Hints / solutions to Practice Problems, No. 1

1 (i) The problem has been printed wrongly. (x_n) is supposed to be:

$$x_n = (\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}}) \cdots (\sqrt{2} - 2^{\frac{1}{2n+1}}).$$

Here $0 \le x_n \le (\sqrt{2}-1)^n$. Therefore, $x_n \longrightarrow 0$.

(ii) $(1+2+\cdots+n)\frac{1}{n^2+n} \le x_n \le (1+2+\cdots+n)\frac{1}{1+n^2}$. Hence $x_n \to \frac{1}{2}$.

(iii) $(b^n)^{\frac{1}{n}} \le x_n \le (2b^n)^{\frac{1}{n}} \Rightarrow x_n \longrightarrow b$.

---

2 (i), (ii) & (iii) : $\frac{x_{n+1}}{x_n} \longrightarrow l < 1$. Therefore, $x_n \longrightarrow 0$.

(iv) $\frac{x_{n+1}}{x_n} \longrightarrow \frac{b}{2}$. Hence, for $b<2$, $x_n \to 0$ & for $b>2$, $x_n \longrightarrow \infty$.

---

3 (i) & (ii). $\frac{x_{n+1}}{x_n} < 1 \ \forall n$. Hence (x_n) is decreasing. Since $0 \le x_n \ \forall n$, the sequence converges.

---

4 (i) First note that $x_n > 4 \ \forall n$. Therefore $\frac{x_{n+1}}{x_n} = \frac{1}{2} + \frac{2}{x_n} < 1$.
The sequence is bounded below and decreasing. Hence (x_n)
converges. Suppose $x_n \longrightarrow l$. This implies that $l = \frac{1}{2}l + 2$.
Therefore $l = 4$.

(ii) This is just the constant sequence $1, 1, 1, \ldots$.
In case, $x_1 = 2$, note that (x_n) is bounded below by 0 and
decreasing: $x_1 > x_2 > x_3$ and suppose $x_n < x_{n-1}$. Then

$$x_{n+1} = 2 - \frac{1}{x_n} < 2 - \frac{1}{x_{n-1}} = x_n.$$

---

5 (i) $|x_{n+2} - x_{n+1}| = \frac{1}{(2+x_{n+1})(2+x_n)} |x_n - x_{n+1}| < \frac{1}{4}|x_n - x_{n+1}|$

By problem 6(ii), (x_n) satisfies Cauchy criterion, hence (x_n)
Converges. Suppose $x_n \longrightarrow l$. Then $l = \frac{1}{2+l}$. Find l.

(ii) $|x_{n+2} - x_{n+1}| = \frac{1}{x_n \cdot x_{n+1}} |x_n - x_{n+1}| < \frac{1}{4}|x_n - x_{n+1}|$.

(x_n) satisfies Cauchy criterion, hence (x_n) converges.
Suppose $x_n \longrightarrow l$. Then $l = 2 + \frac{1}{l}$. Find l.