear approximation to $f(x)$ near x_0. Geometrically, this is clear because we approximate the curve near $(x_0, f(x_0))$ by the tangent line at $(x_0, f(x_0))$. The following result provides an estimation of the size of the error $E_1(x) = f(x) - P_1(x)$.

Theorem 10.1: (Extended Mean Value Theorem) *If f and f' are continuous on $[a, b]$ and f' is differentiable on (a, b) then there exists $c \in (a, b)$ such that*

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2.$$

Proof (*): This result is a particular case of Taylor's Theorem whose proof is given below.

If we take $b = x$ and $a = x_0$ in the previous result, we obtain that

$$\mid E_1(x) \mid = \mid f(x) - P_1(x) \mid \le \frac{M}{2}(x - x_0)^2$$

where $M = sup\{\mid f''(t) \mid : t \in [x_0, x]\}$. The above estimate gives an idea "how good the approximation is i.e., how fast the error $E_1(x)$ goes to 0 as $x \to x_0$".

Naturally, one asks the question: Can we get a better estimate for the error if we use approximation by higher order polynomials. The answer is yes and this is what Taylor's theorem talks about.

There might be several ways to approximate a given function by a polynomial of degree ≥ 2, however, Taylor's theorem deals with the polynomial which agrees with f and some of its derivatives at a given point x_0 as $P_1(x)$ does in case of the linear approximation.

The polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

has the property that $P_n(x_0) = f(x_0)$ and $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $k = 1, 2, .., n$ where $f^{(k)}(x_0)$ denotes the k th derivative of f at x_0. This polynomial is called Taylor's polynomial of degree n (with respect to f and x_0).

The following theorem called Taylor's Theorem provides an estimate for the error function $E_n(x) = f(x) - P_n(x)$.

Theorem 10.2: *Let $f : [a, b] \to \mathbb{R}, f, f', f'', ..., f^{(n-1)}$ be continuous on $[a, b]$ and suppose $f^{(n)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + ... + \frac{f^{(n-1)}(a)}{(n-1)!}(b - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b - a)^n.$$

Proof (*): Define

$$F(x) = f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 - ... - \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1}.$$

We will show that $F(a) = \frac{(b-a)^n}{n!}f^{(n)}(c)$ for some $c \in (a,b)$, which will prove the theorem. Note that

$$F'(x) = -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}. \tag{1}$$

Define $g(x) = F(x) - (\frac{b-x}{b-a})^n F(a)$. It is easy to check that $g(a) = g(b) = 0$ and hence by Rolle's theorem there exists some $c \in (a,b)$ such that

$$g'(c) = F'(c) + \frac{n(b-c)^{n-1}}{(b-a)^n}F(a) = 0. \tag{2}$$

From (1) and (2) we obtain that $\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1} = \frac{n(b-c)^{n-1}}{(b-a)n}F(a)$. This implies that $F(a) = \frac{(b-a)^n}{n!}f^{(n)}(c)$. This proves the theorem. $\square$

Let us see some applications.

Problem 1 : Show that $1 - \frac{1}{2}x^2 \leq cosx$ for all $x \in \mathbb{R}$.

Solution : Take $f(x) = cosx$ and $x_0 = 0$ in Taylor's Theorem. Then there exists c between 0 and x such that

$$cosx = 1 - \frac{1}{2}x^2 + \frac{sinc}{6}x^3.$$

Verify that the term $\frac{sinc}{6}x^3 \geq 0$ when $\mid x \mid \leq \pi$. If $\mid x \mid \geq \pi$ then $1 - \frac{1}{2}x^2 < -3 \ \leq cosx$. Therefore the inequality holds for all $x \in \mathbb{R}$.

Problem 2 : Let $x_0 \in (a,b)$ and $n \geq 2$. Suppose $f', f'', .., f^{(n)}$ are continuous on (a,b) and $f'(x_0) = .. = f^{(n-1)}(x_0) = 0$. Then, if n is even and $f^{(n)}(x_0) > 0$, then f has a local minimum at x_0. Similarly, if n is even and $f^{(n)}(x_0) < 0$, then f has a local maximum at x_0.

Solution : By Taylor's theorem, for $x \in (a,b)$ there exists a c between x and x_0 such that

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n. \tag{3}$$

Let $f^{(n)}(x_0) > 0$ and n is even. Then by the continuity of $f^{(n)}$ there exists a neighborhood U of x_0 such that $f^{(n)}(x) > 0$ for all $x \in U$. This implies that $\frac{f^{(n)}(c)}{n!}(x-x_0)^n \geq 0$ whenever $c \in U$. Hence by equation (3), $f(x) \geq f(x_0)$ for all $x \in U$ which implies that x_0 is a local minimum.

Problem 3 : *Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that*

$$x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k}x^{2k} \quad < \ \log(1+x) \quad < \ x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution : By Taylor's theorem, there exists $c \in (0,x)$ s.t.

$$\log(1+x) = x - \frac{1}{2}x^2 + ... + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}.$$

Note that, for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} > 0$ if $n = 2k$ and < 0 if $n = 2k+1$.