Hints and Solutions to Practice Problems, NO. 2.

2. Let $x_0 \in \mathbb{R}$. Then whenever $x_n \longrightarrow x_0$,

$$|f(x_n) - f(x_0)| \leq K|x_n - x_0| \longrightarrow 0.$$

Hence $f(x_n) \longrightarrow f(x_0)$. Therefore f is continuous at x_0.

Aliter: Let $\epsilon > 0$ be given. choose $\delta = \frac{\epsilon}{K}$, then, whenever $|x_n - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$, because,

$$|f(x) - f(x_0)| \leq K|x - x_0| < K.\delta = K.\frac{\epsilon}{K} = \epsilon.$$

Therefore f is continuous at x_0.

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1. Note that $||x| - |x_0|| \leq |x - x_0|$. Take $K = 1$ in Problem 2.

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3. Suppose f is continuous at some x_0. Let (x_n) be a sequence of rational numbers and (y_n) be a sequence of irrational numbers such that $x_n \to x_0$ and $y_n \longrightarrow x_0$.
Since f is continuous at x_0, $f(x_n) \longrightarrow f(x_0)$ and $f(y_n) \longrightarrow f(x_0)$.
However, $f(x_n) = 2x_n \longrightarrow 2x_0$ & $f(y_n) = x_n + 3 \longrightarrow x_0 + 3$.
Therefore $2x_0 = x_0 + 3$. This implies that $x_0 = 3$. (This doesn't prove that f is cont at $x_0 = 3$)
Note that f is continuous at $x_0 = 3$, because,

$$|f(x) - f(x_0)| = |f(x) - 6| < 3|x - 3|. \text{ (Apply Problem 2).}$$

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4 Since $f(c) > 0$, choose some $\epsilon > 0$ s.t. $f(c) - \epsilon > 0$. Therefore, $f(c) \in (f(c) - \epsilon, f(c) + \epsilon)$. By continuity, $\exists\ \delta$ such that whenever $x \in (c - \delta, c + \delta)$ we have $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$.

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5 Note that $p(0) < 0$, $p(2) > 0$ and $p(-10) > 0$. Apply Intermediate value Prop.

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6 Let $p(x) = x - \cos x$. Then $f(0) < 0$ and $f(\frac{\pi}{2}) > 0$. Apply IVP.

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7 If $g(a) = a$ or $g(b) = b$, then take $c = a$ or b respectively.
Suppose $g(a) > a$ and $g(b) < b$, then define $f(x) = g(x) - x$.
Then $f(a) > 0$ and $f(b) < 0$. By IVP, $\exists\ c \in [a, b]$ s.t. $f(c) = 0$, ie $g(c) = c$.

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8 Let $g(x) = f(x) - f(x + 1/2)$. Note that $g(0) = -g(1/2)$
Apply IVP for g on $[0, 1/2]$.