Solutions to Practice Problems, No 4.

1. Let $x > 0$. By MVT, $\exists\, c \in (0,x)$ s.t. $\sin x - \sin 0 = \cos c\,(x-0) \le x$.

2. Done in the lecture.

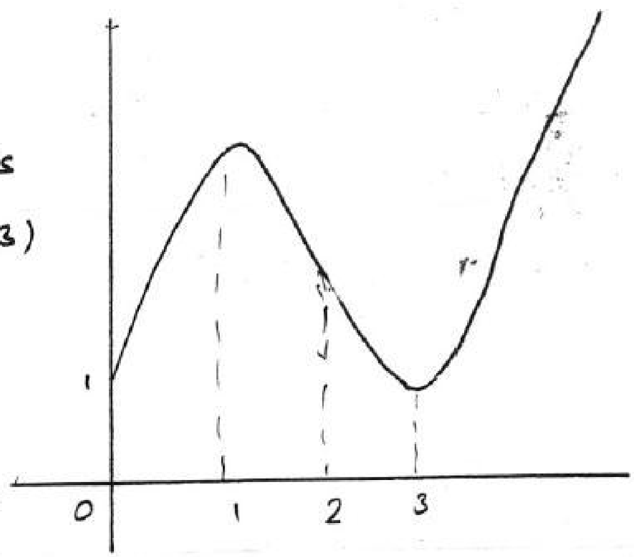
4. Let $x, y \in [a,b]$ s.t $y > x$ & $|f'(x)| \le M$ $\forall\, x \in [a,b]$. Then by MVT, $\exists\, c \in (x,y)$ s.t. $|f(y) - f(x)| = |f'(c)|\,|x-y| \le M|x-y|$.

5. Note that $(g-f)'(x) \geq 0$ $\forall\, x \geq 0$ and $(g-f)(0) = 0$. Since $g-f$ is increasing on $[0,\infty)$, $(g-f)(x) \geq (g-f)(0)$ $\forall x \geq 0$.

6 By MVT, $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a} > 0$.

7. Let $f(x) = x^3 - 6x^2 + 9x + 1$. Then

$$f'(x) = 3(x-1)(x-3) \;\&\; f''(x) = 6(x-2).$$



- f is strictly increasing on $(-\infty,1)$ and $(3,\infty)$, and f is strictly decreasing on $(1,3)$
- $x=1$ is a local max & $x = 3$ is a local min.
- f is concave on $(-\infty, 2)$ & f is convex on $(2,\infty)$.
- $x = 2$ is a point of inflection.

3. There are corrections in the statement of Problem 3.

<u>Problem 3:</u> Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be differentiable and assume that $f'(x) \longrightarrow b$ as $x \longrightarrow \infty$.

(a) Show that for any $h > 0$, we have $\displaystyle\lim_{x\to\infty} \frac{f(x+h)-f(x)}{h} = b$.

(b) Show that if $f(x) \longrightarrow a$ as $x \longrightarrow \infty$, then $b = 0$.

(c) Show that if f is bounded then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x} = b$.

Solution to Problem 3: (of Practice Problem, No. 4)

(a) Let $x, h \in [0, \infty)$. By MVT, $\exists\ c_x \in (x, x+h)$

s.t $\dfrac{f(x+h) - f(x)}{h} = f'(c_x)$.

$\therefore \lim \dfrac{f(x+h) - f(x)}{h} = \lim_{x \to \infty} f'(c_x) = b$.

(Note that, here h is fixed & x is varying.
Also, when $x \to \infty$, we have $c_x \to \infty$).

(b) Assume that $\lim_{x \to \infty} f'(x) = b$ and $\lim_{x \to \infty} f(x) = a$ for some $a \in \mathbb{R}$.

By MVT $\exists\ c_n \in (n, n+1)$ s.t $f(n+1) - f(n) = f'(c_n)$.
Note that $c_n \to \infty$ as $n \to \infty$. Hence $f'(c_n) \to b$.
Since $f(c_n) \to a$, we have $f(n+1) - f(n) \to 0$.
Therefore, $b = 0$.

(c) Since f is bounded $\lim_{x \to \infty} \dfrac{f(x)}{x} = 0$. Therefore, it is sufficient to show that, in this case, $b =$ which infact, follows from Problem 3 of Assignment No. 4.

Remark: 1. The proof of (b) infact follows directly from (a).