Model-theoretic Grothendieck rings of Dense Linear Orders without end points

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1 Introduction

Grothendieck laid the foundations of *K-theory* in 1950's for generalising Reimann-Roch theorem in algebraic geometry. The zeroth K group denoted by K_0 is also called the *Grothendieck group*. In most of the cases K_0 can also be given a commutative ring structure. Later various connections of Grothendieck group were found with different areas of mathematics such as *homological algebra*. In 1995 Kontsevich invented the concept of *motivic integration* associated with a Grothendieck ring. In this report we study a certain type of Grothendieck rings known as model theoretic Grothendieck ring. Model theory is a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas. Motivated by motivic integration Krajiček and Scanlon [1] introduced the concept of model theoretic Grothendieck ring. They associate various interesting combinatorial aspects to the model theoretic Grothendieck ring of structures.

The main objective of this work is to compute model theoretic Grothendieck ring associated with a structure having dense linear order. In section 2 we provide model theoretic definitions of structures and signatures. Moving ahead from section 3 to section 7 are dedicated to preliminaries of model theory following [2]. In section 8 we provide the formal definition of Grothendieck ring associated with a given structure and also give a brief survey about known Grothendieck rings. From section 9 to section 11, the focus is on computation of Grothendieck ring of *Dense Linear Orders (DLOs)* without end points. Finally we conclude the report by talking about some of the interesting combinatorial properties of Grothendieck ring of DLOs without end points.

2 Structures and Signatures

We confront many algebraic structures in mathematics, like groups, rings, fields, posets, etc. One can observe that all these have some *underlying set*, and some *functions* and *relations*. Building on this intuition we define abstract structures but before that let us define signature :

Definition 1. *A* signature $\boldsymbol{L}$ *consists of constant symbols (c), function symbols (F) and relational symbol (R) which are* n-ary *for all* n.

Definition 2. *A* structure $\mathcal{A}$ *is an entity with the following ingredients -:*

- *A set of elements which is called . Cardinality of the structure is the cardinality of A. Note that A should be non-empty.*
- *The collection* $\{c^{\mathcal{A}} \in A \mid c \in L$ *where* c *is a constant symbol*$\}$.
- *The collection* $\{F^{\mathcal{A}} \in A \mid F \in L$ *where* F *is a function symbol*$\}$.
- *The collection* $\{R^{\mathcal{A}} \in A \mid c \in L$ *where* R *is a constant symbol*$\}$.

Note that the signature for groups is $< 1, \cdot, ^{-1} >$, where $\cdot$ is a 2-ary function symbol, $^{-1}$ is a unary relational symbol and 1 is constant symbol. So with an underying set G, we have a structure $(G, 1, \cdot, ^{-1})$.

Another example is of linear order. Suppose $\leq$ is an order relation on A. Then we can make $(A, \leq)$ into a structure $\mathcal{A}$, where A is the domain of $\mathcal{A}$ and $\leq$ is 2-ary relation symbol.

Another interesting example of an structure is a vector space for a fixed field. Here the usual additon, subtraction and 0 are there, but for field, we add a 1-ary function symbol for multiplication by that scalar. Thus, we have a structure vector space over a fixed field.

3 Homomorphisms and Substructures

We define a homomorphism between two structure which have same signature. A homomorphism is a map from one structure to another structure preserving quantities related to the signature.

Definition 3. *Formally, a map $f : A \longrightarrow B$ is a homomorphism if following conditions are satisfied:*

- $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$ *for each constant symbol* c *in* $\boldsymbol{L}$*.*
- $f(F^{\mathcal{A}}(\bar{a})) = F^{\mathcal{B}}(f(\bar{a}))$*, for each function symbols* $\boldsymbol{F} \in L$*.*
- *if* $\bar{a} \in R^{\mathcal{A}}$ *, then* $f(\bar{a}) \in R^{\mathcal{B}}$ *for each relation symbols* $\boldsymbol{R} \in L$*.*

If a homomorphism is *injective* and the following property hold, then it is called an *embedding.*

- $\bar{a} \in R^{\mathcal{A}}$ iff $f(\bar{a}) \in R^{\mathcal{B}}$.

An **isomorphism** is a surjective embedding. For two isomorphic $L-structures$ $\mathcal{A}, \mathcal{B}$, we write $A \cong B$. With all this we have the following theorem.

Theorem 1. *Let L be a signature.*

1. *If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are $L-structures$ and $f : A \to B$ and $g : B \to C$ are homomorphisms, then gf is also a homomorphism.*
2. $1_{\mathcal{B}} f = f = f 1_{\mathcal{A}}$.
3. *If f and g are ismomorphisms then so is gf. $\cong$ is an equivalence relation.*
4. *If $gf = 1_{\mathcal{A}}$ and $fh = 1_{\mathcal{B}}$ then f is an isomorphism and $g = h = f^{-1}$.*

Proof. All of the above statements are just verification and follow from properties listed for homomorphism. For part (e), $g = g1_{\mathcal{B}} = gfh = 1_{\mathcal{A}}h = h$. Since $gf = 1_{\mathcal{A}}$, f is an embedding. Since, $fh = 1_{\mathcal{B}}$, f is surjective.

3.1 Substructures

If $\mathcal{A}$ and $\mathcal{B}$ are $L-structures$ such that $A \subseteq B$ and the inclusion map is an embedding, then we say that A is a substruture of B or symbolically $\mathcal{A} \subseteq \mathcal{B}$. Since inclusion is an *embedding*, it follows that for every constant c in L $c^{\mathcal{B}} \in A$; for every $n-ary$ function symbol F in L and every $n-tuple$ $\bar{a}$ of elements of X, $F^{\mathcal{B}}(\bar{a}) \in X$; for every $n-ary$ relational symbol R, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap (A)^n$. Thus we have the following lemma:

Lemma 1. *Let $\mathcal{B}$ be an $L-structure$ and X a subset of B, then the following are equivalent :*

1. *$X = A$ for some $\mathcal{A} \subseteq \mathcal{B}$*
2. *For every constant c of L $c^{\mathcal{B}} \in A$; for every $n-ary$ function symbol F of L and every $n-tuple$ a of elements of X, $F^{\mathcal{B}}(a) \in X$*

Proof. Assume (1), so it is easy to see that (2) is true by using the fact that the inclusion map has to be an embedding. Conversely, assume (2), we can define $\mathcal{A}$ by putting $A = X$, and for each constant c in L, $F^{\mathcal{A}} = F^{\mathcal{B}}|X^n$, for each $n-ary$ functional symbol F in L, and now define $R^{\mathcal{A}} = R^{\mathcal{B}} \cup X^n$ for each $n-ary$ realational symbol R in L. Hence the lemma.

3.2 Generation of structures

Now we ask a question : given a set $Y \subseteq A$, where $\mathcal{A}$ is an $L-structure$ does there exists a $L-structure$ such that its domain is Y and it is a substructure of $\mathcal{A}$? In general the answer is NO. But it turns out we can have one whose domain contains Y. Let us look at the construction.
We will define Y_is using some set of definitions and then take their union and that will be our required structure, which will be the smallest substructure containing Y. So,

1. $Y_0 = Y \cup \{c^B$ for all c in L$\}$
2. $Y_m = Y_{m-1} \cup \{F^A(\bar{a})$ for all n-tuples $\bar{a} \in Y_m$ and for all n-ary function symbols $F \in L\}$

Finally form a new structure, such that $\cup_{i=0}^{\infty} Y_i$ is the domain of the required substructure.

4 Terms and Atomic Formulas

Every language has a stock of variables. These are symbols written v, x, y, z an one of their purposes is to serve as temporary labels for elements of a structure. So, *terms* are basically just expressions and they get meaning when they are interpreted in the structure. So, for a given signature L,

1. Every variable is a term of L.

2. Every constant is a term of L.
3. For all $n - ary$ function symbol F, $F(s_1, s_2, s_3, \ldots, s_n)$ is a term of L, where each s_i is a terms of L.
4. Nothing else is a term. So, if t is a term, it is just an expression in the language.

4.1 Some definitons

1. Closed - A term without any variables.
2. Complexity of a term - Number of symbols appearing in it, counting each occurence separately. *Important point is that a term having another terms in it will have more complexity than all the terms present in it.*
3. The meaning of $t(\overline{x})$ - Variables in term t are from the sequence $\overline{x} = (x_1, \ldots, x_n)$.

Now the following statements relate the terms of a given signature to its structure.

1. If t is the variable x_i, then $t^{\mathcal{A}}[\overline{a}]$ is a_i.
2. If t is a constant c, then $t^{\mathcal{A}}[\overline{a}]$ is the element $c^{\mathcal{A}}$.
3. If t is of the form $F(s_1, s_2, s_3, ..., s_n)$, where each s_i is a term $s_i(\overline{x})$, then $t^{\mathcal{A}}[\overline{a}]$ is the element $F^{\mathcal{A}}(s_1^{\mathcal{A}}(\overline{a}), ..., s_n^{\mathcal{A}}(\overline{a}))$.

4.2 Atomic formulas

The **atomic formulas** of L are the strings of symbols given by the rules below:

1. If s and t are terms of L then the string $s = t$ is an atomic formula of L.
2. Given R an $n - ary$ relation symbol and $t_1, \ldots, t_n$ are terms of L then the expression $R(t_1, \ldots, t_n)$ is an atomic formula of L.

Atomic sentence is an atomic formula in which there are no variables.
Note that if the variables are interpreted as elements of the structure, it makes a statement about the structure. If this statement holds, we say ϕ is *true of* $\overline{a} \in A$ or in symbols we have,

$$\mathcal{A} \vDash \phi[\overline{a}] \iff \mathcal{A} \vDash \phi(\overline{a})$$

So, formally

1. If ϕ is the formula $s = t$ where $s(\overline{x}), t(\overline{x})$ are terms, then $\mathcal{A} \vDash \phi[\overline{a}]$ *iff* $s^{\mathcal{A}}[\overline{a}] = t^{\mathcal{A}}[\overline{a}]$
2. If ϕ is the formula $R(s_1, ..., s_n)$ where $s_1(\overline{x}), ..., s_n(\overline{x})$ are terms, then $\mathcal{A} \vDash \phi[\overline{a}]$ iff the ordered couple $(s_1(\overline{x}), ..., s_n(\overline{x}))$ is in $R^{\mathcal{A}}$

If ϕ is true in $\mathcal{A}$, then we say that $\mathcal{A} \vDash \phi$. Let T be a set of atomic sentence, we say that $\mathcal{A}$ is a *model* of T (in symbols, $\mathcal{A} \vDash T$) if $\mathcal{A}$ is a model of every atomic sentence in T.

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be $L-$structures and f a map from A to B.*

1. *If f is a homomorphism then for every term $t(\overline{x})$ of L and tuple $\overline{a}$ in A $f(t^{\mathcal{A}}[\overline{a}]) = t^{\mathcal{B}}[f(\overline{a})]$.*
2. *f is a homomorphism* iff *for every atomic formula $\phi(\overline{x})$ of L and tuple $\overline{a}$ in A*

$$\mathcal{A} \vDash \phi[\overline{a}] \Rightarrow \mathcal{B} \vDash \phi[f\overline{a}].$$

3. *f is an embedding iff for eery atomic formula $\phi(\overline{x})$ of L and tuple $\overline{a}$ in A*

$$\mathcal{A} \vDash \phi[\overline{a}] \text{ iff } \mathcal{B} \vDash \phi[f\overline{a}].$$

Proof. 1. The proof of is easy by induction on the complexity of t.
2. When f is homomorphism then if $s^A[\overline{a}] = t^A[\overline{a}]$ means that $f(s^A[\overline{a}]) = f(t^A[\overline{a}])$ and using (1), that gives $s^B[f(\overline{a})] = t^B[f(\overline{a})]$, which is $\phi[f(\overline{a})]$. Similarly we do the same thing for relation symbol as well. For the converse part also argue using the conditions required for homomorphism.
3. It has the same proof as (b). Consider the symbol $\neg$ and we define

$$\mathcal{A} \vDash \neg\phi[\overline{a}] \iff \mathcal{A} \vDash \phi[\overline{a}] \text{ does not hold}$$

5 Canonical Structures

In previous sections we saw some definitions of model of an atomic sentence and hence we can come up with the set of all atomic sentence such that given L structure $\mathcal{A}$ is a model for each of them. Further we also saw how we can translate $\mathcal{A}$ into a set of atomic sentences. We build up the converse here i.e. given a signature L and a set of atomic sentences T of L, we now construct a L structure $\mathcal{A}$ such that every element of $\mathcal{A}$ is a closed term of L further $\mathcal{A}$ is a model for each atomic sentence in T.

Theorem 3. *Given a signature L and a set of atomic sentences T of L, $\exists$ a L structure $\mathcal{A}$ (say) such that following holds.*

1. *$\mathcal{A} \vDash T$.*
2. *Every element of $\mathcal{A}$ is $t^{\mathcal{A}}$ for some closed term t of L..*
3. *There exists a unique homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{B}$ is a L structure and $\mathcal{B} \vDash T$.*

We go via the following definition and lemma for proof of above theorem.

Definition 4. *A set of atomic sentences T of signature L satisfying following two properties is called a* -closed set*.*

1. *For each closed term t of L, $t = t \in T$.*
2. *if $\phi(x)$ is a atomic formula and $s = t \in T$ then $\phi(s) \in T$ iff $\phi(t) \in T$.*

Lemma 2. *Given a signature L and a -closed set T of L then there exists a L structure $\mathcal{A}$ such that the following holds*

1. *Set of all atomic sentences of L true in $\mathcal{A}$ is exactly T*
2. *Every element of $\mathcal{A}$ is $t^{\mathcal{A}}$ for some closed term t of L*

Proof. (Theorem) The proof of the first two parts of the theorem is now straightforward. We just take the minimal -closed set containing T, U (say) and hence as $\mathcal{A} \vDash U$, hence $\mathcal{A} \vDash T$. For the third part we use the Diagram lemma and show that every atomic sentence ϕ of L true in $\mathcal{A}$ is also true in $\mathcal{B}$. To see this take the set of all true atomic sentences in $\mathcal{B}$, U' (say), by definition of U, U' contains U, and noting that U' is a -closed set, the theorem follows.

We now give a proof sketch of Lemma 2.

Proof. (Lemma) We explicitly construct the L structure $\mathcal{A}$. Consider the set of all closed terms of L. We say a closed term t is related to closed term s by relation $\sim$, $(s \sim t)$ *iff* $(s = t) \in T$. Using conditions of -closed set one can show that this is an equivalence relation. Represent equivalence class of t by $\tilde{t}$. We now construct the structure $\mathcal{A}$. Define A to be set of equivalence classes of $\sim$ and for each named constant symbol c of L define $c^{\mathcal{A}} = \tilde{c}$. For each n and each n-ary function symbol F of L and a tuple $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ define

$$F(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n) = \tilde{F}(t_1, t_2, \ldots, t_n)$$

. Further for each n and n-ary relation symbol R and a tuple $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ define

$$(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n) \in R^{\mathcal{A}} \textit{ iff } R((t_1, t_2, \ldots, t_n)) \in T$$

. Using the construction and the conditions for -closed set one can easily show the lemma.

The L structure constructed in $\mathcal{A}$ is called a canonical L structure of T and theorem guarantees that two such structures would be isomorphic.

6 Inserting Logical Operators and Definable Sets

The set of all atomic formulae of a given signature L is not enough at times to define a certain useful property or subset of A^n. We also need to take their logical combinations in a given interpretation as a formula usually refers to a property that a set of elements of A satisfy but we might need intersection or union or complement of one or more such sets. For this purpose we insert logical connectives and enlarge our language formulas which initially consisted of only atomic formulas. We would restrict ourselves to first order language which is main focus of this work.

Definition 5. *For a given signature L define the set $\mathcal{L}$ as:*

1. *All atomic formulas are in $\mathcal{L}$.*
2. *If ϕ and ψ are in $\mathcal{L}$ then $\phi \wedge \psi$ and $\phi \vee \psi$ are in $\mathcal{L}$.*
3. *If $\phi \in \mathcal{L}$ then $\neg\phi \in \mathcal{L}$.*

4. *If y is a free variable in ϕ then $\exists\ y\ \phi$ and $\forall\ y\ \phi$ is in $\mathcal{L}$* (free variable is in the first order logic sense).
5. *Nothing else is in $\mathcal{L}$.*

Elements in $\mathcal{L}$ would be called as *formulas* and a formula without any free variables is called a *sentence*. Now we talk about similar concepts like a formula being true in an structure as we did for atomic formulas. Given a L structure $\mathcal{A}$, formula ϕ and a tuple $\bar{a}$ of appropriate length we say that $\mathcal{A} \vDash \phi(\bar{a})$ if the following happens:

1. If ϕ is of the form $\phi_1 \vee \phi_2$ then $\mathcal{A} \vDash \phi(\bar{a})$ *iff* $\mathcal{A} \vDash \phi_1(\bar{a})$ or $\mathcal{A} \vDash \phi_2(\bar{a})$.
2. If ϕ is of the form $\phi_1 \wedge \phi_2$ then $\mathcal{A} \vDash \phi(\bar{a})$ *iff* $\mathcal{A} \vDash \phi_1(\bar{a})$ and $\mathcal{A} \vDash \phi_2(\bar{a})$.
3. If ϕ is of the form $\neg\psi$ then $\mathcal{A} \vDash \phi(\bar{a})$ *iff* its false that $\mathcal{A} \vDash \psi(\bar{a})$.
4. If ϕ is of the form $\exists\ y\ \psi$ then $\mathcal{A} \vDash \phi(\bar{a})$ *iff* there is a $b \in A$ such that $\mathcal{A} \vDash \psi(b, \bar{a})$.
5. If ϕ is of the form $\forall\ y\ \psi$ then $\mathcal{A} \vDash \phi(\bar{a})$ iff for each $b \in A$ $\mathcal{A} \vDash \psi(b, \bar{a})$.

$\mathcal{A}$ is said to be a model for a sentence ϕ if $\mathcal{A} \vDash \phi$. Further given a n-ary formula $\phi(\bar{x})$, by $\phi(\mathcal{A}^n)$ we mean all the tuples $\bar{a}$ such that $\mathcal{A} \vDash \phi(\bar{a})$. Further if $\phi(\bar{x}, \bar{y})$ is a formula and let $\bar{b}$ be of length of $\bar{y}$ then by $\phi(\mathcal{A}^n, \bar{b})$ we mean all the tuples $\bar{a}$ such that $\mathcal{A} \vDash \phi(\bar{a}, \bar{b})$. A set $\phi(\mathcal{A}^n)$ for some formula is called a *definable set without parameters*. Further if X is a subset of A and $\bar{b}$ be a sequence of elements from X then the set $\phi(\mathcal{A}^n, \bar{b})$ is called a *definable set with parameters from X* i.e. we include X in our signature as we did previously for atomic formulas.

Now with respect to definable sets the above logical operations equivalently mean union, intersection, complement, projection. With all this we have the following theorem:

Theorem 4. *Let $\mathcal{L}$ be a first order language as in Definition 5 and let $\mathcal{A}$ be a L structure. Further if Y is a definable subset of A defined by some formula $\phi(x)$ with parameters from X. If f is an automorphism of $\mathcal{A}$ such that f fixes X point-wise then f fixes Y set-wise.*

Proof. The proof is via induction on complexity (number of symbols appearing in the formula with base case as an atomic formula) of a formula and is a mechanical exercise.

Corollary 1. *If L is an empty signature then $\mathcal{A}$ is just a set and automorphism are just set bijections of A onto A. If Y is subset of A defined by some formula with parameters from X and f is an automorphism as in the previous theorem. Then either $Y \subseteq X$ or $\overline{X} \subseteq Y$ where $\overline{X}$ means the complement of X in A.*

Proof. The proof is via contradiction. Suppose this is not the case then there exist an element $\alpha \in A \setminus (X \cup Y)$ and choose any $\beta \in Y \setminus X$ and let f be such that it fixes every point of A except $f(\alpha) = \beta$ and $f(\beta) = \alpha$. Now using the previous theorem f should fix Y but $f(\beta) \notin Y$.

We further recall few more terminologies from model theory. Given a first order language $\mathcal{L}$ let T be a set of sentences in $\mathcal{L}$. We get the class of all L structures K such that every structure in K is a model for each sentence in T. We say that T *axiomatises* K. Further for a given class of L structures form the set of all $\mathcal{L}$ sentences T (say) such that each structure in K is a model for each sentence in T. Such a T is called $\mathcal{L}-$*theory of* K. These correspondences when looked upon as operation on T and K respectively have their interesting properties but we leave those things for future exploration as they are not currently relevant to the discussion. For now we move on to the concept of *quantifier elimination* in a theory.

7 Quantifier Elimination

Definition 6 (Eliminating set). *We say that a set Φ of formulas of L is an elimination set for K if for every formula $\phi(\overline{x})$ of L, there is a formula $\phi^*(\overline{x})$ such that $\phi*$ is equivalent to ϕ in every structure and is a boolean combination of formulas from a set. That set is called eliminating set.*

So, quantifier elimination refers to finding an eliminating set for a given structure K. Also, if a theory has such a set which works for all models of the theory, then theory is said to have the *property of quantifier elimination.*

Lemma 3. *Suppose*

1. *every atomic formula of L is in Φ.*
2. *for every atomic formula $\theta(\overline{x})$ of L which is of form $\exists y \bigwedge_{i<n} \psi_i(\overline{x}, y)$, where each $\psi(\overline{x}, y)$ is in Φ or Φ^-, there is a formula $\theta^*(\overline{x})$ of L which*
   (a) *is a boolean combination of formulas in Φ.*
   (b) *is equivalent to θ in every structure in K.*

Then Φ is an elimination set for K.

Proof. Use the fact that $\forall\ y\ \phi$ is equivalent to $\neg\exists x \neg\phi$. And then use induction on complexity of $\theta(\overline{x})$.

Theorem 5. *Let L be a language whose signature consists of $2-ary$ relational symbol $<$, and let K be the class of all dense linear orderings. If Φ consists of formula $x < y$, $x = y$, it is an elimination set.*

Proof. For proving quantifier elimination in dense linear order without end-points, we will find an eliminating set using the lemma. So, obviously the eliminating set contains at least all the atomic formulas. Now, there are only two possible atomic formulas : $x = y$ and $x < y$ and also their negation. Note that there $\exists$ distributes over disjunction of formulas and we know that every first order quantifier-free formula can be converted to Disjunctive Normal Form. Hence every formulas of form $\exists y \bigwedge_i \phi_i(x, y)$ can be converted to $\bigvee_i \psi_i(x, y)$, where psi_i is conjunction of literals. So, now let's have a look at the possibilities of ψ_i.

First of all if a literal doesn't have variable y in it, that can be pulled out of the bracket. Also, θ and $\neg\theta$ can't be together, where θ is an atomic formula. So, the only possibilities are that ψ_i is

- $x = y$
- $x > y$
- $x < y$

and their negations with conjunctions between them. So, it is easy to see that either the formulas define the whole set or the empty set and nothing else, which means that we have a formula logically equivalent to $\exists \bigvee_i \psi_i(\overline{x}, y)$ and thus, by use of lemma, only the atomic formulas form eliminating set.

8 Grothendieck Ring of a Structure

Having all the model theoretic terminologies in hand we now look at the definable sets in aggregate. We would like to define something like all the n-ary definable subsets i.e. all the definable subsets of A^n. But we must realize that till now we haven't removed the redundancy of formulas. We do it by fixing a fixed countable sequence of variables as $\{x_1, x_2, \ldots, x_n, \ldots\}$. This still doesn't remove the redundancy of formulas as for two formulas $\phi(A^n)$ and $\psi(A^n)$ may be same sets but they may be having different variable sets. But we see that this wouldn't matter if we are only concerned with the n-ary definable sets rather than the n-ary formulas. Now by *Def*(A^n) denote the set of all n-ary definable subsets of A^n (by definable set here we mean all definable set with or without parameters). Further for each n aggregate these sets to form *Def*($\mathcal{A}$) i.e. we take a direct limit under inclusion. Hence

$$Def(\mathcal{A}) = \varinjlim_{n\to\infty} Def(A^n)$$

Definition 7. *For definable sets D_1 and D_2 in Def($\mathcal{A}$), and a bijective map $f : D_1 \longrightarrow D_2$ is called a* definable bijection *if Graph*$(f) = \{(\bar{a}, f(\bar{a})) \mid \forall \bar{a} \in D_1\} \in Def(\mathcal{A})$

Now we relate elements of $Def(\mathcal{A})$ by definable bijections, i.e. we say $D_1 \sim D_2$ if there is a definable bijection between D_1 and D_2. One easily checks that $\sim$ is an equivalence relation. Further we look at the quotient space of *Def*($\mathcal{A}$) with respect to $\sim$ and call it $\widetilde{\mathrm{Def}}(\mathcal{A})$. With $[D]$ we would represent the equivalence class of D. Now we would put some algebraic structure on the quotient space and study it.

8.1 Formal construction

Put the following algebraic structure on $(\mathcal{A})$.

1. $0 := [\{\Phi\}]$ (equivalence class of empty set)
2. $1 := [\{\bar{a}\}]$ (equivalence class of singleton)
3. $[A] + [B] := [A' \cup B']$ where $A' \cap B' = \Phi$ (note that such A' and B' will always exist)
4. $[A].[B] := [A \times B]$

It is easy to verify that '$+$' and '$.$' are commutative and '$.$' distributes over '$+$'. The above definitions define a commutative *semiring* structure on $\widetilde{Def}(\mathcal{A})$.

Definition 8 (Cancellative semiring). *A semiring S is said to be cancellative if for each $A, B, C \in S$, $A + C = B + C \;\Rightarrow\; A = B$.*

Definition 9. *A binary relation $\sim$ on a semiring S is called a* congruence relation *if*

1. *$\sim$ is an equivalence relation.*
2. *For $a, b, c, d \in S$, with $a \sim b$ and $c \sim d$ we have $a + c \sim b + d$ and $ac \sim bd$.*

Given a semiring S one can construct a cancellative semiring from it using a standard procedure described in the following theorem [3]:

Theorem 6. *Let S be a semiring and $\sim$ be a binary relation defined as follows.*

$$\text{for } a, b \in S,\; a \sim b \iff \exists c \in S,\; a + c = b + c$$

Then $\sim$ is a congruence relation. Denote the quotient space of S by $\tilde{S}$. $\tilde{S}$ forms a cancellative semiring under induced multiplication and addition operation. Further the natural map $f : S \longrightarrow \tilde{S}$ is a surjective homomorphism of semirings. Moreover given a cancellative semiring T and a surjective homomorphism $g : S \longrightarrow T$ there is a unique semiring homomorphism $\overline{g} : \tilde{S} \longrightarrow T$ such that the diagram S [dr,"g"] [r, "f"] [d,"$\exists!\overline{g}$"]
T commutes.

Further given a cancellative semiring S we now form the ring of differences of the semiring. Consider the set $S \times S$ with a binary relation E defined as follows.

$$For\ (a, b),\ (c, d) \in S \times S,\ (a, b)E(c, d) \iff a + d = b + c$$

One can easily verify that E is an equivalence relation and hence we look the the quotient space

$$(S \times S)/E := \{(a, b)_E \mid (a, b) \in S \times S\}$$

where $(a, b)_E$ represents the equivalence class of (a, b). The quotient space $(S \times S)/E$ is also a cancellative semiring with '$+$' and '$.$' operations induced from S as follows:

1. $(a, b)_E + (c, d)_E = (a + c, b + d)_E$.
2. $(a, b)_E.(c, d)_E = (ac + bd, ad + bc)_E$.

Theorem 7. *Assuming the above notations the quotient space $(S \times S)/E$ form a ring with $-(a, b)_E = (b, a)_E$. Further there is an injective semiring homomorphism $i : S \longrightarrow (S \times S)/E$ such that for $a \in S$, $i(a) = (a, 0)_E$. Moreover given a ring T with an injective semiring homomorphism $g : S \longrightarrow T$, there is a unique ring homomorphism $\overline{g} : (S \times S)/E \longrightarrow T$ such that the diagram S [dr,"g"] [r, "i"] $(S{\times}S)/E$[d,"$\exists!\overline{g}$"]*
T commutes.

Using standard notation of *K-theory* we denote the quotient $(S \times S)/E$ by $K_0(S)$. This is called the *Grothendieck ring* associated with cancellative semiring S. In respect to model theory one notes that we can define a Grothendieck ring associated with a structure $\mathcal{A}$ using the semiring $(\mathcal{A})$ by using Theorem 6 and Theorem 7. We would denote the Grothendieck ring associated with $(\mathcal{A})$ by $K_0(\mathcal{A})$.

8.2 PHP and ontoPHP

In this section following [1] we define the concepts of *Pigeon Hole Principle(PHP)* and *onto Pigeon Hole Principle(ontoPHP)* for a structure. A structure A is said to satisfy ontoPHP if for every $D \in Def(\mathcal{A})$ and $\overline{a} \in D$, there does not exists a definable bijection between D and $D \backslash \{\overline{a}\}$. Further a structure is said to satisfy PHP if for every $D_1, D_2 \in Def(\mathcal{A})$ such that $D_1 \subsetneq D_2$, there does not exists a definable bijection between D_1 and D_2. With all this we have following.

Lemma 4. *Given a structure $\mathcal{A}$, $K_0(\mathcal{A})$ is non trivial iff $\mathcal{A}$ satisfies ontoPHP.*

Proof.

$$\begin{aligned} K_0(\mathcal{A}) = 0 &\iff 0 = 1 \\ &\iff ([D], 0)_E = ([D] + [\{a\}], 0)_E \textit{ for some } D \in Def(\mathcal{A}), \textit{ and } \overline{a} \in A^n \end{aligned}$$

where $(.,.)_E$ represent the equivalence class as defined in previous section.

Definition 10 (Partially ordered ring). *A commutative ring R with unity is said to be* partially ordered *if there exists $P \subseteq R$ such that the following conditions are satisified:*

1. $0 \in P$
2. $1 \in P$
3. $P + P \subseteq P$
4. $P.P \subseteq P$
5. *For $x \neq 0$ and $x \in P$ then $-x \notin P$*

P as above is usually referred as the positive part of ring R. With this we have the following result.

Theorem 8. *A structure $\mathcal{A}$ satisfies PHP iff $K_0(\mathcal{A})$ is a partially ordered ring.*

Proof. The proof is easy. One can look at [1] theorem 4.3.

8.3 A brief survey on known Grothedieck rings

Only a few examples of Grothendieck rings are known. If M is a finite structure, then $K_0(M) \cong \mathbb{Z}$. Krajiček and Scanlon showed in [1, Example 3.6] that $K_0(\boldsymbol{R}) \cong \mathbb{Z}$ using dimension theory and cell decomposition theorem for o-minimal structures, where $\boldsymbol{R}$ denotes a real closed field. Cluckers and Haskell [4], [5] proved that the fields of p-adic numbers and $F_q((t))$, the field of formal Laurent series, both have trivial Grothendieck rings, by constructing definable bijections from a set to the same set minus a point. Denef and Loeser [6], [7] found that the Grothendieck ring $K_0(\mathbb{C})$ of the field $\mathbb{C}$ of complex numbers regarded as ring admits the ring $\mathbb{Z}[X, Y]$ as a quotient. Krajiček and Scanlon obtained a strong result that $K_0(\mathbb{C})$ contains an algebraically independent set of size c of the continuum, and hence the ring $\mathbb{Z}[X_i : i \in \mathfrak{c}]$ embeds into $K_0(\mathbb{C})$. Perera showed in [8, Theorem 4.3.1] that $K_0(M) \cong \mathbb{Z}[X]$ whenever M is an infinite module over an infinite division ring. Prest conjectured [8, Chapter 8, Conjecture A] that $K_0(M)$ is nontrivial for all nonzero right R-modules M. In his PhD. thesis, Kuber [3], [9] solved the conjecture by showing that $K_0(M)$ is actually a quotient of a monoid ring and, furthermore, it is nontrivial.

9 Model Theory for DLOs Without End Points

9.1 Theory of DLOs without end points

Fix the signature $(<)$ and let $\mathbb{Q}$ be the structure of the signature, then $\mathbb{Q} = (Q, <)$. Further define the theory of DLO without end points with following axiomatization:

1. Linear Order: $\forall x\ \forall y\ (x < y \vee x < y \vee x = y)$.
2. Linear Order: $\forall x\ \neg(x < x)$.
3. Transitivity: $\forall x\ \forall y\ \forall z\ (x < y \wedge y < z \Rightarrow x < z)$.
4. Without end points: $\forall x\ \exists y\ (y < x)$
5. Without end points: $\forall x\ \exists y\ (x < y)$
6. Density: $\forall x\ \forall y\ (x < y \Rightarrow \exists z(x < z \wedge z < y))$

Further Theorem 5 implies that theory of DLOs without end points admits quantifier elimination. Examples of DLOs without end points include the structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$. Our aim in the following sections would be to compute the Grothendieck ring of DLOs without end points denote by $K_0(\mathbb{Q})$. Here are few notations which we fix beforehand for ease.

1. We denote the variable set $\{X_{n+1}, X_{n+2}, \ldots, X_{n+m}\}$ by $\overline{X}[n+1 : n+m]$ for $n \in \mathbb{N}$.
2. With $\overline{X}$ we refer to $\{X_1, X_2, \ldots, X_n\}$ for some $n \in \mathbb{N}$. We would use $\overline{X}$ and $\overline{X}[1 : n]$ interchangeably though they have the same meaning.
3. $\mathcal{C}_\Phi$ represents set of conjunctive clauses for a propositional formula $\Phi(\overline{X})$ written in DNF.
4. For a conjunctive clause γ, $f \in \gamma$ means an atomic formula f of the language which appears in γ.

9.2 Definable sets in dimension n

Let $\mathcal{D}_n^{\emptyset}$ represent the set of definable sets in X^n by formulas containing no parameters :

$$\mathcal{D}_n^{\emptyset} := \{\Phi(\mathbb{Q}^n) \mid \Phi(\overline{X}[1:n] \textit{ is a formula with } n \textit{ variables only})\}.$$

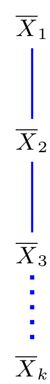
Given a definable set $D \in \mathcal{D}_n^{\emptyset}$ such that $D = \Phi(\mathbb{Q}^n)$, we convert $\Phi(\overline{X})$ into DNF with conjunctive clauses containing only positive literals (due to strict order). Now we show that $\mathcal{D}_n^{\emptyset}$ forms a finite Boolean algebra. For proving this we devise a pictorial representation of atomic formulas present in a conjunctive clause of $\Phi(\overline{X})$.

For $X_i > X_j$

For $X_i = X_j$

Definition 11. *Call a definable set $D \in \mathcal{D}_n^{\emptyset}$ to be* related set *if $D = \Phi(\mathbb{Q}^n)$ such that every pair of variables (X_i, X_j) appearing in Φ are comparable (via the order relation of the structure) and the comparablity is unique throughout D i.e. $\forall \overline{a} \in D$ we have unique relation between (a_i, a_j) for all (i, j).*

Clearly they will correspond to the diagrams of the form



where $\overline{X}_i$ refers to a tuple, $(X_{i_1}, X_{i_2}, \cdots, X_{i_{l_i}})$ corresponding to the diagrams of the form

where $X_{i_t} \in \overline{X}[1:n]$ for $t \in \{1, 2, \ldots, l_i\}$ such that each X_i appears exactly once in the diagram. For a related set R let $H(R)$ *represent the diagram associated to* R.

Definition 12. *[Height of a related set] Define the height of a related set* R *to be* k *in* $H(R)$.

Proposition 1. *The related sets in* $\mathcal{D}_n^{\emptyset}$ *form atoms for boolean algebra structure on* $\mathcal{D}_n^{\emptyset}$.

Proof. To begin with we show that any two distinct related sets are mutually disjoint. Say R_1 and R_2 are distinct related sets in $D_n^{\emptyset}$ then there exists variable pair (X_i, X_j) which differs in their relation in R_1 and R_2. WLOG assume that $X_i < X_j$ in R_1 and $X_j < X_i$ in R_2 and if $\overline{a} \in R_1 \cap R_2$ then

$$\mathbb{Q} \models \Phi_1(\overline{a})$$

$$\mathbb{Q} \models \Phi_2(\overline{a})$$

but we cannot have $a_i < a_j$ and $a_j > a_i$ both in strict dense linear order simultaneously. Hence a contradiction.
For showing that any definable set can be written as their disjoint union we do a conjunctive clause wise analysis. Fix a clause $\gamma \in \mathcal{C}_{\Phi}$, if $X_i > X_j$ for some i, j appears in the clause then this means that we just look at the set of all related diagrams with $X_i < X_j$ (similarly with $X_i = X_j$). Every such atomic formula fixes the relation between two variables. Let for every $f \in \gamma$, S_f^{γ} represent the set of all the related sets which satisfy f. The quantity

$$S^{\gamma} = \bigcap_{f \in \gamma} S_f^{\gamma}$$

is the set definable by the conjunctive clause (by definition). Now for the formula $\Phi(\overline{X})$ we have

$$D = \bigcup_{\gamma \in \mathcal{C}} S^{\gamma}$$

Now the set defined by the union of elements in D is set definable by $\Phi(\overline{X})$.

We fix the following notations before proceeding.

1. By $At_n^{\emptyset}$ we would mean the set of related sets of $\mathcal{D}_n^{\emptyset}$.
2. By $A(\overline{X}[1:n-1], a)$ for some $a \in Q$ we mean evaluation at $X_n = a$ and similarly for any definable set D.
3. Let $\mathcal{D}_n^{\overline{a}}$ denote the set of all the definable sets defined by formulas of n variables and $\overline{a}$ parameters.

We abuse the notation slightly by using $At_n^{\emptyset}$ for set of equivalence classes of formulas which represent same atom in $\mathcal{D}_n^{\emptyset}$. One also notes that the pictorial representation of formula will essentially be putting a in place of X_n. With all this we have following proposition.

Proposition 2. *$\mathcal{D}_n^{\overline{a}}$ forms a boolean algebra with*

$$\{A(\overline{X}[1:n],\overline{a}) \neq \emptyset \mid A \in At_{n+m}^{\emptyset}\}$$

as the set of atoms.

Proof. The disjointness of $A_1(\overline{X},\overline{a})$ and $A_2(\overline{X},\overline{a})$ follows from the fact that $A_1(\overline{X},\overline{X}[n+1:n+m])$ and $A_2(\overline{X},X_{n+1},\ldots,X_{n+m})$ are disjoint. To show that they generate any definable set in $\mathcal{D}_n^{\overline{a}}$, we show that the evaluation map

$$\begin{aligned} \mathrm{eval}_{\overline{X}[n+1:n+m]}^{\overline{a}} : \mathcal{D}_{n+m}^{\emptyset} &\to \mathcal{D}_n^{\overline{a}} \\ D(\overline{X}[1:n+m]) &\mapsto D(\overline{X}[1:n],\overline{a}). \end{aligned}$$

The above map is clearly a surjection. If A is an atom below D in the boolean algebra $\mathcal{D}_{n+m}^{\emptyset}$ then $A(\overline{X},\overline{a})$ lies below $D(\overline{X}[1:n],\overline{a})$ in $\mathcal{D}_n^{\overline{a}}$. To see this if $A(\overline{X},\overline{a}) = \emptyset$ then its trivial otherwise if $X \models A(\overline{b},\overline{a})$ for some $\overline{b}$ then $X \models D(\overline{b},\overline{a})$ as $A(\overline{X}[1:m+n])$ is below $D(\overline{X}[1:m+n])$ which also implies that $A(\overline{X},\overline{a})$ is below $D(\overline{X},\overline{a})$. Hence the result.

Definition 13 (Decomposition into related sets). *We define the map*

$$\begin{aligned} At_{n;\overline{a}} : \mathcal{D}_n^{\overline{a}} &\to \mathcal{P}(At_n^{\overline{a}}) \\ D &\mapsto \{A \in At_n^{\overline{a}} | A \cap D \neq \emptyset\} \end{aligned}$$

This maps gives us the atomic decomposition of definable set $D \in \mathcal{D}_n^{\overline{a}}$. We define *related set* in $\mathcal{D}_n^{\overline{a}}$ to be an element which is defined by a formula in which any pair of the form (X_i, X_j) or (X_i, a_j) are related for all (i, j) and the relation between (X_i, X_j) or (X_i, a_j) is unique. Clearly every element in $At_n^{\overline{a}}$ is a related set by because any element in $At_{n+m}^{\emptyset}$ is related. We claim the converse to be true. To see this just construct the formula corresponding to related set $R \in \mathcal{D}_n^{\overline{a}}$ and with slight abuse of notation let $R(\overline{X},\overline{a})$ represent the formula for R. As noted in the proof, the map $\mathrm{eval}_{\overline{X}[n+1:n+m]}^{\overline{a}}$ is a surjection. We obtain a representative formula $R'(\overline{X}[1:n+m])$ such that

$$\mathrm{eval}_{\overline{X}[n+1:n+m]}^{\overline{a}}(R'(\overline{X}[1:n+m])) = R(\overline{X},\overline{a})$$

where $R'(\overline{X}[1:n+m])$ is obtained just by replacing each X_{n+i} by a_i and now use the previous proposition. Further we order $\overline{a}$ in descending order for our convenience. One notes that ordering has no affect.

$$\begin{tikzcd}
\overline{X}_{p_1} \arrow[d, dash] \\
\overline{X}_{p_2} \arrow[d, dash] \\
\overline{X}_{p_1} \arrow[d, dash, dotted] \\
\overline{X}_{p_k}
\end{tikzcd}$$

where $p_i \in \{a_i, e_i\}$ and $\overline{X}_{a_i}$ refers to a tuple, $(X_{i_1}, X_{i_2}, \cdots, X_{i_{l_i}})$ corresponding to diagrams of the form

$$\begin{tikzcd}
a_i \arrow[r, dash] & X_{i_1} \arrow[r, dash] & X_{i_2} \arrow[r, dash] & X_{i_3} \arrow[r, dash, dotted] & X_{i_{l_i}}
\end{tikzcd}$$

and $\overline{X}_{e_i}$ refers to a tuple $(X_{i_1}, X_{i_2}, \cdots, X_{i_{l_i}})$ corresponding to diagrams of the form

$$\begin{tikzcd}
X_{i_1} \arrow[r, dash] & X_{i_2} \arrow[r, dash] & X_{i_3} \arrow[r, dash, dotted] & X_{i_{l_i}}
\end{tikzcd}$$

Definition 14 (Components of a related set). *Define the set of* components *of a related set* $R \in At_n^{\overline{a}}$ *as*

$$\text{Comp}(R) := \{\overline{X}_{p_i} \mid \textit{for each } \overline{X}_{p_i} \textit{appearing in H(R)}\}.$$

Similar to Definition 12 we define *Height* of related set $R \in At_n^{\overline{a}}$ as

$$Height(R) = \text{card}\left\{\overline{X}_{p_i} \in \text{Comp}(R) \;\middle|\; p_i = e_i\right\}.$$

It's clear how to construct a formula from an related set. Hence we induce order relation between $\overline{X}_{p_i}$'s from the order relation on $\overline{X}[1:n]$ naturally. For $\overline{X}_{p_i}, \overline{X}_{p_j} \in \text{Comp}(R)$, we say $\overline{X}_{p_i} < \overline{X}_{p_j}$ if in $H(R)$ $\overline{X}_{p_j}$ appears above $\overline{X}_{p_i}$. Further extend the definition of $Height()$ to $\mathcal{D}_n^{\overline{a}}$ by defining

$$Height(D) = \max\{Height(R) \mid R \in (D)\}$$

for any $D \in \mathcal{D}_n^{\overline{a}}$.

Definition 15. *Assume $\bar{a}$ is sorted in descending order then for $a_i \in \bar{a}$ let*

$$\#(\overline{X}_{a_i}, \overline{X}_{a_{i+1}}) = \text{card}\left(\left\{\overline{X}_{e_k} \in \mathcal{C}(A_1) \;\middle|\; \overline{X}_{a_i} < \overline{X}_{e_k} < \overline{X}_{a_{i+1}}\right\}\right).$$

Definition 16 (Equivalence up to permutation). *For any $n, \overline{a}$ and for $A_1, A_2 \in At_n^{\overline{a}}$ we say $H(A_1)$ and $H(A_2)$ are* equivalent up to permutation, *denoted as $H(A_1) \sim_\sigma H(A_2)$ if for every consecutive a_i, a_j such that $a_i > a_j$*

$$\#(\overline{X}_{a_i}, \overline{X}_{a_j}) = \#(\overline{Y}_{a_i}, \overline{Y}_{a_j})$$

where A_1 is represented in dummy variables $\overline{X}[1:n]$ and A_2 by $\overline{Y}[1:n]$.

The above definition directly implies $Height(A_1) = Height(A_2)$ and

$$\mathrm{card}(\mathrm{Comp}(A_1)) = \mathrm{card}(\mathrm{Comp}(A_2)).$$

With slight abuse of terminology we further call two related sets A_1 and A_2 to be *equivalent up to permutation* denoted by $A_1 \sim_\sigma A_2$ if $H(A_1) \sim_\sigma H(A_2)$. With all this we have the following proposition :

Proposition 3. *If $A_1 \sim_\sigma A_2$ then there is a definable bijection between A_1 and A_2.*

Proof. We construct a bijection using definitions. Let $A_1, A_2 \in At_n^{\overline{a}}$. Assuming that $\overline{a}$ is ordered in descending order. Renaming the variables $\overline{X}[n+1:2n]$ by $\overline{Y}[1:n]$ just for ease of notation. The following formula

$$f(\overline{X}, \overline{Y}) = A_1(\overline{X}) \wedge A_2(\overline{Y}) \bigwedge_{\overline{X}_p \in \mathrm{Comp}(A_1)} \overline{X}_p = \overline{Y}_p$$

Note that the formula f is valid because $A_1 \sim_\sigma A_2$. To see that its graph of an bijection we evaluate $f(\overline{X}, \overline{Y})$ on $\overline{b} \in A_1$ we have $\overline{Y}_p = \overline{b}_p$, $\forall \overline{Y}_p \in \mathrm{Comp}(A_2)$. This implies each $Y_i = b_{j_i}$ for some $b_{j_i} \in \overline{b}$. Clearly the tuple $(Y_i)_{i=1}^n = (b_{j_i})_{i=1}^n \in A_2$. One can easily extract a bijection $g : A_1 \to A_2$ out of $f(\overline{X}, \overline{Y})$.

10 Computation of $\widetilde{\mathrm{Def}}$

10.1 Local and global characteristic

In previous section we defined a relation $\sim_\sigma$ on $At_n^{\overline{a}}$ (Definition 16). Proposition 3 makes this relation an equivalence relation as if $A_1 \sim_\sigma A_2$ then there is a definable bijection between A_1 and A_2. We look at the quotient $= / \sim_\sigma$ as we want to construct . For $A \in$ we represent its equivalence class in $/ \sim_\sigma$ by $[A]$. Define for each $[A] \in / \sim_\sigma$, the map

$$\begin{aligned} : \mathcal{D}_n^{\overline{a}} &\to \mathbb{N} \\ D &\mapsto \mathrm{card}\big(\{B \in At_n^{\overline{a}}(D) \mid B\sigma A\}\big). \end{aligned}$$

This definition gives rise to following map :

$$: \mathcal{D}_n^{\overline{a}} \to \bigoplus \mathbb{N}$$

$$D \mapsto \bigoplus_{T \in At_n^{\bar{a}}/\sigma} \chi_T D.$$

We call to be *local characteristic* and to be *global characteristic* in $\mathcal{D}_n^{\overline{a}}$.

Proposition 4. *Given $D_1 \in \mathcal{D}_n^{\bar{a}}$ and $D_2 \in \mathcal{D}_n^{\bar{b}}$ if $\exists$ a parameter set $\bar{c}$ such that $\chi_n^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(D_1) = \chi_n^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(D_2)$ then there is a definable bijection between D_1 and D_2.*

Proof. The proof is easy and directly follows from Proposition 3.

Let $\mathcal{D}_n^Q := \sqcup_{\bar{a}} \mathcal{D}_n^{\bar{a}}$. Motivated from above proposition we give the following definition.

Definition 17. *For D_1, $D_2 \in \mathcal{D}_n^Q$. We say $D_1 \sim_\sigma D_2$, if $\exists$ $\bar{a}$ such that $(D_1) = (D_2)$. Clearly this is an equivalence relation because of above proposition. Let $D \in \mathcal{D}_n^{\bar{a}}$, so its equivalence class is denoted by $[D]$ and*

$$\mathcal{D}_n^Q/\sigma := \{[D] \mid D \in \mathcal{D}_n^Q\}.$$

Proposition 5. *Let $A_1, A_2 \in At_n^{\bar{a}}$ and $A_1 \neq A_2$, we have $\chi_n^{\bar{a}}(A_1 \sqcup A_2) = \chi_n^{\bar{a}}(A_1) + \chi_n^{\bar{a}}(A_2)$.*

Proof. Obvious by definition of $\chi_n^{\bar{a}}$.

10.2 Main lemma

Lemma 5. *Given a definable bijection $\Phi \in \mathcal{D}_{2m}^{\overline{a}\cup\bar{b}\cup\overline{c}}$ between two definable sets - $P_1 \in \mathcal{D}_m^{\overline{a}}$, $P_2 \in \mathcal{D}_m^{\bar{b}}$. Then $\exists$ $\bar{c}$ such that*

$$\chi_m^{\overline{a}\cup\bar{b}\cup\overline{c}}(P_1) = \chi_m^{\overline{a}\cup\bar{b}\cup\overline{c}}(P_2)). \tag{1}$$

Proof. Let $\Phi = \sqcup_{i=1}^m A_i$. Consider A_i and fix i. Note that A_i is the graph of a definable bijection. Let the variables be $\overline{X}[1:2m]$. Let $\overline{q_i} \in A_i$ denote a $2m$-tuple. Compare $(\overline{q_i})_{m+1}$ with $(\overline{q_i})_j (j \in [1:m])$. Assume that each $(\overline{q_i})_j (j \in [1:m])$ is either strictly less or strictly greater than $(\overline{q_i})_{m+1}$. It means that in the related set Φ the relations are either

$$X_{m+1} > X_j \text{ or } X_{m+1} < X_j \text{ for } j \in [1:m].$$

Also, there will be relations between $\overline{X}[m+1:2m]$. Form three sets

$$(\overline{X}_{m+1})_<,\ (\overline{X}_{m+1})_=,\ (\overline{X}_{m+1})_>$$

as described earlier and find the least element in $(\overline{X}_{m+1})_<$, call it X_u; find the greatest element in $(\overline{X}_{m+1})_>$, call it X_v. Note that if $X_j \in \overline{X}_{m+1})_= \Rightarrow j > m$. Clearly

$$(\overline{q_i})_u < (\overline{q_i})_{m+1} < (\overline{q_i})_v$$

and we can find q such that

$$(\overline{q_i})_u < q < (\overline{q_i})_v \quad and \quad q \neq (\overline{q_i})_{m+1}$$

as universe Q is dense. Now consider another tuple $\overline{q_i}'$ such that $(\overline{q_i}')_j = q$ for j such that $X_j \in (\overline{X}_{m+1})_=$ and for other j, it remains unchanged. Note that $\overline{q_i}' \in A_i$, as all relations are still satisfied. But this is a contradiction to the fact that A_i is a bijection as we get two image of one m-tuple. Thus $\exists\ j < m+1$ such that $j \in (\overline{X}_{m+1})_=$. This whole process can be carried out for any X_j for $j > m$.

Form a related set involving variables X_j for $j \in [1:m]$ and their relations between variables and parameters of $\bar{a} \cup \bar{b} \cup \bar{c}$ as that in Φ. Thus, we get for each related set of Φ we have one-one correspondence between related sets of P_1 and P_2, where the parameter set is $\bar{a} \cup \bar{b} \cup \bar{c}$.

Also, these two related sets are equivalent up to permutations. Hence proof of the lemma is complete.

Corollary 2. *Let $D_1, D_2 \in \mathcal{D}_n^{\overline{a}}$ be such that $Height(D_1) \neq Height(D_2)$ then there is not definable bijection between them.*

Proof. The proof directly follows from above lemma.

10.3 Aggregation with respect to n

Let $D \in \mathcal{D}_n^Q$. Also, assume that Ψ_D denotes a defining formula for D then

$$\begin{aligned} \Delta_{n,n+1}^Q : \mathcal{D}_n^Q &\to \mathcal{D}_{n+1}^Q \\ D &\mapsto D' \end{aligned}$$

where $D' \in \mathcal{D}_{n+1}^Q$ and has the defining formula $\Psi_{D'} = \Psi_D \wedge (X_{n+1} = X_1)$. For a general $n < m$ (in case of $n = m$, define $\Delta_{n,n}^Q$ to be identity), $\Delta_{n,m}^Q$ is defined as

$$\mathcal{D}_n^Q \hookrightarrow D_{n+1}^Q \hookrightarrow D_{n+2}^Q \ldots . \hookrightarrow D_m^Q. \tag{2}$$

Lemma 6. *$\Delta_{n,m}^Q : \mathcal{D}_n^Q \to \mathcal{D}_m^Q$ induces a map $\Delta_{n,m}^Q/\sigma : \mathcal{D}_n^Q/\sigma \to \mathcal{D}_n^Q/\sigma$, which is defined as $[D_1] \mapsto [\Delta(D_1)]$.*

Proof. The only thing to check is that map is well defined. Note that the height and relative arrangement of parameters is not changed under the map Δ_n^Q. Thus the lemma is true.

Let $\mathbb{N}$ denote the set of natural numbers. Clearly $\langle \mathbb{N}, \leq \rangle$ forms a directed set. Let $\mathcal{D}_n^Q/\sigma$ be the family of sets indexed by $\mathbb{N}$ and $\Delta_{n,m}^Q/\sigma : \mathcal{D}_n^Q/\sigma \to \mathcal{D}_m^Q/\sigma$ be function defined as above. Note that the following two properties are satisfied :

- $\Delta_{n,n}^Q/\sigma$ is the identity map.
- $\Delta_{n,p}^Q/\sigma = (\Delta_{m,p}^Q/\sigma) \circ (\Delta_{n,m}^Q/\sigma)$ for $n \leq p \leq m$.

Both are obvious. Hence the pair $\langle \mathcal{D}_n^Q/\sigma, \Delta_{n,m}^Q/\sigma\rangle$ forms a directed system over $\mathbb{N}$ and direct limit of the system is defined :

$$\widetilde{\mathrm{Def}} := \frac{\bigsqcup_n \mathcal{D}_n^Q}{\simeq}$$

where if $[D_1] \in \mathcal{D}_n^Q/\sigma$ and $[D_2] \in \mathcal{D}_m^Q/\sigma$, $[D_1] \simeq [D_2]$ if there is some p such that $\Delta_{n,p}^Q/\sigma([D_1]) = \Delta_{m,p}^Q/\sigma([D_2])$.

10.4 $\widetilde{\mathbf{Def}}(\mathbb{Q})$

In this section we show that $(\mathbb{Q}) =$ constructed in the above section.

Theorem 9. *Two definable sets are in bijection if and only if they are in the same equivalence class in .*

Proof. $(\Rightarrow)$: Assume that $D_1 \in \mathcal{D}_n^{\bar{a}}$ and $D_2 \in \mathcal{D}_n^{\bar{b}}$ are in bijection. Then by Lemma 5, there exists a parameter set $\bar{c}$ containing $\bar{a}$ and $\bar{b}$, such that $\chi_n^{\bar{a}\cup\bar{b}\cup\bar{c}}$ is equal. Hence they are in same equivalence class in $\mathcal{D}_n^Q/\sigma$ and consequently in . Now, if $D_1 \in \mathcal{D}_n^Q$ and $D_2 \in \mathcal{D}_m^Q$ are in bijection (and WLOG assume that $n < m$). Let $\Delta_{n,m}^Q(D_1) = D_1'$ and hence $[D_1]_\simeq = [D_2]_\simeq$ and D_1 and D_2 are in definable bijection with each other and hence D_1' and D_2 are. By the previous argument, $[D_1'] \simeq [D_2]$. Thus we have $[D_1]_\simeq = [D_2]_\simeq$.

$(\Leftarrow)$: Assume that two sets D_1 and D_2 are in same equivalence class in . Thus

$$[\Delta_{n,m}^Q(D_1)] = [\Delta_{p,m}^Q(D_2)] \tag{3}$$

Clearly D_1 is in bijection with $\Delta_{n,m}^Q(D_1)$ and similarly for D_2. Thus D_1 and D_2 are in bijection.

In the next section we will study $(\mathcal{Q})$ by endowing the usual semi ring structure on it as discussed in subsection 8.1.

11 Grothendieck ring of dense linear orders $(\mathbf{K_0}(\mathbb{Q}))$

Let us denote the element of a ring as $[D]_\simeq$. Let us define the ring structure as previously done in section 8.

- $0 := [\Phi]_\simeq$
- $1 := [*]_\simeq$
- $[A]_\simeq + [B]_\simeq := [A' \sqcup B']_\simeq$,
  where $A' \cap B' = \emptyset$ and $[A]_\simeq = [A']_\simeq$, $[B]_\simeq = [B']_\simeq$.
- $[A] * [B] := [A \times B]$,which is the cartesian product of A and B.

Theorem 10 (Cancellativity). $[A]_\simeq + [C]_\simeq = [B]_\simeq + [C]_\simeq \Rightarrow [A]_\simeq = [B]_\simeq$.

Proof. $[A]_{\simeq}+[C]_{\simeq}=[A'\sqcup C']_{\simeq}=[B'\sqcup C'']_{\simeq}$. By Theorem 9, $A'\sqcup C'$ and $B'\sqcup C''$ are in bijection with each other. Let $D_1=A'\sqcup C'\in\mathcal{D}_n^{\bar{a}}$ and $D_2=B'\sqcup C''\in\mathcal{D}_m^{\bar{b}}$, and WLOG assume that $n<m$. Let $\Delta_{n,m}^{\bar{a}}(D_1)=\Delta_{n,m}^{\bar{a}}(A')\sqcup\Delta_{n,m}^{\bar{a}}(C')=D_1'$, where $D_1'\in\mathcal{D}_m^{\bar{a}}$. Since D_1' is in bijection with D_2, using Lemma 5

$$\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(D_1')=\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(D_2)$$

Using Proposition 3, we have

$$\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(A')\sqcup\Delta_{n,m}^{\bar{a}}(C'))=\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(A'))+\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(C'))$$

and

$$\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(B')\sqcup\Delta_{n,m}^{\bar{a}}(C''))=\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(B'))+\chi_m^{\bar{a}\sqcup\bar{b}\sqcup\bar{c}}(\Delta_{n,m}^{\bar{a}}(C''))$$

C' and C'' are bijective image of same set C. Thus there exists some parameter set $\bar{p}$ in which their characteristic is same. Thus from above two relations we have that $\exists$ some parameter set $\bar{P}$ such that

$$\chi_m^{\bar{P}}(A')=\chi_m^{\bar{P}}(B') \tag{4}$$

So B' and A' are in definable bijection and consequently B and A. Hence proof is complete.

The above theorem shows that $\widetilde{Def}$ is a semiring under '$+'$ operation. Now our task is to convert the semiring into ring by defining meaning of additive inverse.
Given the cancellative semiring we follow the standard procedure of Theorem 7 is $\widetilde{Def}$, now we will define an equivalence relation E on $\widetilde{\mathrm{Def}}\times\widetilde{\mathrm{Def}}$ such that

$$([D_1]_{\simeq},[D_2]_{\simeq})\ E\ ([D_3]_{\simeq},[D_4]_{\simeq})\Longleftrightarrow[D_1]_{\simeq}+[D_4]_{\simeq}=[D_2]_{\simeq}+[D_3]_{\simeq}$$

Then E is an equivalence relation. Let $([D_1]_{\simeq_1/\sim_2},[D_2]_{\simeq_1/\sim_2})_E$ denote the equivalence class then the quotient structure

$$(R\times R)/E\ :=\{([D_1]_{\simeq_1/\sim_2},[D_2]_{\simeq_1/\sim_2})_E\ :([D_1]_{\simeq_1/\sim_2},[D_2]_{\simeq_1/\sim_2})\in\widetilde{\mathrm{Def}}\times\widetilde{\mathrm{Def}}\}$$

forms a ring with respect to standard operations. The ring $R\times R/E$ as discussed in section 8 is called Grothendieck ring of the structure $\mathbb{Q}$ denoted by $K_0(\mathbb{Q})$.

Let $[D]$ represent the equivalence class of a definable set D in to ease the notations. Due to Theorem 7 we abuse the notation slightly by using $[D]$ for $[([D],0)]_E\in K_0(\mathbb{Q})$. The meaning of $[D]$ would be clear from the context and in case of an ambiguity the context would be stated exclusively. Further for definable set D' we say that $[D']$ is contained in $[D]$ if there is definable set $D''\in[D]$ such that $D'\subseteq D$. Further one notes that for D_1 and D_2 such that $[D_1]=[D_2]$ we have $Height(D_1)=Height(D_2)$. Hence we extend the definition of $Height()$ to .

11.1 Multiplication of two related sets

It suffices to study multiplication in with semi ring structure on it, as it is embedded inside $K_0(\mathbb{Q})$. Let $\overline{a}$ be the ordered set of k elements of $Q \cup \{\pm\infty\}$ sorted in descending order and $\overline{n} = (n_1, n_2, \ldots, n_k)$ be k tuple of non negetive integers. Let ${}^{\overline{a}}R_{\overline{n}}$ be the related set corresponding to the formula

$$(X_1 > X_2) \wedge \ldots \wedge (X_{n_1} > a_1) \wedge \ldots \wedge (X_{n_1+n_2} > a_2) \wedge \ldots \wedge (X_{n_1+\ldots+n_k} > a_k)$$

and $\left[{}^{\overline{a}}R_{\overline{n}}\right] \in$ be its equivalence class. With R^a_n we would mean ${}^{(\infty,a)}R_{(0,n)}$ to simplify the notation. With all this we have following proposition.

Proposition 6. *With the above notations we have the following:*

1. *Let $a_1, a_2, a_3 \in Q \cup \{\pm\infty\}$ be such that $a_1 > a_2 > a_3$, then*

$$\left[{}^{(a_1,a_2,a_3)}R_{(0,n,0)}\right].\left[{}^{(a_1,a_2,a_3)}R_{(0,0,m)}\right] = \left[{}^{(a_1,a_2,a_3)}R_{(0,n,m)}\right] .$$

2. *Let $m \leq n$ then*

$$\left[R^a_m\right]\left[R^a_n\right] = \sum_{i=0}^{m} \binom{n+i}{i}\binom{n}{m-i}\left[R^a_{n+i}\right] .$$

3. *Given $\overline{a} \in Q \cup \{\pm\infty\}$ a k tuple we have*

$$[R^{a_i}_n] = \bigoplus_{R \in (R^{a_i}_n)} [R]$$

where each R has form ${}^{\overline{a}}R_{(n_1,n_2,\ldots,n_i,0,0,\ldots,0)}$ for some non-negetive integers n_j and $n_1 + n_2 + \ldots + n_i \leq n$.

Proof. 1. The proof follows from the definition of '.' operation on .

2. The idea is to show equality by introducing relations between variables of R^a_m and R^a_n. Let us name variables of R^a_m as $\overline{X}[1:m]$ and of R^a_n as $\overline{Y}[1:m]$. So in the cartesian product there is no relation to be satisfied between any of X_i and Y_j for all i and j i.e. they are free to take values irrespective of value taken by other variable. Hence it is easy to see that we will have cases in which i variables of R^a_m are in equality relations with i variables of R^a_n, for $i \in [0:m]$. Clearly every such set is a subset of $R^a_m \times R^a_n$. Also, let $\bar{q}$ be a tuple in $R^a_m \times R^a_n$, then it is obvious that the relations in this tuple will fall in one of the $m+1$ cases. Now our task is reduced to find number of possible related sets in each case.
For a fixed i look at the coefficient of R^a_{m+i} , that means $m-i$ variables are equal to $Y_{i's}$. Others are strictly placed between two variables. One should take care of the already existing order relations between $X_{i's}$. Note that there are $m!$ different ordering of $\overline{X}[1:m]$. Since every ordering gives same number of sets after multiplying, thus we ignore ordering and will divide the final answer by $m!$. Now, choose $m-i$ variables which are to be made equal

and then choose $m-i$ variables out of n ; these can be arranged in $(m-i)!$ ways, also the left over i variables will have $(n+1)(n+2)(n+3)\dots(n+i)$ possible arrangements. Thus combining all these we have

$$\frac{1}{m!}\left(\binom{m}{m-i}\binom{n}{m-i}(m-i)!(n+1)(n+2)\dots(n+i)\right). \tag{5}$$

On simplifying we get the required term in the summation. Further using the fact that $[R^a_n] \neq [R^a_m]$ for $m \neq n$ (Corollary 2), we get the required sum.

3. The proof is clear.

Lemma 7. *With the above notations and $R^{a_i}_{n_i} \in$ for $i \in \{1, 2, \dots, k\}$, and for*

$$[D] = \prod_{i=0}^{k} \left[R^{a_i}_{n_i}\right]$$

we have $\left[{}^{\overline{a}}R_{\overline{n}}\right]$ contained in $[D]$ where $\overline{n} = (n_i)^k_{i=1}$ and $\overline{a} = (a_i)^k_{i=1}$. Further $\left[{}^{\overline{a}}R_{\overline{n}}\right]$ has the property that among all other $[R]$ contained in $[D]$ it has maximum value for n_k with $Height(R) = n$.

Proof. The proof is easy and follows the Proposition 6 as follows. From part 3 of the proposition we know that each $\left[R^{a_i}_{n_i}\right]$ contains ${}^{\overline{a}}R_{0,0,\dots,n_i,0,,0}$ and for any other $R \in (R^{a_i}_{n_i})$ with height $R = n_i$, we have $(X_{a_{i-1}}, \overline{X}_{a_i}) < n_i$. Using the part 1,2 of the proposition distributivity property of we have the lemma.

Theorem 11. *With all above notations we have*

$$\mathcal{O} := \frac{\mathbb{Z}\left[{}^aX_n \mid n \in \mathbb{N}; a \in Q \cup \{-\infty\}\right]}{({}^aX_k{}^aX_l) - \sum_{i=0}^{l} \binom{k+i}{i}\binom{k}{l-i}{}^aX_{k+i} \mid k, l \in \mathbb{N}; a \in Q \cup \{-\infty\}} \simeq K_0(\mathbb{Q}),$$

where ${}^aX_n$ for $n \in \mathbb{N}$ and $a \in Q$, represents a formal variable.

Proof. With all this define the association map

$$\begin{aligned} \Psi : \{{}^aX_n \mid a \in Q, n \in \mathbb{N}\} &\longrightarrow K_0(\mathbb{Q}) \\ {}^aX_n &\longmapsto [R^a_n] \end{aligned}$$

Extend the association map Ψ naturally to a ring homomorphism from $\mathcal{O}$ to $K_0(\mathbb{Q})$ and represent it by Ψ. We will show that Ψ is an isomorphism. Ψ is well defined from Part 2 of Proposition 6. Surjectivity of Ψ is easy to show and follows from Part 1 and 3 of Proposition 6. Hence we would directly show that Ψ is an injection. Let

$$I := ({}^aX_k{}^aX_l) - \sum_{i=0}^{l} \binom{k+i}{i}\binom{k}{l-i}{}^aX_{k+i} \mid k, l \in \mathbb{N}; a \in Q \cup \{-\infty\} \tag{6}$$

Further for $f \in \mathbb{Z}\big[{}^{a}X_n \mid n \in \mathbb{N}; a \in Q \cup \{-\infty\}\big]$, let $\overline{f}$ represent the corresponding element in $\mathcal{O}$. Let $g = c \prod_{i=1}^{k} {}^{a_i}X_{n_i}^{m_i}$ be general monomial of f where $n_i, m_i \in \mathbb{N}$ and $c \in \mathbb{Z}$. For any monomial g of f define

$$Height(g) = Height(\Psi(\overline{g})) = \sum_{i=1}^{k} n_i m_i.$$

Further define

$$Height(f) = \max\{Height(g) \mid g \textit{ is a monomial of } f\}$$

Now each $({}^{a}X_n)^m \equiv c\,{}^{a}X_{nm} +$ *lower height terms (mod I)* for some positive integer c. Hence for every monomial

$$g \equiv c' \prod_{i=1}^{k} {}^{a_i}X_{n_i m_i} + \textit{ lower height terms (mod I)}$$

for some $c \in \mathbb{Z}$. We want to show that $Ker(\Psi) = 0$. One notes that if $f \equiv f'$ *(mod I)* then $Height(f) = Height(f')$. Hence we define $Height(\overline{f}) = Height(f)$. Let $\overline{f} \in Ker(\Psi)$ be such that $Height(f) = n$ is minimum. Let $f' \equiv f$ *(mod I)* be such that every monomial g of f with $Height(g) = n$ has the form

$$c' \prod_{i=1}^{k} {}^{a_i}X_{n_i m_i}$$

Such a f' exists from previous analysis. Let

$$k = card\big(\{a_j \mid {}^{a_j}X_m \textit{ appears in any monomial of } f \textit{ for some } n\}\big) > 1$$

otherwise the proof is trivial. Let

$$f' = f_1 + f_2\ ,$$

where f_1 has all coefficients of monomials positive and f_2 has all coefficients of monomials negetive. We have $\Psi(\overline{f'}) = 0$ hence $\Psi(\overline{f_1}) = \Psi(\overline{f_2})$. WLOG assume that the set $\{a_j \mid {}^{a_j}X_m \textit{ appears in any monomial of } f' \textit{ for some } m\}$ is ordered in descending order. Let

$$g_{max} = \prod_{i=1}^{k} {}^{a_i}X_{n_i}$$

with $Height(g) = n$ be the monomial of f such that n_k is maximum. If there exist two or more such monomials then from amongst them choose the one in which n_{k-1} is maximum and so on. It is easy to see that this process will render us a unique g. Note that g will either be in f_1 or f_2. WLOG assume that g is in f_1. We claim that $\big[{}^{\overline{a}}R_{\overline{n}}\big]$ where $\overline{a} = (a_1, \ldots, a_k)$ and $\overline{n} = (n_1, \ldots, n_k)$ is contained in $\Psi(\overline{g}_{max})$ interpreted in and no other $\Psi(\overline{g})$ with $Height(g) = n$ contains $[R]$. To see this suppose there is $g \neq g_{max}$ and $\Psi(\overline{g})$ contains $[R]$, further let

$$g = \prod_{j=1}^{l} {}^{a_j}X_{n'_j} \prod_{i=l+1}^{k} {}^{a_i}X_{n_i}$$

for $l \leq k$. Clearly by construction we see that $n'_l < n_l$. Hence from Lemma 7 we see that $\Psi\left(\overline{\prod_{j=1}^{l} {}^{a_j}X_{n'_j}}\right)$ contains $[{}^{(a_1,\ldots,a_l)}R_{(n'_1,\ldots,n'_l)}]$ with the property that among all other $[R]$ contained in $\Psi\left(\overline{\prod_{j=1}^{l} {}^{a_j}X_{n'_j}}\right)$ it has maximum value for n'_l. Clearly as for $i > la_i < a_l$ hence by Part 3 of Proposition 6, $\Psi(\overline{g})$ does not contain $\left[{}^{\overline{a}}R_{\overline{n}}\right]$. This implies that $\Psi(\overline{f_1}) = \Psi(\overline{f_2})$ does not hold in . Hence the coefficient of $\overline{g}_{max} = 0$ which is a contradiction, therefore $\overline{f} = 0$. This established the isomorphism between $K_0(\mathbb{Q})$ and $\mathcal{O}$.

12 Some Interesting Combinatorial Properties of $K_0(\mathbb{Q})$

In this section we discuss some interesting combinatorial properties of $\mathcal{O}$ and hence of $K_0(\mathbb{Q})$.

Theorem 12. *For $a, b \in Q \cup \{-\infty\}$ and $a < b$*

$${}^{n}f(b,a) = \Psi^{-1}(\left[{}^{(b,a)}R_{0,n}\right])$$

then

1. *For any $c \in Q \cup \{-\infty\}$ such that $a < c < b$ we have*

$${}^{n}f(b,a) = \sum_{i=0}^{n} {}^{i}f(b,c)^{n-i}f(c,a) + \sum_{i=0}^{n-1} {}^{i}f(b,c)^{n-i}f(c,a)$$

2. $(n!)({}^{n}f(b,a)) = \prod_{i=0}^{n-1} ({}^{1}f(b,a) - i)$

Proof. 1. Since Ψ is an isomorphism hence ${}^{n}f(b,a)$ satisfies the same relation as $[{}^{(b,a)}R_{0,n}]$. Therefore by part 1 and 3 of Proposition 6 we have the result.

2. From part 2 of Proposition 6 putting $m = 1$ and $n = n-1$ we have $n[R^a_n] + (n-1)[R^a_{n-1}] = [R^a_{n-1}][R^a_1]$. One can observe that the same relation holds for $[{}^{(b,a)}R_{(0,n)}]$, i.e.

$$n[{}^{(b,a)}R_{(0,n)}] + (n-1)[{}^{(b,a)}R_{(0,n-1)}] = [{}^{(b,a)}R_{(0,n-1)}][{}^{(b,a)}R_{(0,1)}].$$

Therefore the result follows.

Theorem 13. *The ring $\mathcal{O}$ is partially ordered or equivalently $\mathbb{Q}$ satisfies PHP.*

Proof. We will use Theorem 8. Further Lemma 5 implies that there cannot be a definable bijection between a definable set and its proper definable subset. Hence the result.

Remark 1. Very few structures satisfy PHP.

Theorem 14. *Let $X_k := {}^aX_k$, $\forall k \in \mathbb{N}$. Then*

$$\left(X_k \prod_{i=0}^{l-1}(X_1 - i)\right) = (l!)\left(\sum_{i=0}^{l}\binom{k+i}{i}\binom{k}{l-i}X_{k+i}\right) \tag{7}$$

$\forall\ k, l \in \mathbb{N}$.

Proof. Putting $l = 1$ in Proposition 6 and doing elementary calculations we get :

$$\prod_{i=1}^{l}(k+i))X_{k+l} = X_k(\prod_{j=0}^{l-1}(X - k - j) \tag{8}$$

Let us look at the RHS.

$$(X_k)({}^kP_l) + \Sigma_{i=1}^{l}({}^lC_i)({}^kP_{l-i})(\prod_{s=1}^{i}(k+s))\ (X_{k+i}) \tag{9}$$

Using Equation 8 we get

$$(X_k)({}^kP_l) + \Sigma_{i=1}^{l}({}^lC_i)({}^kP_{l-i})\ (\prod_{j=0}^{i-1}(X_1 - k - j))X_k$$

As X_k can be factored out thus we will focus on the following expression :

$$P_1(X_1) := ({}^kP_l) + \Sigma_{i=1}^{l}({}^lC_i)({}^kP_{l-i})\ (\prod_{j=0}^{i-1}(X_1 - k - j))$$

and show it equal to

$$P_2(X_1) := (X_1)(X_1 - 1)(X_1 - 2)\dots(X_1 - l + 1).$$

We will evaluate $P_1(X)$ at $X_1 = k + t$, where $t = 0, 1, 2, ..., l$ and show that $P_2(k+t) - P_1(k+t) = 0$. And since we are in integral domain, polynomial of degree l can have atmost l zeros, which will force $P_1 - P_2$ to be identically zero. For $t = 0$, checking is straightforward as the summation term vanishes. To prove for $k + t$ in general, we will use induction.

So, for $t = r$, assume that $P_1(r) = P_2(r)$. Now, for $t = r + 1$,

$$P_1(r) = {}^kP_l + \Sigma_{i=1}^{r+1}({}^lC_i)({}^kP_{l-i})(\prod_{j=0}^{i-1}(r+1-j))$$

$$= {}^kP_l + \Sigma_{i=1}^{r}({}^lC_i)({}^kP_{l-i})(\prod_{j=0}^{i-1}(r-j))$$

$$+ \Sigma_{i=1}^{r}({}^lC_i)\ ({}^kP_{l-i})(\prod_{j=0}^{i-1}(r+1-j) - \prod_{j=0}^{i-1}(r-j))$$

$$+ ({}^lC_{r+1})\ ({}^kP_{l-r-1})\prod_{j=0}^{r}(r+1-j)$$

Note that, if $i > 1$, then $(\prod_{j=0}^{i-1}(r+1-j) - \prod_{j=0}^{i-1}(r-j)) = (i)(\prod_{j=0}^{i-2}(r-j))$ and for $i = 1$, it is 1 , as evident from the formula. Using induction hypothesis, the sum of first two terms in the sum is $({}^{k+r}P_l)$. Hence, the remaining part can be written as

$$(l)({}^kP_{l-1}) + (l)(\Sigma_{i=2}^{r}({}^{l-1}C_{i-1})\ ({}^kP_{l-i})\ \prod_{j=0}^{i-2}(r-j)) + \ ({}^lC_{r+1})\ ({}^kP_{l-r-1})\prod_{j=0}^{r}(r+1-j)$$

Re-indexing the summation we get

$$l({}^kP_{l-1}\ +\ \Sigma_{i=1}^{r-1}({}^{l-1}C_i)\ ({}^kP_{l-i-1})\ \prod_{j=0}^{i-1}(r-j)) + \ ({}^lC_{r+1})\ ({}^kP_{l-r-1})\prod_{j=0}^{r}(r+1-j) \tag{10}$$

Note that the term $({}^lC_{r+1})\ ({}^kP_{l-r-1})\ \prod_{j=0}^{r}(r+1-j)$ can be re-written as $({}^{l-1}C_r)(l/r+1)\ ({}^kP_{l-r-1})\ \prod_{j=0}^{r}(r+1-j)$ and hence, we have $({}^{l-1}C_r)(l)\ ({}^kP_{l-r-1})\ \prod_{j=1}^{r}(r+1-j) = ({}^{l-1}C_r)(l)\ ({}^kP_{l-r-1})\ \prod_{j=0}^{r-1}(r-j) = (l)({}^{l-1}C_r)\ ({}^kP_{l-r-1})\ \prod_{j=0}^{r}(r-j)$
Hence, (1) can be re-written as,

$$(l)({}^kP_{l-1}\ +\ \Sigma_{i=1}^{r}({}^{l-1}C_i)\ ({}^kP_{l-i-1})\ \prod_{j=0}^{i-1}(r-j))$$

which is equal to ${}^{k+r}P_{l-1}$. by induction on l. And now, we use the relation ${}^kP_l + l\ {}^kP_{l-1} = {}^{k+1}P_l$ (easily proved using counting arguments). Hence, one side is ${}^{k+r+1}P_l$. Now, evaluating the other side is easy. $P_2(k+r+1) = (k+r+1)(k+r)(k+r-1)\ldots.(k+r+1-(l-1)) = {}^{k+r+1}P_l$. Thus we get the required result.

Corollary 3. *Let I be the ideal as defined in Equation 6 and define :*

$$I' := \left\langle (k!)({}^{a}X_k) - \prod_{i=0}^{k-1}({}^{a}X_1 - i) \mid k \in \mathbb{N}; a \in Q \cup -\infty \right\rangle. \tag{11}$$

Given an element $\alpha \in I$ there exists and integer l_α such that $(l_\alpha!)\alpha \in I'$.

Proof. Follows from the previous theorem.

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