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Dynamics of Entire Functions: A Review

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1. Introduction

In complex dynamics, the iteration theory of rational functions originated around 1920 in the work of Gaston Julia and Piere Fatou. The study of dynamics of entire functions essentially started in 1926 in the work of Fatou [28]. Subsequently, there was not much activity in the field for about sixty years. Around 1980, a renewed interest in the complex analytic dynamics was generated, due to the beautiful computer graphics introduced into the subject. A comprehensive survey of the work on complex dynamics of rational functions and polynomials can be found in [5,14,15,16,20]. Baker extended much of the work of Fatou and Julia to the class of entire functions, showing along the way that a new type of stable behaviour (wandering domain) could occur for entire transcendental functions. The dynamics of entire functions is quite different from the dynamics of polynomials or rational functions, mainly because of the essential singularity at ∞. By Picard's theorem, any neighbourhood of ∞ is mapped over the entire plane infinitely often, missing at most one point which, in the language of dynamical systems, means that an entire map exhibits a tremendous amount of hyperbolicity near ∞.

For polynomials or rational functions, there are several comprehensive reviews and books available for the work on their dynamics. However, no effort has been made to update the work on the dynamics of entire functions after the survey of Devaney [19]. The present work is aimed at to fill this gap. However, this endeavour of the authors has been guided solely by their own research interests and many meritorious research papers might have been omitted just because they do not belong to the main theme of the review.

2. Basic definitions and results

Let $f(z)$ be a non-constant entire function.Let $z_0 = f^0(z_0)$ and $z_n = f(z_{n-1}) =$

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$f^n(z_0)$; $n = 1, 2, \ldots$, where $f^n = fofofo\ldots of$ is the nth iterate of f. The set $O^+(z_0) = \{f^n(z_0) : n = 0, 1, 2, \ldots\}$ is called the orbit or forward orbit of z_0 and the set $O^-(z_0) = \{z : f^n(z) = z_0$ for some positive integer $n\}$ is called the backward orbit of z_0.

A family $\mathcal{F}$ of analytic functions defined on a domain $\Omega \subseteq C$ is said to be normal in Ω if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains either a subsequence which converges (in usual metric) to a limit function f uniformly on each compact subset of Ω, or a subsequence which converges (in usual metric) uniformly to ∞ on each compact subset. A family $\mathcal{F}$ is said to be normal at a point $z_0 \in \Omega$ if it is normal in some neighbourhood of z_0.

Let $C^\infty = C \cup \{\infty\}$ denote the extended complex plane. The Fatou set of an entire function $f(z)$, denoted by $F(f)$, is defined as $F(f) = \{z \in C^\infty : \{f^n\}$ is normal at $z\}$. The Julia set is the complement of Fatou set.

A set S is called invariant under f if $f(S) \subseteq S$. The set S is called completely invariant if $f(S) \subseteq S$ and $f^{-1}(S) \subseteq S$.

The following are some of the elementary properties of the Fatou set and the Julia set of $f(z)$ (See e.g. [5,7,14,35]).

- $F(f)$ is open and $J(f)$ is closed.
- $F(f) = F(f^n)$ and $J(f) = J(f^n)$ $\forall n \geq 2$.
- Either $J(f)$ has empty interior or $J(f) = C^\infty$.
- $J(f)$ is nonempty perfect set.
- $F(f)$ and $J(f)$ are completely invariant.
- If $z_0 \in J(f)$ is not a exceptional value (i.e., $f(z) = z_0$ for some $z \in C$), then $J(f) = \overline{O^-(z_0)}$.

A point z is said to be a periodic point of period p for a function $f(z)$ if $f^p(z) = z$. The least positive integer p for which $f^p(z) = z$ is called the minimal period of z. The number $\lambda = (f^p)'(z)$ is called the multiplier or eigenvalue of the periodic point z.

If the minimal period of z is 1 (i.e. $f(z) = z$) then z is called a fixed point of $f(z)$. For a periodic point z_0 of period p, the orbit $\{z_0, f(z_0), f^2(z_0)\ldots, f^{p-1}(z_0)\}$ is called a cycle or a periodic cycle of z_0. The periodic point z_0 of period p is called attracting, repelling and neutral or indifferent if $|\lambda| < 1$, $|\lambda| > 1$ or $|\lambda| = 1$ respectively. If $\lambda = 0$, the attracting periodic point z_0 is called superattracting. If $|\lambda| \neq 1$, the periodic point z_0 is called hyperbolic. When $\lambda = e^{2\pi i\alpha}$ the indifferent periodic point is further classified as rationally or irrationally indifferent according as α is rational or an irrational number. The cycle of the periodic point z_0 is called attracting, repelling, rationally neutral or irrationally neutral if z_0 is attracting, repelling, rationally neutral or irrationally neutral respectively.

A rational function always has a fixed point. However, a transcendental entire function need not have any fixed point, e.g. consider $f(z) = e^z + z$. Fatou [28] proved that a rational function $f(z)$ (of degree > 1) has periodic point of (not necessarily minimal) period n for all $n \geq 1$, while an entire transcendental function $f(z)$ has at least one periodic point of order 2. However, an entire transcendental function has infinitely many periodic points of period n (not necessarily minimal period) for all $n \geq 2$ [39].

In 1968, Baker [1] conjectured that if $f(z)$ is an entire transcendental function and $n \geq 2$, then f has infinitely many repelling points of minimal period n. The above conjecture was proved in affirmative by Bergweiler [6]. Further, it was conjectured [30] that if $f(z)$ is an entire transcendental function, l a straight line in the complex plane, and $n \geq 2$, then f^n has infinitely many fixpoints that do not lie on l. This conjecture was proved by Bergweiler [11]. Further, Bergweiler [9] combined the above two results, and proved that if f is an entire transcendental function, l a straight line in the complex plane, and $n \geq 2$, then f has infinitely many repelling periodic points of period n that do not lie on l.

The Julia set $J(f)$ is the closure of the set of repelling periodic points of $f(z)$ [1]. It is easily seen that the attracting periodic points are in the Fatou set, while repelling periodic points are in the Julia set. Further, it is well known that rationally indifferent periodic points are in the Julia set. However, the irrationally indifferent periodic points may lie either in the Fatou set or the Julia set.

Let $f(z)$ be either an entire function or a rational function. A maximal connected domain U contained in the Fatou set of a function $f(z)$ is said to be a component of the Fatou set $F(f)$. Since, $F(f)$ is completely invariant and $f^n(z)$ is analytic in U for each n, $f^n(U) = U_n$ (say) is a component contained in $F(f)$. Bergweiler and Rohde [12] proved that if f is a transcendental entire function and U and V are two Fatou components of $F(f)$ such that $f(U) \subset V$, then $V \backslash f(U)$ contains at most one point. This result was also proved by Herring [31] in 1993 independently.

Yang and Hua [44] proved that if f is an entire transcendental function, then

$$F = f^{-1}(F) = f(F) \cup \{PV(f) \cap F\}$$

where $PV(f)$ denotes the set of all finite Picard exceptional values with respect to f.

A component U of $F(f)$ is called periodic with period n if $f^n(U) = U$. The set $\{U_0 = U, f(U), f^2(U), \ldots, f^{n-1}(U)\}$ is called the (periodic) cycle of components. The least positive integer n with this property is called the minimal period of U. A component U of $F(f)$ is called preperiodic if there exist nonnegative integers n and m with $n > m \geq 0$ such that $f^n(U) = f^m(U)$. It is easily seen that the periodic components are preperiodic and if U is preperiodic component then $f^m(U)$ is a periodic component of period $n-m$. A component of $F(f)$ which is not preperiodic is called a wandering component or a wandering domain.

Sullivan [40] proved that if $f(z)$ is a rational function of degree greater than one, then the function $f(z)$ has no wandering domain in its Fatou set. Sullivan [40,41,42] completed a classification scheme studied in parts by Fatou, Julia, Siegel, Arnold, Moser and Herman for the dynamics of a rational function in the periodic components of its Fatou set. A similar classification scheme for the class of entire functions is given by Bergweiler [7]. Thus, we have the following theorem giving the behaviour of iterates of an entire function $f(z)$ in the periodic components :

Theorem 2.1 Let $f(z)$ be an entire function other than linear polynomial. Let U be a periodic component of the Fatou set $F(f)$ having minimal period n and let $S = f^n$. Then, only the following possibilities can occur :

(1) U **is an attracting domain** : In this case the periodic component U contains an attracting periodic point z_0 of period n.
An attracting domain is also called an attractive basin of z_0. Further, $|S'(z_0)| < 1$, $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$. The cycle $\{z_0, f(z_0), f^2(z_0), \ldots$ $\ldots, f^{n-1}(z_0)\}$ is called the attracting cycle for $f(z)$. If $S'(z_0) = 0$, then U is called a superattracting domain.

(2) U **is a parabolic domain** : In this case the boundary ∂U of the periodic component U contains a periodic point z_0 of period n and $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$.
A parabolic domain is also called a leau domain. Further, $S'(z_0) = 1$ the cycle $\{z_0, f(z_0), \ldots, f^{n-1}(z_0)\}$ is called the parabolic cycle for $f(z)$.

(3) U **is Siegel disk** : In this case there exists an analytic homeomorphism $\phi : U \longrightarrow D$ where $D = \{z : |z| < 1\}$ is the unit disk, such that $\phi(S\phi^{-1}(z)) = e^{2\pi i\alpha}z$ for some irrational number α.

(4) U **is a Baker domain** : In this case there exists $z_0 \in \partial U$ such that $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$, but $S(z_0)$ is not defined. If $f(z)$ is an entire transcendental function, $z_0 = \infty$. Thus, for an entire transcendental function U is also called a domain at infinity. However, for a polynomial entire function $P(\infty) = \infty$, and hence Baker domain do not exist for polynomials.

Remark : A periodic component, called Herman ring, that is different from all the periodic components of above theorem occurs only in the dynamics of rational functions. The component U of the Fatou set of a rational function is called a Herman ring if there exists an analytic homeomorphism $\phi : U \longrightarrow A$, A being the annulus $A = \{z : 1 < |z| < r\}$, such that $\phi(S(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some irrational number α. Herman rings do not exist in the dynamics of entire functions [43].

The following examples illustrate each of the classifications of periodic components of the Fatou set $F(f)$ given by the above theorem.

Example 1 **Attracting domain** : Let $f(z) = z^2$ be entire function. the point $z = 0$ is an (super) attracting fixed point for $f(z)$. In this case, $U = A(0) = \{z : |z| < 1\}$ is an attracting domain for $f(z)$.
Example 2 **Parabolic domain** : Let $E(z) = e^{z-1}$. The point $z = 1$ is a rationally indifferent fixed point of $E(z)$. Let $U = interior\{z \in C : E^n(z) \to 1 \ as \ n \to \infty\}$. The fixed point $z = 1$ lies on the boundary of U, since $E^n(x) \to \infty$ for $x > 1$ as $n \to \infty$. Thus, U is a parabolic domain for $E(z)$.
Example 3 **Siegel disk** : Let $P(z) = e^{2\pi i\alpha}z + \cdots + z^d$, where α is an irrational number satisfying the condition $\sum_{n=1}^{\infty} (\log (q_{n+1})/q_n) < \infty$ where p_n/q_n, $n = 1, 2, \ldots$ are the continued fraction approximants to α. The point $z = 0$ is irrationally indifferent fixed point of $P(z)$ with multiplier $e^{2\pi i\alpha}$. Further, there exists an analytic homeomorphism $\phi : U \to D$ where D is the unit disk and U is an open neighborhood containing 0 such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\,\alpha}z$ [35]. Thus, U is a Siegel disk for $P(z)$.
Example 4 **Baker domain** : It is easily seen that a polynomial has no Baker domain, since ∞ is a super attracting fixed point for a polynomial. Let $f(z) = 1+z+e^{-z}$ be an entire transcendental function. Set $H^+ = \{z \in C : \Re(z) > 0\}$ and $U = \{z \in C : f^n(z) \to \infty$ as $n \to \infty\}$. It is observed that $\Re(f(z)) = 1 + \Re(z) + \Re(e^{-z}) > \Re(z)$ for $z \in H^+$. Therefore, all the orbits in the half plane H^+ lie in stable domain U consisting of points whose orbits tend to the essential singularity ∞. Consequently, $H^+ \subseteq U$ is a Baker domain for $f(z)$.
Example 5 **Wandering domain** : Let $f(z) = z + \lambda \sin z$, where $\lambda > 1$ is chosen such that each critical point z_0 of $f(z)$ is mapped to $z_0 \pm 2\pi$, another critical point of $f(z)$. Thus, there are only two distinct orbits corresponding to the critical points. Consequently, all sufficiently small neighborhoods of a critical point lie in the Fatou set, since the iterates contracts these regions and all orbits tend uniformly to ∞ in these neighborhoods. Further, the vertical lines $x = k\pi$ lie in the Julia set of $f(z)$ for any integer k [19]. It therefore follows that each critical point on a critical orbit lies in a distinct component of the Fatou set. Thus, each of these components are wandering domains since they are not preperiodic.

There are points other than periodic points which play important role in the dynamics of a function. The following is a review of the role of such points in the complex dynamics.

A point w is said to be a critical point of $f(z)$ if $f'(w) = 0$. The value $f(w)$ corresponding to a critical point w is called a critical value of f. A point $w \in C^{\infty}$ is said to be an asymptotic value of $f(z)$, if there is a continuous curve $\gamma(t)$ satisfying

$$\lim_{t\to\infty}\gamma(t)=\infty \text{ and } \lim_{t\to\infty} f(\gamma(t))=w$$

Clearly, ∞ is an asymptotic value for every entire function. If a function $f(z)$ has an asymptotic value w, the preimage of any neighbourhood of w is unbounded and has noncompact closure.

The set $SV(f)$ of singular values of an entire function $f(z)$ is defined as the union of the set of all critical values of $f(z)$ and the set of all finite asymptotic values of $f(z)$. Thus,

$$SV(f)=CV(f)\cup AV(f)$$

where, $CV(f)=$ set of all critical values of $f(z)$ and $AV(f)=$ set of all (finite) asymptotic values of $f(z)$.

The following results exhibit the importance of singular values in the dynamics of an entire function. Devaney [22] proved that if $f(z)$ is an entire function other than a linear polynomial and z_0 lies on an attracting cycle or a parabolic cycle of $f(z)$, then the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_0, and also proved that if $f(z)$ is an entire function other than a linear polynomial and the Fatou set $F(f)$ has a Siegel disk, then the forward orbit of some critical point must accumulate in its boundary.

In case of domains which are not preperiodic, the finite limit function of iterates of an entire function in wandering domains are limit points of the forward orbits of the singular values. More precisely, it was proved in [10] that if $f(z)$ is an entire function and U is a wandering domain of $f(z)$ and $S'(f)=Der\{f^n(z):z\in SV(f), n=0,1,2,\ldots\}$ denote the derived set of the forward orbits of all singular values of f, then all limit functions of the sequence $\{f^n(U)\}$ are contained in $S'(f)\cup\{\infty\}$. Let

$$B=\{f: sing(f^{-1}) \text{ is bounded}\}$$
$$S=\{f: sing(f^{-1}) \text{ is finite}\}$$

Eremenko and Lyubich [27] proved that if $f\in B$ is a transcendental entire function and $z\in F(f)$ then the orbit $\{f^n\}$ does not tend to ∞, and if $f\in S$, then $F(f)$ has no wandering components and every orbit in $F(f)$ is absorbed by a cycle of Fatou domains or by a cycle of Siegel disks. Further, the set $sing(f^{-1})$ contains at most q points, then $n_F+n_I\le q$, where n_F is the number of the cycles of Fatou domains and n_I the number of irrational neutral cycles.

3. Complex Dynamics of Critically Finite Fntire Functions

An entire function is said to be critically finite if it has only finitely many asymptotic and critical values. The critically finite entire maps form a class of entire functions whose dynamics prove to be most tractable. We review below some of the recent work on the dynamics of the class of critically finite transcendental entire functions.

It is easily seen that the entire functions λe^z, $\lambda sinz$ and $\lambda cosz$ are in the class of critically finite transcendental entire functions. On the other hand $\lambda(e^z - 1)/z$, and $z + \lambda sinz$ are not in the class of critically finite transcendental entire functions, since they have infinitely many critical values. As a dynamical system, a function in the class of critically finite transcendental entire functions shares many of the property of the polynomials or rational functions. However, there are several significant differences, such as existance of a wandering domains, existance of a Baker domains, existance of unbounded domain of attraction for a finite attracting periodic point in the Fatou sets of entire transcendental functions while any of domains cannot be contained the Fatou sets of polynomials or rational functions. The dynamics of the functions in the class of critically finite transcendental entire functions is mainly studied by Devaney [17,18,21,24,25], Durkin [23], Eremenko [26], Goldberg [29], Lyubich [26], Keen [29], krych [24] and Tangerman [25]. An excellent review of almost all the fundamental results on dynamics of critically finite entire functions is due to Devaney [19].

The "no wandering domain" theorem due to Sullivan [40] gives that if $f(z)$ is a rational function of degree greater than one then $f(z)$ has no wandering domains in its Fatou set. Goldberg and Keen [29] and Eremenko and Lyubich [26] extended Sullivan's result to the entire functions in the class of critically finite transcendental entire functions as follows:

Theorem 3.1 ([26, 29]) Let $f(z)$ be a critically finite entire transcendental function. Then $f(z)$ has no wandering domains.

Theorem 3.2 ([27]) Let $f(z)$ be a critically finite entire transcendental function. Then $f(z)$ has no Baker domains.

It is well known that the Julia set of a polynomial never equals the extended complex plane, since point at infinity is an attracting fixed point for a polynomial. Fatou [28] in 1926, conjectured that the Julia set of the function e^z equals the extended complex plane. Misiurewicz [37] proved the Fatou's conjecture affirmatively in 1981. The following theorems give criteria for the Juila set of an entire function to be the extended complex plane.

Theorem 3.3 ([21]) Let $f(z)$ be a critically finite entire transcendental function and the forward orbits of all its singular values tend to ∞ under iteration of $f(z)$. Then the Julia set $J(f)$ of $f(z)$ equals the extended complex plane C^∞.

Theorem 3.4 ([21]) Let $f(z)$ be a critically finite entire transcendental function and all its singular values are prepriodic (but not periodic). Then, the Julia set $J(f)$ of $f(z)$ equals the extended complex plane C^{∞}.

For critically finite transcendental functions, Devaney and coworkers [17, 18, 21, 23, 24] have taken more interest in the exponential family λe^{z}. They studied the dynamics of entire functions λe^{z}, $(\lambda > 0))$ and exhibited all its beauties. Some of the main results on the dynamics of entire transcendental function $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$ are reviewed in the following:

Theorem 3.5 ([23]) Let $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$. Then, the Julia set of $E_{\lambda}(z)$ is a nowhere subset of the right half plane for $0 < \lambda < \frac{1}{e}$.

Theorem 3.6 ([23]) Let $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$. Then, the Fatou set of $E_{\lambda}(z)$, $0 < \lambda < \frac{1}{e}$, is the attractive basin $A(a_{\lambda}) = \{z : f^{n}(z) \to a_{\lambda}$ as $n \to \infty\}$of the attracting real fixed point a_{λ} of $E_{\lambda}(z)$.

Theorem 3.7 ([25]) Let $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$. Then, the Julia set of $E_{\lambda}(z)$ contains 'Cantor bouquets' for $0 < \lambda < \frac{1}{e}$.

Theorem 3.8 ([21]) Let $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$. Then, for $\lambda > \frac{1}{e}$ Julia set of $E_{\lambda}(z)$, equals the extended complex plane C^{∞}.

Devaney and Durkin [23] proved that the Julia set of $E_{\lambda}(z)$ for $0 < \lambda < \frac{1}{e}$ is a nowhere dense subset entirely contained in the right half plane. As soon as the the parameter λ crosses the value $\frac{1}{e}$, $E_{\lambda}(z)$ suddenly explodes and equals to the extended complex plane. This phenomena is reffered to as explosion or chaotic burst in the Julia sets of functions in one parameter family. This type of explosion occurs as well as in other family of functions like $i\lambda cosz$, $\lambda > 0$. The characterization of the Julia set of $E_{\lambda}(z)$ as the closure of the set of all escaping points (i.e. the points whose orbits tend to ∞ under iteration) is as follows:

Theorem 3.9 ([23, 25]) Let $E_{\lambda}(z) = \lambda e^{z}$, $\lambda > 0$. Then, the Julia set of $E_{\lambda}(z)$ is given by

$$J(E_{\lambda}) = \{z \in C : E_{\lambda}^{n}(z) \to \infty \text{ as } n \to \infty\}$$

At a particular parameter value, the dynamical changes suddenly, after which it again remains the same for the parameter belonging to a large interval. These sudden changes in dynamics are called bifurcations. A bifurcation accuring in the dynamics of $E_\lambda(z) = \lambda e^z$, $\lambda > 0$ is observed by Devaney. Thus,

Theorem 3.10 ([17, 21]) Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$ be a one parameter family of functions. Thus, a bifurcation in the dynamics of $E_\lambda(z)$ occur at the parameter value $\lambda = \frac{1}{e}$.

This type of bifurcation is also found to occur in the families of functions like $\lambda sinz$, $i\lambda cosz$, for $\lambda > 0$.

Jang [32] while describing the dynamics of $ze^{z+\mu}$, proved that if the real parameter μ belongs to the set $(-\infty, 2) \cup (2, \mu_*)$, then the critically finite function $f_\mu(z) = ze^{z+\mu}$ has the Julia set that is not the whole complex plane, where μ_* is root of the equation

$$G(\mu) = \mu + \alpha(\mu) + (-\mu + \alpha(\mu))exp\alpha(\mu) = 0,$$
$$\alpha(\mu) = (\mu^2 - 2\mu + 2)^{1/2}.$$

4. Complex Dynamics of Non-Critically Finite Entire Functions

An entire function is said to be non-critically finite if it has infinitely many asymptotic and critical values or we can say if an entire function is not critically finite then it is said to be non-critically finite.

The dynamics of non-critically entire functions had not been explored lately probably because of non-applicability of Sullivan's theorem to these functions. Also, the presence of infinitely many critical values and the behaviour of the orbits of critical values make it difficult to study the dynamics of non-critically finite entire functions. The work in this direction has been started recentely [33,34], with the studied of the entire transcendental functions $\lambda \frac{e^z - 1}{z}$ and introduction of a class of non-critically finite entire functions. Some of the basic results for the dynamics of such functions are reviewed here.

Let $K = \left\{ f_\lambda(z) = \lambda \frac{e^z - 1}{z} : \lambda > 0 \right\}$ be one parameter family of entire transcendental functions. A function $f_\lambda(z) \in K$ has infinitely many critical values in the disk centered at origin and having radius λ and $f'_\lambda(z)$ has infinitely many zeros in the left half plane. Further f_λ is not periodic. The bifurcation occurs in the dynamics of f_λ at the critical parameter value $\lambda^* \approx 0.64761$. For $0 < \lambda < \lambda^*$, $F(f_\lambda)$ equals the basin of the attraction of the real attracting fixed point and $J(f_\lambda)$ lies only in the right half plane. $J(f_\lambda)$ contains the real line R for $\lambda > \lambda^*$. Figure 1 suggests that the Julia sets of f_λ admits cantor bouquets for $0 < \lambda < \lambda^*$ and there is an explosion in the Julia set of f_λ asλ crosses the value λ^*. Further, the Julia set of $f_\lambda(z)$ is characterized as the closure of the set of escaping points for $\lambda > 0$.

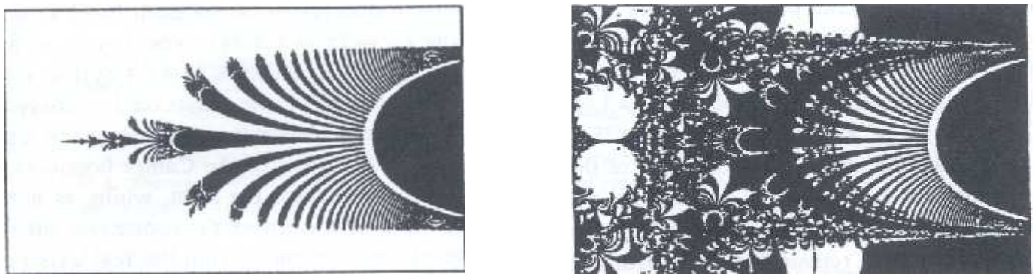

Figure 1: Explosion in the Julia set of $f_\lambda(z)$ (i) $\lambda = 0.64 < \lambda^*$ (ii) $\lambda = 0.65 > \lambda^*$

Let H be the class of functions defined by

$$H = \left\{ f(z) : \begin{array}{ll} (i) & f(z) \text{ is an entire function having order } \rho \text{ with } (1/2) \leq \rho < 1 \\ (ii) & f(z) \text{ has only negative real zeros in the complex plane} \\ (iii) & |f(-x)| \leq f(0) = 1 \text{ for all } x > 0 \\ (iv) & \lim_{x\to\infty} f(-x) = 0 \end{array} \right\}$$

and G be the class of functions defined by

$$G = \{g(z) = f(z^2) : f \in H\}$$

For a function $g \in G$, let

$$S = \{g_\lambda(z) = \lambda g(z) : g \in G \text{ and } \lambda \in R\backslash\{0\}\}$$

be one parameter family of entire transcendental functions. One of the interesting examples of the family S is $\{\ \lambda I_0(z); \lambda \in R\backslash\{0\}$ and I_0 is the well known modified Bessel function of zero order$\}$. The dynamics of $g_\lambda(z) \in S$ has been studied in [34]. Some of the basic dynamical properties of the function $g \in G$ are given in the sequel. If $g \in G$, then $g(z)$ passes infinitely many real critical values and w=0 is the only finite asymptotic value of $g(z)$. If $g_\lambda(z) \in S$ and $A(a_\lambda)$ be the basin of attraction of the real attracting fixed point a_λ of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$, then, for $0 < |\lambda| < \lambda_g^*$, $F(g_\lambda(z))= A(a_\lambda)$. The Julia set of $g_\lambda(z) \in S$ for $0 < |\lambda| < \lambda_g^*$ is characterized as the closure of the set escaping points. Further, if $g_\lambda(z) \in S$ and $|\lambda| > \lambda_g^*$, then Julia set of $g_\lambda(z) = C^\infty$ for $|\lambda| > \lambda_g^*$.

The dynamics of another non-critically finite entire function $f_\lambda(z) = \lambda + z + e^z$ is described in [38]. Let the lines $L_k = \{z = t + k\pi i : t \in R, k \in I\}$ be invariant under f_λ, where I is set of integers. The lines L_k, for k even, belong to the Julia set and divide the plane into parallel strips of width 2π. These strips are denoted by T_m. Inside each strip we define a set M_m, where the derivative is bounded,i.e.,

$$M_m = \{z = x + iy \in T_m : |f'_\lambda(z)| \leq 1\}$$

If $0 < \lambda \leq 1$, then the set M_m is contained in the immediate basin of attraction (i.e., connected component of basin of attraction) of $log(\lambda) + m\pi i$. Figure 2 shows part of the dynamical plane for $\lambda = 0.1$ and $\lambda = 1$ $[-3\pi \leq x \leq 3\pi, -2\pi \leq y \leq 2\pi]$; the black points belong to $J(f_\lambda)$.

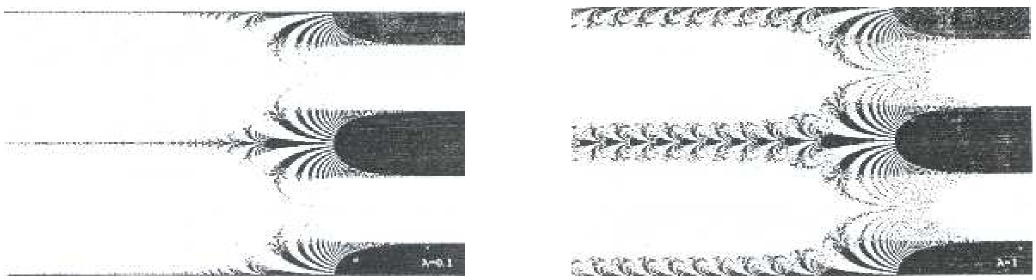

Figure 2: The julia set of $\lambda + z + e^z$ for λ=0.1 and 1.

If semistip $T = \{x + iy : x \geq 3, -\pi/2 \leq y \leq 0\}$ and $\alpha > 0$ and $0 < \beta < \pi$. Then there exists an invariant curve in the semistrip T, which is a graph of a continuous function $y = \phi(x)$. Figure 3 show part of the dynamical plane for $\lambda = 0.9 + 0.2i$, $\lambda = 1 + i$ and $\lambda = 1 + 0.25i$ $[-3\pi \leq x \leq 3\pi, -2\pi \leq y \leq 2\pi]$, respectively; black points belong to the Julia set.

5.Connectivity of Julia Sets

The connectivity of Julia sets for entire transcendental function is described in [36]. The following results are in this direction:.

Theorem 5.1 Let f be a transcendental function. Then the set $J(f) \cup \infty$ in C^∞ is connected if only if f has no multiply connected wandering domain.

It is follows from theorem 5.1 that $J(f) \cup \infty$ is C^∞ connected if one of the following conditions holds;

1. $sing(f^{-1})$ is a bounded set.

2. The Fatou set $F(f)$ has an unbounded component.

3. there exists a curve $L(t)$ $(0 < t < 1)$ with $\lim_{t\to\infty} L(t) = \infty$ such that $f_{|L}$ is bounded. In particular, (3) holds if f has a finite asymptotic value.

Theorem 5.2 Let f be a transcendental entire function. If all the Fatou

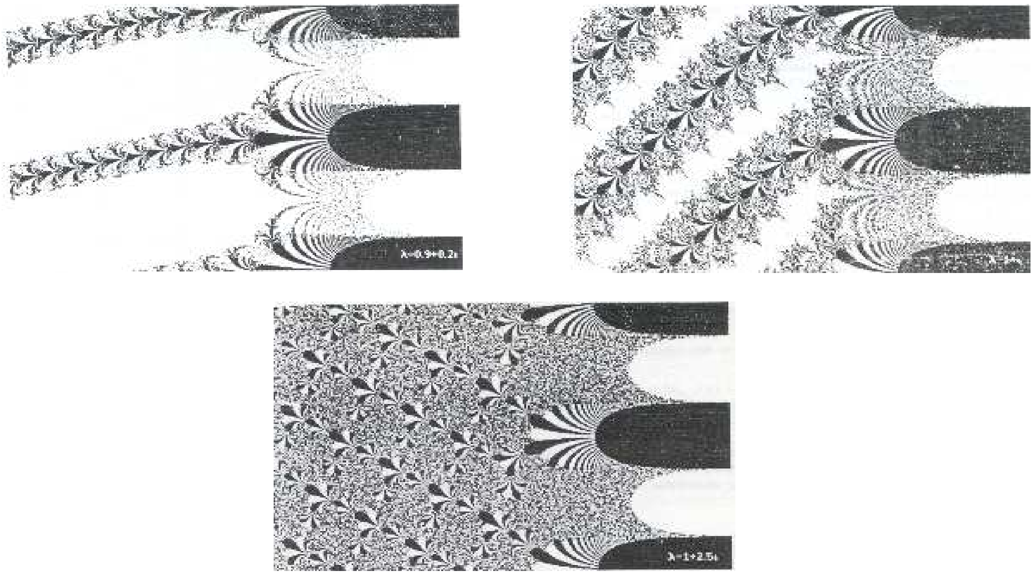

Figure 3: The julia set of $\lambda + z + e^z$ for λ=0.9+0.2i, 1+i and 1+0.25i(left to right).

components of f are bounded and simply connected then $J(f)$ is connected.

The following is an easy consequence of theorem 5.1 and 5.2. If f is a transcendental entire function and all the Fatou components of f are bounded, then $J(f)$ is connected in C if and only if $J(f) \cup \{\infty\}$ is connected in C^∞. The following theorem gives that the connectivity of Julia set $J(f)$ depends on the boundaries of the Fatou components of f.

Thermo 5.3 The boundary of each Fatou components of the function $f(z) = 2 - log2 + 2z - e^z$ is Jorden curve in C^∞. In particular, the Julia set $J(f)$ is connected in C.

Bergweiler found the property in theorem 5.3 independently for the function $f(z) = 2 - log2 + 2z - e^z$ in [8].

6. Growth aspects and complex dynamics

In the dynamics of an entire function the order of an entire function plays an important role. For a polynomial or an entire function of order zero, the basin of attraction of any finite attracting periodic point is bounded. However, this is not necessarily true for entire transcendental functions with non zero order. Bhattacharya [13] showed that the basin of attraction of any finite attracting periodic point is bounded if the entire transcendental function has growth $\left(\frac{1}{2}, 0\right)$, that is, either order

$0 < \rho(f) = \lim_{r\to\infty} \sup \frac{loglog M(r,t)}{logr} \leq \frac{1}{2}$ or $0 < \rho(f) = \frac{1}{2}$ and the type $\tau = \lim_{r\to\infty} sup \frac{logM(r)}{r(f)} = 0$. Thus,

Theorem 6.1([13]) Let $f(z)$ be a non-constant entire function of growth $\left(\frac{1}{2}, 0\right)$ and let $\alpha \in C$ be an attracting periodic point of period n. Then the basin of attraction of α is bounded.

The estimates for the growth of functions with unbounded(e.g including half planes) basins of attraction are obtained in [13]. Baker proved that if f(z) is an entire transcendental function of sufficiently small rate of growth, then $F(f)$ can have unbounded completely invariant component and under suitable slow growth conditions no unbounded component at all. Thus,

Theorem 6.2([3]) If for a transcendental entire function $f(z)$, there is an unbounded invariant component of the Fatou set , then $f(z)$ must be of growth greater that $\left(\frac{1}{2}, 0\right)$.

Theorem 6.3([3]) If a transcendental entire function $f(z)$ is of generalized (α, α)-order $\rho_\alpha(f) = \lim_{r\to\infty} sup \frac{\alpha(logM(r))}{\alpha(logr)}$ with $\alpha(x) = logx$ and $1 < p < 3$, then every component of the Fatou set of $f(z)$ is bounded.

Theorem 6.4([3]) If $f(z)$ is a transcendental entire function of growth not greater than $\left(\frac{1}{2}, 0\right)$, then the Fatou set $F(f)$ has no completely invariant component.

Devaney [19] proved that if a critically finite entire transcendental function f satisfies certain growth conditions(see [25] for specific growth condition), then any point which tends to ∞ under iteration of f lies $J(f)$. Moreover, $J(f)$ is precisely the closure of the set of points which escape to ∞ under iteration.

Baker [2,4] showed the existance of an entire functions of a given order, $0 \leq \rho \leq \infty$, with multiply connected wandering domains. Thus, he proved

Theorem 6.5 For ρ such that $0 \leq \rho \leq \infty$ there is an entire function of order ρ, which has multiply connected wandering domains.

Theorem 6.6 There exists an entire function which has wandering domains of infinite connectivity.

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