Hints / Solutions of Practice Problems, No. 3

1 Note that $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^{\frac{1}{3}}}{x}$ doesn't exist.

2 Note that $\left| \dfrac{f(h) - f(0)}{h} \right| = \left| \dfrac{f(h)}{h} \right| \leq \dfrac{h^2}{h} \to 0$ as $h \to 0$.

This shows that $\lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h - 0} = 0$, therefore, $f'(0) = 0$.

3 $\lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c} = \lim\limits_{x \to c} \dfrac{|f(x)|}{x - c}$ exists iff $\lim\limits_{x \to c} \dfrac{f(x)}{x - c} = 0$.

4 $\lim\limits_{x \to a} \dfrac{x f(a) - a f(x)}{x - a} = \lim\limits_{x \to a} \dfrac{x f(a) - a f(a) + a f(a) - a f(x)}{x - a} = f(a) - a f'(a)$

5 Consider the function $g(x) = \dfrac{f(x)}{x}$. Note that $g(a) = g(b)$.

Apply the Rolles theorem (or MVT) on $[a, b]$

6 Consider the function

$$g(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \cdots + a_n x.$$

Note that $g(0) = g(1) = 0$. Apply the Rolles thm for g on $[0,1]$.