Theory of Modular forms

(with a view towards Modularity)

- Action of $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ is a natural thing to look at, in complex analysis (see St-sak- Chap-3). Hence we go to the action of $SL_2(\mathbb{Z})$ on $\mathbb{C}$. without motivating it much but it some how arises naturally in theory of Riemann Surfaces and also is a conformal map.

- Now we provide some definitions. for the readers who dont have aquantance with Complex analysis. (which is required!)

□ - Holomorphic funtion : Given an open set $\Omega \subseteq \mathbb{C}$, we say a function $f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic if $f'(z)$ exists at every point $z \in \Omega$. In complex plain $f'(z)$ is defined as $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$. Further one can clearly check that the following eqns. are satisfied

▷ Lemma : If $f = u(z) + iv(z)$ then v, u can be modified as such that they are real valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. where $u = u(x,y)$, $v = v(x,y)$ $z = x+iy$. hence $f(\overset{x,y}{z}) = u(x,y) + i v(x,y)$. The following eqns. known as Cauchy-Riemann eqn's are satisfied.

$$\frac{\partial u}{\partial x} = u_x = v_y = \frac{\partial v}{\partial y} \qquad -(\text{I})$$

$$u_y = -v_x \qquad -(\text{II}).$$

▷ Lemma : If f is holomorphic in $\Omega \subseteq \mathbb{C}$ which is open and let D be a disc in Ω with its closure inside Ω then f has an power series expn. in D.

We recall the following famous theorems from the Complex analysis. A reader not aquanted with Complex analysis but with real analysis will not have much problem if we assume them to be true.

* Cauchy Integral formula: Suppose f is holomorphic in an open set that contains the closure of disc D. If C is the boundary circle of this disc with the positive orientation. then

$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\cancel{f(\zeta)}(\zeta - z)} d\zeta \quad \forall\ z \in D.$$

* If f is a holomorphic fn. in $\Omega \subseteq \mathbb{C}$ and $(x_n)_{n\in\mathbb{N}}$ be a seq. of points in Ω with limit in Ω s.t. $f(x_n) = 0\ \forall\ n \in \mathbb{N}$ then $f(z) = 0\ \forall\ z \in \Omega$.

* Liouville's theorem: If f is entire (holomorphic in $\mathbb{C}$) and bounded the f is constant.

* Morera's theorem: If f is continuous in $\Omega \subseteq \mathbb{C}$ open subset of $\mathbb{C}$. ~~then~~ and suppose $\forall$ T a triangle in Ω with interior in Ω ~~then~~ and $\int_T f(z)\,dz = 0$ then f is holomorphic in Ω.

* □ A function f holomorphic in the nbhd of $z_0 \in \Omega$ ~~and let Ω be the open subset of $\mathbb{C}$ in which f is holomorphic~~ then if defining $\frac{1}{f}(z_0) = 0$ makes $\frac{1}{f}$ holomorphic in that nbhd of z_0. including at z_0 then f is said to have a pole of at z_0.

* If f has a zero (pole) at $z_0 \in \Omega$, then in a nbhd of that point $\exists$ a function h which is holomorphic in that nbhd and an integer n s.t. $f(z) = (z - z_0)^n h(z)$ in that nbhd with h non vanishing in that nbhd. and $n \geq 0$ If z_0 is a zero

and $n<0$ if z_0 is a pole. This n is unique, (n) known as order of zero (pole) at z_0.

* If f has a pole at z_0 then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \dots + \frac{a_{-1}}{(z-z_0)} + G(z)$$

where $G(z)$ is holomorphic fn. in a nbhd of z_0.

□ a_{-1} is called (as in above thm.) residue of f at (z_0).

* Residue thm: $\frac{1}{2\pi i}\int_C f(z)dz = \operatorname{res}_{z_0} f$. where C is circle with centre at z_0. inside the nbhd of ~~f~~ z_0. in which f is holomorphic in z_0's deleted nbhd. (i.e. nbhd obtained after removing z_0).

* Corollary of above thm: Suppose f is holomorphic in an open set containing any contour γ. and its interior except at the poles, $z_1, \dots, z_n$ in Interior of γ. then

$$\int_\gamma f(z)\,dz = 2\pi i \left(\sum_{i=1}^{n} \operatorname{res}_{z_i}(f)\right).$$

* if z_0 is the isolated singularity ~~at the point~~ z_0 of f. then z_0 is a pole of f iff $\lim_{z\to z_0}|f(z)| \Rightarrow \infty$. ~~and~~

□ - A function f is called "meromorphic" on an open set $\Omega \subseteq \mathbb{C}$. If $\exists$ $(z_0, z_1, \dots z_n, \dots)$ with no limit point in Ω s.t f is holomorphic in $\Omega \setminus \{z_0, z_1, \dots\}$ and f has poles at $\{z_0, \dots\}$.

* Meromorphic functions in the extended complex plane are the rational functions. (Properties of a function f at ∞ can be defined by looking at its properties of $F(z) = f(1/z)$ at $z \to 0$.

and $n<0$ if z_0 is a pole. This n is unique, $|n|$ known as order of zero (pole) at z_0.

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☐ Group Action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ (upper half plane) is defined naturally as a fractional linear transformation as follows:

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \gamma \in SL_2(\mathbb{Z}). \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$z \in \mathbb{H}.$$

— For a reader who is not aware of $SL_2(\mathbb{Z})$ here is a definition. $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{Z} \ \& \ ad-bc=1 \right\}$. and upper half plane, $\mathbb{H} := \{ z \in \mathbb{C} \mid Im(z) > 0 \}$. One verifies that $\gamma(z) = \frac{az+b}{cz+d}$ is a valid group action. and also following identities.

$$\triangleright \quad \cancel{Im(z)} \ Im(\gamma(z)) = \frac{Im(z)}{|cz+d|^2} \quad — (1).$$

We next move on show something interesting about $SL_2(\mathbb{Z})$. which is that $SL_2(\mathbb{Z})$ is generated by just two elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. To show this one looks at the following. $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ (verify!). and let any matrix α in $SL_2(\mathbb{Z})$ be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. then $\alpha T^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' = an+b \\ c & d' = cn+d \end{pmatrix}$
$\forall n \in \mathbb{Z}$.
and $\alpha S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$. Now let Γ be the subgroup of $SL_2(\mathbb{Z})$ generated by S & T. Then one can show that we can make $\alpha\gamma \in \Gamma$ for some $\gamma \in \Gamma$ and $\alpha \in SL_2(\mathbb{Z})$. The trick is to reduce the value of c' and d' in $\alpha\gamma$ such that $(c', d') = (0, \pm 1)$ which will imply that $\alpha\gamma$ belongs to Γ as then the matrix is merely $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix}$ cushy!! Hence here is what we do, we find an n s.t. $|d'| \le \frac{|c|}{2}$ in $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. and then interchange the columns picking up a negative sign using ~~fa~~ S and hence again. we find an n' s.t. ~~d~~ for matrix $\begin{pmatrix} b' & -a \\ d' & -c \end{pmatrix}$ $|c'| \le \frac{|d'|}{2}$ and we again interchange and move on.

So that we reach a situation of one of them becoming zero. Which is surely the last entry becoming zero.

hence we are ~~that~~ at the matrix $\begin{pmatrix} a^* & b^* \\ c & 0 \end{pmatrix} \Rightarrow b^*c^* = 1$

hence $b^* = c^* \in \{+1, -1\}$ hence applying S again on this we have

$\begin{pmatrix} \pm 1 & -a^* \\ 0 & \pm 1 \end{pmatrix} \in \Gamma$ as a^* is an integer hence

~~we can reduce it~~ belongs to the Subgroup $\pm I \langle T^n | n \in \mathbb{Z} \rangle$

where $\pm I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

This shows $\Gamma = SL_2(\mathbb{Z})$. We will see that this is going to be a very useful property. As we would see that certain properties we will just need to check for the generators. (the property which would be central part in the whole study of modular forms).

Having done with that we look at the following important definition of ~~for~~ modular functions ~~for~~ on $\mathbb{C}$.

□ Let f be a meromorphic function on $\mathbb{C}$ or for our case $\mathbb{H}$. then f is said to be weakly modular of wt. $k \in \mathbb{Z}$ if

$$f(\gamma(z)) = (cz+d)^k f(z) \qquad \forall\, z \in \mathbb{H} \text{ and } \gamma \in SL_2(\mathbb{Z}).$$

clearly as $z \in \mathbb{H}$ what we can say is that ~~f~~ the action preserves ~~that~~ the zero's and poles of a modular function as $(cz+d) \neq 0$. if $f(z)$ has a pole at z_0 (resp zero) then f also has a pole (resp. zero) at z_0. One quickly verifies the following $\dfrac{d(\gamma(z))}{d(z)} = (cz+d)^{-2}$. hence we have for $k=2$

$$f(\gamma(z))\, d(\gamma(z)) = f(z)\,dz. \quad \text{i.e.} \quad \cancel{f(\gamma z)}$$

the path integral $f(z)dz$ is invariant under the action of $SL_2\mathbb{Z}$ for weakly modular functions of wt. 2. For others the situation slightly changes as

$$f(\gamma(z))\,(d(\gamma(z)))^{k/2} = f(z)\,(dz)^{k/2}.$$

Given the definitions of modular (weakly) function of wt k. Its interesting to see what happens when $\gamma = S$ or T as they are the generators.

$$f(T(z)) = f(z) = f(z+1) \qquad -2$$
$$f(S(z)) = f(-1/z) = z^k f(z). \qquad -3$$

One can show that we only need to check that f satisfies (2,3) to show f is weakly modular of wt k. We would soon develop a general technique which will show this directly. But for now lets assume this. Equation 2 is particularly interesting as it shows that f is periodic with period in the real coordinate irrespective of what wt is. Hence as f is meromorphic from complex analysis we know that it has a fourier expn. i.e. if $q = e^{2\pi i z}$. then $\exists$ a function $\tilde{f}(q)$ s.t. $\tilde{f}(q) = f(z)$ $\forall z$. and $\tilde{f}$ has the following expn $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n(f) q^n$ and also.

One can clearly see that ~~ϕ : $\mathbb{C} \to \mathbb{D}$~~ $z \to e^{2\pi i z}$ takes $\mathbb{H} \to D'$ where D' is the unit disk around origin punctured at origin. From now on we omit the notation

$\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n(\tilde{f}) q^n$ but directly write

$$\boxed{f(z) = \sum_{n \in \mathbb{Z}} a_n(f) q^n}$$

Further $e^{2\pi i (x+iy)} = q(z)$. when $z = x+iy$.

hence. $|q| = e^{-2\pi \operatorname{Im}(z)} < 1$ and

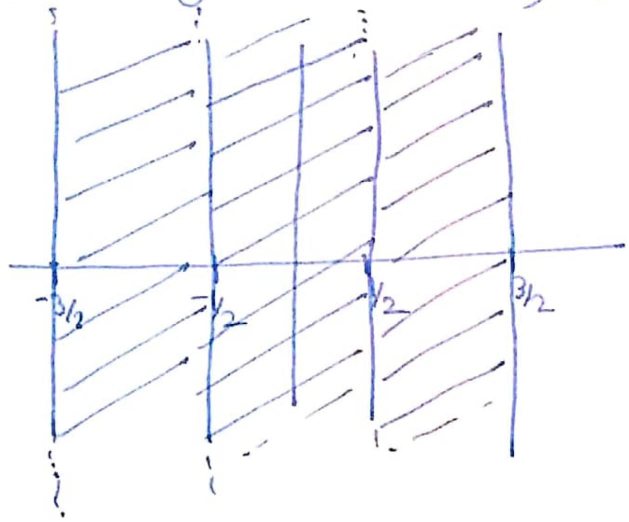
as $\operatorname{Im}(z) \to \infty$, $|q| \to 0. \Rightarrow q \to 0.$

Hence if as $\operatorname{Im}(z) \to \infty$ $f(z)$ is holomorphic at ∞. i.e. $F(z) = f(1/z)$ is holomorphic at $z \to 0$. then $\tilde{f}(q) = f(z)$ will extend holomorphically at $q = 0$. This will mean

$$f(z) = \sum_{n \in \mathbb{N}} a_n(f) q^n.$$

and we say $f(z) = \sum_{n \in \mathbb{N}} a_n(f) q^n$ as fourier expn. of f.

Having this in mind we see that holomorphic function ~~i.e.~~ on $\mathbb{H} \cup \{\infty\}$ are important. Note that ~~by~~ by $z \to \infty$ we mean that $\text{Im}(z) \to \infty$ as we see that in every width 1 strip in $\mathbb{C}$ ~~#~~ parallel to Y axis we have. f taking the same values. Hence it is enough to look at any particular strip.



☐ A holomorphic function f in $\mathbb{H} \cup \{\infty\}$ is said to be a modular form of wt k. if f is weakly modular of wt k.

☐ * If f_1 is a modular form of wt k the cf_1 is $\forall\ c \in \mathbb{C}$ ~~and~~ of wt k

* If f_1, f_2 are modular forms of wt k then $f_1 + f_2$ is also a modular form of wt k.

* If f_1, f_2 are modular form of wt k_1, k_2 (resp.) then $f_1 f_2$ ~~are~~ is modular form of wt. $k_1 + k_2$ (verify!)

This makes the space of wt k ~~for~~ modular forms a V.S. over $\mathbb{C}$. and we denote it by M_k. and clearly $M = \bigoplus_{n \in \mathbb{Z}} M_n$ forms a graded ring because of the third condition.

☐ define cusp form to be a modular form which takes value zero at ∞. i.e. $q \to 0 \Rightarrow f(z) = 0$. where f is modular form.

▷ One quickly sees that space of wt. k cusp forms is a subspace of M_k. We call it S_k.

▷ Using the fact that if g is wt k_1 cusp form and f is a wt k_2 modular form then

gf is wt k_1+k_2 cusp form. hence

makes $S = \bigoplus_{k\in\mathbb{Z}} S_k$ an ideal of $M = \bigoplus_{k\in\mathbb{Z}} M_k$.

Till now we have done enough of abstract theory and lets now appreciate with some examples of modular forms of wt. k. What we are about to do will be important in later part of this series. We first define what is known as lattices in $\mathbb{C}$. Let $\omega_1, \omega_2 \in \mathbb{C}$ (linearly independent in $\mathbb{R}^2$) then define the set

$$\Lambda(\omega_1,\omega_2) := \{ m\omega_1 + n\omega_2 ,\ m,n \in \mathbb{Z} \}.$$

We call $\Lambda(\omega_1,\omega_2)$ as the lattice associated with (ω_1,ω_2). Clearly $\Lambda(\omega_1,\omega_2)$ is a $\mathbb{Z}$-module. with ω_1, ω_2 as generators. But we would like to explore the relation for two different generating pairs of same lattice. say $\Lambda(\omega_1,\omega_2) = \Lambda(\omega_1',\omega_2')$ then we have, $(a,b) \in \mathbb{Z}^2$ s.t. & $(c,d) \in \mathbb{Z}^2$

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2$$

$$\Rightarrow \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

clearly $\exists\ a', b', c', d' \in \mathbb{Z}$ s.t

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and inverse is also in $M_{2\times2}(\mathbb{Z}) \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

And if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then $\omega_1' = a\omega_1 + b\omega_2$, $\omega_2' = c\omega_1 + d\omega_2$ form another generating set for $\Lambda(\omega_1,\omega_2)$.

What we just saw was that the lattice remains unchanged by the action of $SL_2(\mathbb{Z})$. Let $\omega_1/\omega_2 \in \mathbb{H}$ if not then $\frac{1}{\omega_2/\omega_1}$ will belong to $\mathbb{H}$. as $a\omega_1 + b\omega_2 \neq 0$ ~~$\Rightarrow \omega_1, \omega_2$~~ $\forall\ a, b \in \mathbb{Z}$.

<u>Remark</u>! Through out this write up I have assumed most obvious facts. or assumptions about the choices or in the definition : example $\omega_1, \omega_2 \neq 0$ and ω_1 is not a multiple of ω_2. etc.

~~We define~~ G Its natural to look at the following function

$$G_k(\omega_1, \omega_2) = \sum_{m,n \neq (0,0)} \frac{1}{(n\omega_1 + m\omega_2)^k}$$

which is a lattice function. but if $\omega_2 = 1$ then we have

$$G_k(\tau, 1) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k}$$

~~the form~~

if we restrict ourselves to ~~this~~ the lattice of the form $\Lambda(\tau, 1)$ then $G_k(\omega_1, \cancel{\omega_2}) = G_k(\omega_1)$.

It would be shown latter or infact can be shown that assuming the lattices to be of the form $\Lambda(\tau, 1)$ for $\tau \in \mathbb{H}$ will not have much affect on $G_k(\tau, 1)$ ~~exp~~ except for multiplication ~~of~~ by a factor (non-zero). Infact we would later show that $\Lambda(\tau, 1) \cong \Lambda(\omega_1, \omega_2)$ where $\tau = \omega_1/\omega_2$ and $\tau \in \mathbb{H}$. Equivalence is in "some sense" we will see later.

now for $k > 2$ we can show that this series converges (absolutely) $\forall z \in \mathbb{H}$. as an analogue of zeta function in two dimension. We assume this fact. Interested person can look it up in any standard text. Now for k = odd.

$$G_k(\tau) = \sum_{m,n \neq (0,0)} \frac{1}{(m\tau+n)^k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(-m\tau-n)^k} \qquad k \geq 3$$

$$= (-1)^k \left(\sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k} \right)$$

$$\Rightarrow G_k(\tau) = 0 \quad \forall\ \tau \in \mathbb{H}. \quad k = \text{odd}.$$

In fact one can show a more general ~~thing~~ result any weakly modular function of wt. odd is identically zero. We first not that $-I(z) = z$ and $f(-I(z)) = f(z) = (-1)^k f(z)$

$\Rightarrow$ $f(z) = 0 \ \forall z \in \mathbb{H}$. if f is weakly modular and k is odd

So from now on we just focus on even Integers k. But not that this happened because $-I \in SL_2(\mathbb{Z})$. Now suppose if we take a subgroup of $SL_2(\mathbb{Z})$ not containing $-I$. then we might have odd wt. modular funtions. ~~but~~ (this ~~is in~~ statement is clearly vague as we have not yet defined what does it mean to be modular wrt. a subgroup of. $SL_2\mathbb{Z}$). Anyways leaving such matters for later discoveries we move forward and focus on $G_k(\tau)$.

As we would have guessed the $G_k(\tau)$ is a modular form of wt k. We show that if $\gamma \in SL_2(\mathbb{Z})$ then $G_k(\gamma(\tau)) = (c\tau+d)^{+k} G_k(\tau)$

$$G_k(\gamma(\tau)) = (c\tau+d)^k \sum_{\substack{(m,n) \\ \neq (0,0)}} \frac{1}{[(ma+nc)\tau + (mb+nd)]^k}.$$

Pf: now we first show that if $(m,n) \neq (0,0)$ and $ad-bc=1$ then $((ma+nc), (mb+nd)) \neq (0,0)$

as if not then

$ma+nc=0 \Rightarrow ma=-nc$

and

$mb=-nd$

but as $ad-bc=1$ then. $\frac{m}{-n} = \frac{c}{a} = \frac{d}{b} \Rightarrow ad-bc = 0$.

We assume $a,b \neq (0,0)$ but $a=0$ or $b=0$ cases can be shown ~~as an easy exercise as~~ directly as $(a,b) \neq (0,0)$ hence exactly one is 0. so suppose $a=0$

$\Rightarrow$ $ma = -nc \Rightarrow n=0$ and hence $mb=0 \Rightarrow m=0$. as $b \neq 0$. Similarly if $b=0$.

Now we look at the following. As we saw ~~that~~ or better assumed that instead of showing for every element in $SL_2(\mathbb{Z})$ we can show weakly modular behaviour only for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence for T

1)
$$G_k(T(\tau)) = \sum_{m,n}{}' \frac{1}{(m\tau + m + n)^k} = \sum_{(m,n)}{}' \frac{1}{(m\tau + (m+n))^k} = \sum_{(m,n-m)}{}' \frac{1}{(m\tau+n)^k} = G_k(\tau)$$

(Note:) primed summation means denominator is not zero. i.e. $(m,n) \neq (0,0)$

2)
$$G_k(S(\tau)) = \sum_{m,n}{}' \frac{(\tau)^k}{(-m + n\tau)^k} = \tau^k \sum_{n,-m}{}' \frac{1}{\cancel{m+n\tau}}$$

$$= \tau^k \sum_{-n,+m}{}' \frac{1}{(n+m\tau)^k}$$

$$= \tau^k \, G_k(\tau).$$

$\Rightarrow$ $G_k(\tau)$ is a modular form. as absolute convergence & also uniform convergence on compact subsets of $\mathbb{H}$ ~~implies~~ combined with the fact that if it has an isolated singularity then $|G_k(\tau)| \uparrow \infty$ at that point but ~~#~~ $G_k(\tau)$ ~~diverges~~ converges at every point in $\mathbb{H}$. We still need to show that $\lim_{Im(\tau) \to \infty} G_k(\tau) < \infty$.. For this we compute the fourier series of $G_k(\tau)$ on $\mathbb{H}$ and then show that $G_k(\tau)$ has a removable singularity at $q = 0$. Lets ~~to~~ look at that.

$$G_k(\tau) = \sum_{\substack{(m,n) \\ \neq (0,0)}} \frac{1}{(m\tau+n)^k} \qquad k \geq 3$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{n \in \mathbb{Z}} \sum_{m \neq 0} \frac{1}{(m\tau+n)^k}$$

$$= 2\zeta(k) + \sum_{n\in\mathbb{Z}} \sum_{m \neq 0} \frac{1}{(m\tau+n)^k}$$

$$= 2\zeta(k) + 2\sum_{m=1}^{\infty} \left(\sum_{n \in \mathbb{Z}} \frac{1}{(m\tau+n)^k} \right)$$

We need to estimate $\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$ as a fourier expn.

let $f(\tau) = \sum_{m\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$, $k \geq 3$

we find that $\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}$ converges using a proof ~~like~~ by comparing with zeta function. Hence as $f(\tau)$ is $\mathbb{Z}$ periodic hence its holomorphic. Further converges uniformly.

$$\Rightarrow \lim_{\tau\to\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} = \sum_{n\in\mathbb{Z}} \lim_{\tau\to\infty} \frac{1}{(m\tau+n)^k} = 0. = \lim_{\tau\to\infty} f(\tau) = 0$$

We write fourier expn of $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ $q = e^{2\pi i\tau}$.

We know by the ~~argument~~ residue formula.

$$2\pi i(\mathrm{Res} f) = \int_C g(z)\,dz$$

We take the circle (boundary of an open Disc). ~~unit~~ ~~punctured~~

now ~~g=~~ g here we take to be. $\frac{f(\tau)}{q^{m+1}} = \frac{g(q)}{q^{m+1}}$

$$\cancel{2\pi i(a_m) = \int_C \frac{g(q)}{q^{m+1}}\,dq = \int_C \frac{f(\tau)(2\pi i)e^{2\pi i\tau}}{2\pi e^{2\pi i(m+1)\tau}}\,d\tau}$$

$$\cancel{\Rightarrow a_m = \int_{0+iy}^{1+iy} f(\tau)\, e^{-2\pi i k\tau}\,d\tau}$$

$$\cancel{= \int_{0+iy}^{1+iy} \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi i k\tau}}{(m\tau+n)^k}\,d\tau = \int_{-\infty+iy}^{\infty+iy} \frac{\tau^{-k}\, e^{-2\pi i k\tau}}{(m\tau+n)^k}\,d\tau}$$

$$\cancel{a_m = \int_{-\infty+iy}^{\infty+iy} \tau^{-k} e^{-2\pi i k\tau}}$$

$$2\pi i(a_l) = \int_C \frac{g(q)\,dq}{q^{l+1}} = \int_C \frac{f(\tau)(2\pi i)e^{2\pi i\tau}}{e^{2\pi i(l+1)\tau}}\,d\tau$$

$$\Rightarrow a_l = \int_{0+iy}^{1+iy} \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi i l\tau}}{(\tau+n)^k}\,.d\tau = \int_{-\infty+iy}^{\infty+iy} \frac{e^{-2\pi i l\tau}}{\tau^k}\,d\tau$$

$$= \frac{(-2\pi i l)^{k-1}}{(k-1)!} = a_l.$$

now we turn to the fourier expn of $G_k(\tau)$.

$$G_k(\tau) = 2\zeta(k) + 2\sum_{c\in\mathbb{N}}\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^k}$$

~~$= 2\zeta(k) + 2$~~

from previous computation we already know.

$$f(\tau) = \sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \sum_{l\in\mathbb{N}} a_l q^l \quad \text{where } a_l = \frac{(-2\pi i)^{k-1} l^{k-1}}{(k-1)!}$$

$$q = e^{2\pi i\tau}, \quad \text{for } f(m\tau), \; q^m = e^{2\pi i m\tau}$$

$$f_c(\tau) = \sum_{d\in\mathbb{Z}} \left(\frac{1}{c\tau+d}\right)^k = \sum_{l\in\mathbb{N}.} a_l q^{lc}$$

$$G_k(\tau) = 2\zeta(k) + 2\sum_{c\in\mathbb{N}}\sum_{l\in\mathbb{N}} a_l q^{lc} = 2\zeta_k + 2\sum_{n\in\mathbb{N}}\Big(\sum_{l|n} a_l\Big)q^n$$

$$= 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!}\sum \sigma_{k-1}^{(n)} q^n$$

$k>2$, k even

$$\boxed{\sigma_{k-1}^{(n)} = \sum_{d|n.} d^{k-1}}$$

$$\boxed{G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n}$$

is the fourier expn for $G_k(\tau)$.

- Hence the Eisenstein series of wt k provides us with our first example of modular form of wt k.

We move forward to see an example of cusp form. define

$$E_k(t) = \frac{G_k(t)}{2\zeta(k)}$$ (normalized ~~Eigen seri~~ Eisenstien series)

Now for $k = 4$ and 6.

$$\Delta' = (E_4(\tau))^3 - (E_6(\tau))^2 = \left(\frac{G_4(\tau)}{2\zeta(4)}\right)^3 - \left(\frac{G_6(\tau)}{2\zeta(6)}\right)^2$$

$\lim_{q\to 0} \Delta'(q) = 0$ at $q\to 0$.

$$\Delta' = \frac{(2\zeta(6))^2 (G_4(\tau))^3 - (2\zeta(4))^3 (G_6(\tau))^2}{8(\zeta(4))^3 \times 4\,(\zeta(6))^2}$$

Clearly Δ' is a cusp form of wt 12.

& $G_4(\tau) = a_0 + a_1 q + \dots$

$G_6(\tau) = a_0' + a_1' q + \dots$

$$(G_4(\tau))^3 = (a_0^2 + 2q(a_1 a_0) + \dots)(a_0 + a_1 q + \dots)$$
$$= (a_0^3 + (a_0^2 a_1 + 2a_1 a_0^2) q + \dots)$$
$$(G_6(\tau))^2 = (a_0'^2 + 2a_1' a_0' q + \dots)$$

now coeff of q in $\Delta' = a_1''$ as $\Delta' = a_0'' + a_1'' q + \dots$

$$a_1'' = (2a_1' a_0' a_0^3 - 3a_0^2 a_1 a_0'^2)$$

$$= (a_0' a_0^2)(2a_1' a_0 - 3a_1 a_0')$$

$$= (a_0' a_0^2)\left(2\left(2\frac{(2\pi i)^6}{5!}(\sigma_5(1))\, 2\zeta(4)\right) - 3\left(\frac{2(2\pi i)^4}{3!}(\sigma_3(1))\, 2\zeta(6)\right)\right)$$

$\sigma_5(1) = 1$, $\sigma_3(1) = 1$, $(2\pi i)^6 = -(2\pi)^6$

$$= a_0' a_0^2 \underbrace{\left(-\frac{4(2\pi)^6}{5!} 2\zeta(4) - \frac{6}{3!}(2\pi)^4\, 2\zeta(6)\right)}_{<0}$$

$\Rightarrow$ $a_1'' \neq 0$. hence Δ' has a simple zero at $q=0$.

$\Rightarrow$ Δ' is not identically zero

☐ Define $j(\tau) = \dfrac{c\,(G_4)^3}{\Delta'}$, where constant c is suitably chosen so that $\operatorname{res}(j(\tau))\big|_{\tau=0} = 1$.

We can clearly see that $j(\tau)$ is a weakly modular function with a simple pole at $q=0$.

We will call $\Delta = \Delta'/c$ to make representation of $j(\tau)$ simpler. $j(\tau)$ is called "modular invariant." Its name will become clear as we move to elliptic curves.

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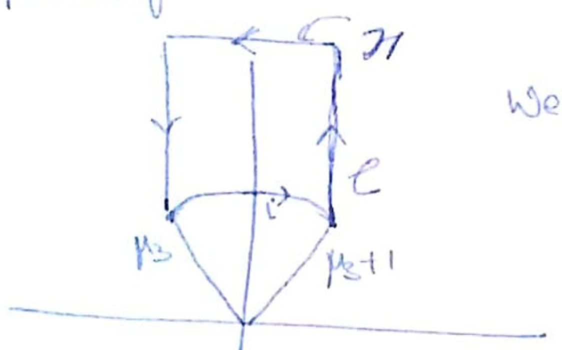
But before that we show a property of j.

<u>Prop:</u> $j : \mathbb{H} \to \mathbb{C}$ is a surjection.

<u>Pf:</u> Suppose not. then $\exists\, c \in \mathbb{C}$ s.t $j(\tau) \neq c$. We do the following. $j(\tau) - c$ is not zero. obviously. Look at the following contour.



We move along the arrows.

Here we evaluate the integral. $\displaystyle\int_{\mathcal{C}} \frac{f'(z)}{f(z)-c}\, dz$ with $f = j$

by argument principle this evaluates to.
$2\pi i$(number of zeros − number of poles) inside the contour.
But we know that $f(z) - c$ ~~z~~ $= j(z) - c$ is a holomorphic function except at $\lim \tau \to \infty$. Hence the integral $= 0$.
Note that f is a modular function of wt 0. hence is invariant under the action of $SL_2\mathbb{Z}$. this implies. that integral on the arch cancels and integral on two vertical lines cancels as j is periodic in Real axis. for the horizontal ~~arc~~ line. we have.

$$0 = \int_{\mathcal{C}} \frac{j'(z)\, dz}{(j(z) - c)} = \int_{\mathcal{H}} \frac{j'(z)\, dz}{j(z) - c}$$

$$= 2\pi i \int_{C} \frac{\left(-\frac{1}{q^2} + \cdots\right) dq}{\left(1/q + \cdots\right)} = 2\pi i \neq 0$$

circle of radius $e^{-y 2\pi}$

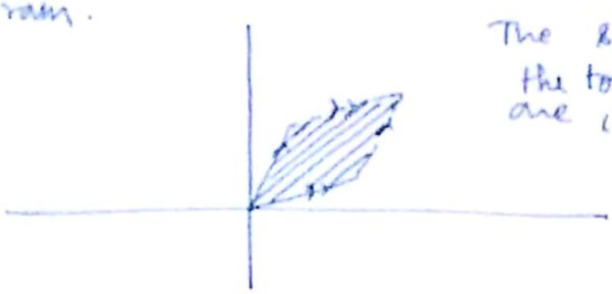
justification for change of variables

$$\left[\begin{array}{l} j'(z) = \dfrac{d}{dz}(j(z)) \\ j(z) = g(q) \to \dfrac{d}{dz}(j(z)) = g'(q)\cdot\dfrac{dq}{dz} \Rightarrow j'(z)\,dz = g'(q)\,dq \end{array}\right.$$

Hence we conclude that j is a surjection of $\mathbb{H} \to \mathbb{C}$

We now move an important class of structures known as Complex torus. Using a bit of manifold theory one can show that complex torus is a compact Riemann surface, meaning it is connected and roughly looks like complex plane (locally) ~~at~~ around every point along with some compatibility condition. But we wont get into those details. but assume that its a compact Riemann surface. Hence if we have two compact ~~C~~ Riemann surfaces and a continous map from $f: S_1 \to S_2$ where S_1, S_2 are compact riemann surface this will imply using open mapping thrm. that f is either constant else surjective. But before all this we define complex torus.

Complex torus: Given a lattice $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$ the complex torus associated with the lattice is defined by $\mathbb{C}/\Lambda$. the quotient represents the fundamental parallelogram.



The shaded region represents the torus as the opp. sides are identified.

▷ Thm: Given $f: \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ an holomorphic function

- f is surjective
- ~~$f(z) = m$~~ $\forall z$ $\exists m, b \in \mathbb{C}$ s.t $f(z) = mz + b$ and $m\Lambda \subseteq \Lambda'$
- f is a bijection if $m\Lambda = \Lambda'$

The ~~proof~~ goes via topology, via lifting of map on quotient space to original space.

At the same time f can also be seen as ~~acting on~~ map from an abelian additive group $\mathbb{C}/\Lambda$. If f is a non zero homomorphism then we call f to be an "Isogeny".

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We would like to see if there are specific conditions which we can apply which will make sure that f in hand is an isogeny. Clearly the conditions would be applied on m, b associated with f as f is completely defined by them.

▷ Prop: Let f be an isogeny then $b = 0 \cdot \bmod \Lambda'$

Pf: Let $f(z) = (mz+b)+\Lambda'$ now

$$f(z_1+z_2+\Lambda) = \cancel{f}\, m(z_1+z_2)+b+\Lambda'$$
$$= (mz_1+\Lambda')+(mz_2+b)+\Lambda'$$
$$= f(z_1+\Lambda)+f(z_2+\Lambda) \quad (f \text{ is homomorphic})$$
$$= (mz_1+\Lambda')+(mz_2+\Lambda')$$
$$\Rightarrow b \in \Lambda' \Leftrightarrow b = 0 \bmod \Lambda'$$

• The converse for the above ~~thm~~ proposition is trivial.

Clearly f is an isomorphism will imply $m\Lambda = \Lambda'$ and $b \in \Lambda'$.

Now as told earlier we will show the "equivalence" of lattices $\Lambda_\tau := \Lambda(\tau, 1)$ with $\Lambda(\omega_1, \omega_2)$ where $\tau \in \mathbb{H}$ & $\tau = \omega_1/\omega_2$.
The equivalence is in the terms that the corresponding complex tori obtained are isomorphic. This is clear that $\mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda$ as $\tau = \omega_1/\omega_2$. $\Rightarrow$ $\Lambda_\tau = \frac{1}{\omega_2}\Lambda(\omega_1, \omega_2)$

and $\phi : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda_\tau$

$$z+\Lambda \longrightarrow \frac{z}{\omega_2}+\Lambda_\tau$$ is an isomorphism (trivially by construction)

Clearly the τ obtained is not unique another τ' with such a property i.e. $\Lambda_{\tau'} \cong \Lambda(\omega_1, \omega_2) = \Lambda(\omega_1', \omega_2')$ is

$$\Lambda(\omega_1,\omega_2) \cong \Lambda_\tau$$

$\tau' = \omega_1'/\omega_2' = \gamma(\tau)$, where $\gamma \in SL_2(\mathbb{Z})$.

Examples of Isogeny.:

Clearly all endomorphisms of $\mathbb{C}/\Lambda$ are also an isogeny. now one trivial such is the identity. But a bit non trivial and perhaps an important one is multiplication by n map. For a ~~natu~~ $n \in \mathbb{N}$ denote

$$[n]: \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$$
$$z+\Lambda \to nz+\Lambda$$

Clearly this map has got a non zero kernel. further from complex analysis we know kernel map must be descrete and finite as o/w one can get an infinite sequence of element which converges to a point in $\mathbb{C}/\Lambda$ (compact) making $f = 0$. map.

One can also say quickly that the Kernel is isomorphic to $\frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}$ as $\left(\frac{\omega_1}{n}\mathbb{Z} \oplus \frac{\omega_2}{n}\mathbb{Z}\right)/\Lambda$ is ~~generate~~ the Kernel.

Further let $\phi: \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be an isogeny then $\phi(z+\Lambda) = mz+\Lambda'$. the set $m^{-1}\Lambda'/\Lambda$ forms the Kernel ~~form~~ the map ϕ and now looking at $m^{-1}\Lambda'$ as a supperlattice of Λ. Hence we define another isogeny which would be of importance as quotient isogeny.

choose a cyclic suproup $C \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$ of the ker $[n]$) map on $\mathbb{C}/\Lambda$ $C = \langle c+\Lambda \rangle$ hence look at the supperlattice of Λ formed by the cosets $\bigcup_{i=1}^{n} ic+\Lambda$. We define a projection map from the ~~latt~~ $\mathbb{C}/\Lambda$ to $\mathbb{C}/C$ as.

$$\pi: \mathbb{C}/\Lambda \to \mathbb{C}/C$$
$$z+\Lambda \mapsto z+C.$$

this map clearly has the Kernel as C.. which is cyclic.

from now on we denote the ~~ma~~ Ker [n] by E[n].
where $E := \mathbb{C}/\Lambda$.

Now here is a theorem which for me is still a surprising result..

Theorem: Every isogeny $\phi : \mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$ with $m\Lambda \subseteq \Lambda'$ and $\phi(z+\Lambda) = mz+\Lambda'$ is a composition of above mentioned isogenies.

Pf.: We just saw that any isogeny is primarily defined how its Kernel looks like as a supperlattic. Now ~~before~~ we have $m^{-1}\Lambda'$ a supper lattice of Λ and $m^{-1}\Lambda'/\Lambda$ is the Kernel for ϕ. Clearly the Kernel is discrete and finite and let N be its order. Hence denoting $\mathrm{Ker}(\phi) = K$

$$K \subseteq E[N] \cong \frac{\mathbb{Z}}{N\mathbb{Z}} \oplus \frac{\mathbb{Z}}{N\mathbb{Z}}$$

$$\Rightarrow \quad K \simeq \frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{nn_1\mathbb{Z}}$$

hence $\quad nK \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \subseteq E[n_1]$

↑ cyclic suproup. Hence we have

$$\mathbb{C}/\Lambda \xrightarrow{[n]} \mathbb{C}/\Lambda \xrightarrow{\pi} \mathbb{C}/nK \xrightarrow{\sim} \mathbb{C}/\Lambda' \qquad (nK \text{ is taken as a supper-lattice})$$

$$z+\Lambda \longrightarrow nz+\Lambda \longrightarrow nz+nK \longrightarrow mz+\Lambda'$$

for seeing the isomorphism b/w. $\quad nK = nm^{-1}\Lambda'$

$$\Rightarrow \left(\frac{m}{n}\right)(nK) = \Lambda' \quad \text{hence} \quad z+nK \rightarrow \frac{m}{n}z+\Lambda'$$

is an isomorphism

hence $\quad \phi = (\sim \circ \pi \circ [n])$.

Till Now things in this writeup have been a lot hapHazard but this is just a motivation to study modular forms and how they connect to elliptic curves. We have shown the connection b/w modular forms and lattices via the two basic theorems that are the Eisenstein series ($G_k(\tau)$) is a modular form. and can be viewed as a lattice function or better homogeneous lattice funct. meaning $G_k(m\Lambda) = m^{-k} G_k(\Lambda)$ on lattices which are isomorphic to Λ_τ. We would very much like to generalize this theory - i.e. we would like to see if other homogeneous lattice functions also extend to modular forms. One can check that they do in some sense

Eg. let $F_k(\Lambda)$ is a lattice funct. ~~i.e. is~~ and homogeneous of wt k then. we can form. $F_k(\tau) = m^{-k} F_k(\Lambda)$

where $m\Lambda = \Lambda_\tau$ is an isomorphism of $\Lambda(\omega_1, \omega_2)$ to Λ_τ. Since $F_k(\Lambda)$ converges $\forall \Lambda$ hence $F_k(\tau)$ converges ~~and is ho~~ as Λ is invariant under $SL_2\mathbb{Z}$ so is $F_k(\tau)$ but obviously with a factor of $(c\tau + d)^{+k}$. We will come to these things later again. But for now we want to know that lattices were fine to study but why the complex tori for modular forms. We now connect the complex tori to very interesting and widely studied objects called Elliptic curves. We will talk about $E(\mathbb{C})$ i.e Elliptic curves over complex. An elliptic curve is an equation of the form. $Y^2 = X^3 + AX + B$, $A, B \in \mathbb{C}$ for our case

We would show that every complex tori gives rise to an elliptic curve. and vice-versa every elliptic curve in $\mathbb{C}$ is associated with ~~a~~ complex tori.

□ <u>Wierstrauss $\wp$-function</u>: Define $\wp : \mathbb{C} \to \mathbb{C}$ for a given lattice Λ as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

One can see the $\wp(z)$ converges as $\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$ makes is approximately $\simeq \frac{1}{|\omega|^3}$. We formalize it as follows

Prop: $\wp(z)$ converges $\forall z \in \mathbb{C}/\{0\}$.

Pf. $A_{\omega} = \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2 - z^2 - \omega^2 + 2z\omega}{(z-\omega)^2(\omega^2)} = \frac{2z\omega - z^2}{(z-\omega)^2\omega^2}$

$\Rightarrow |A_\omega| = \frac{|2z\omega - z^2|}{|(z-\omega)^2\omega^2|} = \frac{|z|\,|2\omega - z|}{|(z-\omega)^2\omega^2|} \leq \frac{|z|^2\,|2|\frac{\omega}{z}| - 1|}{|(z-\omega)^2\omega^2|}$

$\leq \frac{3\left|\frac{\omega}{z}\right|}{\left|-1+\frac{\omega}{z}\right|^2|\omega|^2}$

$\leq \left|\frac{C}{z}\right| \frac{1}{|\omega|^2|\omega|}$ after finitely many ω

$\leq \frac{C'(z)}{|\omega|^3}$ for some constant

$\Rightarrow$ $\wp(z)$ converges $\forall z \neq 0$

Prop: $\wp(z)$ is Λ periodic i.e. $\forall \omega \in \Lambda$ $\wp(z+\omega) = \wp(z)$

Pf: $\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$

~~$\wp(z+\omega) = \frac{1}{(z+\omega)^2} + \sum \frac{1}{z-(\omega}$~~

$\wp(z+\omega_0) = \frac{1}{(z+\omega_0)^2} + \sum_{\omega\in\Lambda} \left(\frac{1}{(z-(\omega-\omega_0))^2} - \frac{1}{\omega^2}\right)$

let $\omega_0 = m_0\omega_1 + n_0\omega_2$, (ω_1, ω_2) are basis for latti

$\wp(z+\omega_0) = \frac{1}{(z+m_0\omega_1+n_0\omega_2)^2} + \sum_{\substack{m,n \neq \\ (0,0)}} \frac{1}{[z-((m-m_0)\omega_1+(n-n_0)\omega_2)]^2} - \frac{1}{\omega}$

$= \frac{1}{z^2} + \sum_{\substack{m,n \neq \\ (m_0,n_0)}} \frac{1}{(z-(m-m_0)\omega_1+(n-n_0)\omega_2)^2} - \sum_{m,n\neq(0,0)} \frac{1}{\omega^2}$

$= \frac{1}{z^2} + \sum_{\substack{m,n \\ \neq(m_0,n_0)}} \frac{1}{[z-((m-m_0)\omega_1+(n-n_0)\omega_2)]^2} - \sum_{\substack{(m,n) \\ \neq(m_0,n_0)}} \frac{1}{(m-m_0)\omega_1 + (n-n_0)\omega_2}$

$= \frac{1}{z^2} + \sum_{m',n'\neq(0,0)} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$ where $m', n' = (m-m_0, n-n_0)$ & $\omega = m'\omega_1 + n'\omega_2$

we now look at the ~~function~~ derivative of the function.
or before lets investigate a bit more in this function.

as $\wp(z)$ is Λ periodic and converges uniformly on every compact subset of ~~&~~ $\mathbb{C}$. We may constrain ourselves as well on the torus $\mathbb{C}/\Lambda$.

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq(0,0)} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

$\left(\wp(z) - \frac{1}{z^2}\right) = 0$ at $z=0$ hence $\left(\wp(z) - \frac{1}{z^2}\right) = z\,g(z)$.

where $g(z)$ is complex analytic at $\left(\wp(z) - \frac{1}{z^2}\right)$ is a holomorphic function.

now we know that $\wp(z)$ is even & $1/z^2$ is even hence

$\left(\wp(z) - \frac{1}{z^2}\right)$ is an even function $\Rightarrow g(z)$ is odd $\Rightarrow g(0) = 0$

$g(z) = z\,m_1(z)$. hence

$\wp(z) = \frac{1}{z^2} + z^2 m_1(z)$ where $m_1(z)$ is complex analytic.

$$\wp'(z) = \frac{-2}{z^3} + 2z\,m_1(z) + z^2 m_1'(z) = \frac{-2}{z^3} + z\,m_2(z)$$

where $m_2(z)$ is a complex analytic function.

Now from def. one can see that $\wp'(z) = \frac{-2}{z^3} - 2\sum'_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}$

An obvious thing to look at is the diff of some powers of $\wp'(z)$ & $\wp(z)$ so that we annihilate the negative z power and make the function complex analytic at $z=0$. Now. $4\wp^3(z) - \wp'(z)^2 = f(z)$

$$4\left(\frac{1}{z^6} + z^6 m_1^3 z + 3z^2 m_1^2(z) + \frac{3}{z^2} m_1(z)\right)$$
$$- \left(\frac{4}{z^6} + z^2 m_2^2(z) - \frac{4}{z^2} m_2 z\right)$$

$$\Rightarrow f(z) = \frac{12 m_1(z) + 4 m_2(z)}{z^2} + z^2(m_4(z))$$ where $m_4(z)$ is complex analytic.

$$f(z) = \frac{m_3(z)}{z^2} + z^2 m_4(z).$$

we choose a_2 s.t ~~$f - a_2\wp(z)$~~ $f(z) - a_2\wp(z)$

$$f(z)-a_2\wp(z)= \frac{M_3(z)-a_2}{z^2} + z^2(M_4(z)) - z^2(M_1(z))$$

choose a_2 s.t $M_3(z)-a_2=0$ at $z=0$ $\Rightarrow$ as M_3 is even by construction. ~~hence~~

$M_3(z)-a_2 = z\,g(z)$ for some complex analytic function $g(z)$. Note that when we write for some complex analytic function $g(z)$ then its implicit that we are talking about in some nbhd about 0.

now $M_3(z)-a_2$ is again even function.

$\Rightarrow$ $g(z)$ is odd $\Rightarrow$ $g(0)=0 \Rightarrow g(z)=z\,h(z)$

hence

$M_3(z)-a_2 = z^2 h(z)$ where $h(z)$ is complex analytic. hence.

$$\frac{M_3(z)-a_2}{z^2} = h(z) \text{ is complex analytic hence}$$

holomorphic ~~$z^2(M_4(z)-M_1(z))$~~

now $f(z)-a_2\wp(z) = h(z)+z^2M_5(z)$

this will imply ~~since~~ as for $z\in \mathbb{C}/\Lambda$ ~~is~~ which is a compact Reimann surface ~~hence~~. is ~~not~~ $f(z)-a_2\wp(z)$ is holomorphic hence we have. $f(z)-a_2\wp(z)$. is ~~not~~ constant. as bondedness of $f(z)-a_2\wp(z)$ coming from its restriction ~~of~~ on $\mathbb{C}/\Lambda$ as $f(z)-a_2\wp(z)$ being Λ periodic.

This implies.

$$f(z)-a_2\wp(z)=a_3 \text{ (constant)}.$$

$$\Rightarrow \boxed{f(z)-a_2\wp(z)-a_3=0.}$$

$$\Rightarrow \quad (\wp'(z))^2 = 4(\wp(z))^3 - a_2\wp(z)-a_3.$$

implying ~~$\wp'(z)$~~ $(\wp(z), \wp'(z))$ satisfy elliptic curve equation $Y^2=4X^3-a_2X-a_3$. We note that a_2 and a_3 are lattice dependent quantities. hence from a lattice Λ_τ we can come up with an elliptic curve eq.n. which is satisfied by $(\wp(z), \wp'(z))$. Similarly

We had an intuition that we can come up with modular forms using the homogeneous lattice functions of respective wts. Hence lattice connects to both elliptic curves and also to modular forms.

Theory of Elliptic curves — Lattices in $\mathbb{C}$ $\longrightarrow$ Theory of Modular for

(Lattice function homogeneous) $\longrightarrow$ modular form

Elliptic curves. $\longleftarrow$ ($\mathbb{C}/\Lambda$ the quotient)

Natural to expect link b/w these two

Soon enough we would find a link b/w these two not the final one tho but quite interesting and natural also..

We move ~~forward~~ in ~~explaing~~ exploring the properties of $\wp$-function.

Since we know that $\wp(z) = \frac{1}{z^2} + z^2(m_1(z))$ about zero. we would like to know how does $m_1(z)$ looks like.

$m_1(z)$ is clearly complex analytic about 0. hence we have. Let $|z| < |w_0|$ where w_0 is such that $|w| \geq |w_0|$ $\forall$ $w \in \Lambda$ and $w_0 \in \Lambda$. hence we have..

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda}{}' \frac{1}{(z-w)^2} - \frac{1}{w^2} = \frac{1}{z^2} + \sum_{w\in\Lambda} \frac{1}{w^2}\left(\frac{1}{(1-(z/w))^2} - 1\right)$$

$$= \frac{1}{z^2} + \sum_{w\in\Lambda} \frac{1}{w^2}\left(\sum_{n\in\mathbb{N}} (n+1)\frac{z^n}{w^n}\right)$$

$$= \frac{1}{z^2} + \sum_{n\in\mathbb{N}} \sum_{w\in\Lambda} (n+1)\frac{z^n}{w^{n+2}}$$

$$= \frac{1}{z^2} + \sum_{n\in\mathbb{N}} (n+1)\, G_{n+2}(\Lambda)\, z^n$$

Clearly. the ~~seq.~~ series $\sum_{n\in\mathbb{N}} (n+1)\left(\frac{z}{w}\right)^n$ converges iff $\frac{|z|}{|w|} < 1$

$$\Rightarrow \boxed{\wp(z) = \frac{1}{z^2} + \sum_{n\in\mathbb{N}} (n+1)\, G_{n+2}(\Lambda)\, z^n}$$

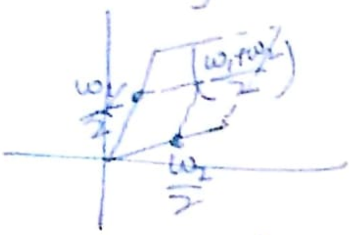
Now we look at the equation. $Y^2 = 4X^3 - a_2X - a_3$.
and with the laurent expn of $\wp(z)$ at hand one can show that $a_2 = 60G_4(\Lambda)$, $a_3 = 140G_6(\Lambda)$ and hence we define.

$$\underbrace{a_2}_{:= g_2(\Lambda)} \qquad \underbrace{a_3}_{=: g_3(\Lambda)} \quad \text{(by standard notation)}$$

Now we have very important thing to show.
Define $\Delta := g_2^3 - 27g_3^2$ as the discriminant of the cubic equation ~~with~~ which dictates whether $4X^3 - a_2X - a_3$ has got common roots or not. Now. we have. any $\omega \in \Lambda$ & $\omega \neq 0$. then

$$\begin{aligned} \wp'(-\omega/2) &= \wp'(-\tfrac{\omega}{2} + \omega) \\ &= \wp'(\omega/2) \\ &= -\wp'(\omega/2) \quad (\text{odd function}) \end{aligned}$$

$\Rightarrow$ $\wp'(\omega/2) = 0$. hence. one can show now that, ~~that~~ as ~~$\wp'(\omega/2)$~~ or $(\wp(\omega/2), \wp'(\omega/2))$ satisfy $Y^2 = 4X^3 - g_2X - g_3$ $\forall\ z \in \mathbb{C}/\Lambda$ hence we have $\wp(\omega/2)$ as roots of $4X^3 - g_2X - g_3$. now. clearly we have three distinct roots inside $\mathbb{C}/\Lambda$



$\Rightarrow$ $\Delta = g_2^3 - 27g_3^2$ is not zero for any lattice.
g_2^3, g_3^2 are lattice (homogeneous functions) of same wt. hence. we can get to an equivalent lattice Λ_τ from $\Lambda(\omega_1, \omega_2)$ by multiplication of $\frac{1}{\omega_2}$. so we may aswell present it by just Λ_τ. i.e working with an isomorphic lattice.
Now Remember $\Delta' = (E_4)^3 - (E_6)^2$ defined ~~in chapter~~ previously. One can with some computation show that $\Delta' = c\Delta$. $\Rightarrow$ Δ' has got no roots. except at ∞.

But we already have seen that Δ' is a modular form in particular a cusp form. ~~I~~nstead of looking at lattice $\Lambda(\omega_1,\omega_2)$ we look at the following, How about Λ_τ, $\tau = \omega_1/\omega_2 \in \mathbb{H}$. One quickly sees that $\Lambda_\tau = \frac{1}{\omega_2}\Lambda$ and sine Δ' is also a lattice function (homogeneous) ~~o~~ wt ~~k~~ 12. hence ~~Δ_τ on $\Delta'(\Lambda_\tau)$~~ on Λ_τ, $\Delta'(\tau) = \left(\frac{1}{\omega_2}\right)^{-12}\Delta'(\Lambda) = \omega_2^{12}\Delta'(\Lambda)$. We want a quantity that represents an equivalence partition in the ~~latt~~ space of all complex tori. So the best thing is to get a modular ~~form~~ function on Λ. ~~or~~ or corresponding homogenous lattice function which has wt 0. Hence we ~~make~~ make use of

$$j = \frac{c\,(G_4)^3}{\Delta'}$$

as previously. Since j is a modular function of wt 0 with only pole at ~~0. and~~ infinity and holomorphic everywhere. Now we show some exciting properties of j. ~~o~~ One can comput that

$$j = \frac{1728(g_2)^3}{(g_2)^3 - 27(g_3)^2} = \frac{1728(g_2)^3(\tau)}{\Delta(\tau)}$$

This shows that for every lattice one can associate an elliptic curve corresponding to the ~~equi~~ $\mathbb{C}/\Lambda$. and the j invariant is invariant in the equivalente class of $\mathbb{C}/\Lambda$. But at the same time. we saw that j is a surjection ~~and~~ from $\mathbb{H} \to \mathbb{C}$. Hence for every element c in $\mathbb{C}$ there exists an element τ in $\mathbb{H}$ s.t $j(\tau) = c$. Hence for every point in $\mathbb{C}$ we can get element in $\mathbb{H}$ and hence the corresponding lattice.

An interesting question however to ask would be that given an elliptic curve $Y^2 = 4X^3 - a_2X - ~~E~~a_3$, $a_2^3 - 27a_3^2 \neq 0$. Can we come up with a lattice. Λ s.t $a_2 = g_2(\Lambda)$ and $a_3 = g_3(\Lambda)$. We already know we can definitely come up with a τ s.t $j(\tau) = \frac{(a_2)^3}{\Delta}$. If this is true then it will hold ~~an~~ a 1-1 correspondance b/w equivalence class of lattices.

and "equivalence" class of Elliptic curves. Equivalence in some sence. (for now if the corresponding lattices are equivalent or equivalently both have same j invariant.

Proof: Given an elliptic curve $E := Y^2 = 4X^3 - a_2X - a_3$ with $\Delta(E) \neq 0$. then. $\exists$ a lattice Λ s.t. $g_2(\Lambda) = a_2$ and $g_3(\Lambda) = a_3$.

Pf: Now we have $\tau \in \mathbb{H}$ s.t

$$j(\tau) = \frac{1728\,(g_2(\tau))^3}{(g_2(\tau))^2 - 27(g_3(\tau))^2} = \frac{a_2^3\,(1728)}{a_2^3 - 27a_3^2}$$

Now if $j(\tau) = 0 \Leftrightarrow g_2(\tau) = 0 \Rightarrow \tau = \mu_3$, (third root of unity)

choose ω, s.t $\omega^{12} = \left(\frac{a_3}{g_3(\mu_3)}\right)^2$

if $j(\tau) = 1728 \Leftrightarrow g_3(\tau) = 0 \Rightarrow \tau = i$ (4th root of unity)

$j(\tau) \neq 0, 1728$ then we have choose $\omega^{12} = \left(\frac{a_2}{g_2(\tau)}\right)^3$.

$$(g_2(\tau))^3(a_2^3 - 27a_3^2) = a_2^3(g_2(\tau))^2 - 27a_2^3(g_3(\tau))^2$$

$$\Rightarrow (g_2(\tau))^3(a_3)^2 = a_2^3(g_3(\tau))^2$$

$$\Rightarrow \left(\frac{g_2(\tau)}{a_2}\right)^3 = \left(\frac{g_3(\tau)}{a_3}\right)^2$$

now. choose ω_2. s.t. $\omega_2^{+12} = \left(\frac{g_2(\tau)}{a_2}\right)^3 = \frac{(g_3(\tau))^2}{a_3^2}$

now. let $\omega_2^{+4} = \frac{g_2(\tau)}{a_2}$, $\pm\frac{g_3(\tau)}{a_3} = \omega_2^{-6}$

$\Rightarrow$ if $\omega_2^{+6} = \frac{g_3\tau}{a_3}$ then okay

o/w take $\omega_2 = i\omega_2$. hence we have

~~$\omega_2(\tau)$~~ ~~$\omega_2^{-4} g_2(\tau) = a_2 \Rightarrow$~~

$g_2(\Lambda) = \omega_2^{-4} g_2(\tau) = a_2$ when $\Lambda = \omega_2\Lambda_\tau = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ where $\omega_1 = \tau\omega_2$

This establishes the correspondence b/w equivallence of elliptic curves and equivalence class of ~~En~~ Lattices.

Till now we have been talking about equivalence of Elliptic curves.

We saw that an elliptic curve can be looked as $\mathbb{C}/\Lambda$ under the isomorphism (complex tor) $z \longmapsto (\wp(z), \wp'(z))$ now we already know that an elliptic curve $\mathbb{C}/\Lambda$ is group isomorphic to $\mathbb{C}/\Lambda'$ (holomorphically) if $\exists\, m \in \mathbb{C}$ s.t. $m\Lambda = \Lambda'$ now this implies as $\Lambda \cong \Lambda_\tau$ and $\Lambda' \cong \Lambda_{\tau'}$ then $m\Lambda = \Lambda' \cong \Lambda_{\tau'}$ now. this implies that $\begin{pmatrix} m\omega_1 \\ m\omega_2 \end{pmatrix} = \gamma \begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix}$

$\Rightarrow$ $\frac{\omega_1}{\omega_2} = \tau = \gamma(\tau')$. hence the equivalence class or orbit space of $\mathbb{H}/SL_2\mathbb{Z}$ i.e. under action of $SL_2\mathbb{Z}$ ~~forms a~~ goes in one to one correspondance with elliptic curves isomorphisms (class).

We noticed that the torsion data i.e $E[N]$ of an elliptic curve ~~...~~ is specifically very important as it is a Kernel to the isogeny $[n]$. of $\mathbb{C}/\Lambda$. But how does the isomorphism act on them. we ~~so~~ define a relation on the space $\mathcal{C} \times T(N)$. ~~as the subs~~ where $\mathcal{C} = \{\mathbb{C}/\Lambda \mid \Lambda \text{ a lattice in } \mathbb{C}\}$ and

$T(N)$ ~~an~~. be the ~~p N torsion sub~~ set of order N cyclic subgroup of $E[N]$ of a given elliptic curve. now a typical element in ~~$\mathcal{E}\times T$~~ $\mathcal{C} \times T(N)$ looks like (E, C) where C is the order N cyclic subgroup of $E[N]$. Now we define $(E_1, C_1) \cong (E_2, C_2)$ if $\exists$ an isomorphism $\phi: E_1 \to E_2$ and $\phi|_{C_1} = C_2$. Note the this is an equivalence ~~class~~. relation ~~the~~ and by $[E, C]$ we denote the equivalence element ~~class~~ of $[E, C]$. Let now denote the Equivalence class of relation $\cong$ by $S_0(N)$. At first it might seem that this will turn out to be same as equivalence class of elliptic curves. but it doesn't as we want the ϕ to take ~~the~~ the subgroup to a particular cyclic subgroup which might not always be possible. We show that by a simple illustration.

Suppose Λ_τ and $\Lambda_{\tau'}$ be two equivalent lattices and let $\langle \frac{1}{N}+\Lambda_\tau\rangle$, $\langle \frac{1}{N}+\Lambda_{\tau'}\rangle$ be the cyclic subgroups of ~~E~~ $\mathbb{C}/\Lambda_\tau[N]$, $\mathbb{C}/\Lambda_{\tau'}[N]$

We have the following clearly $\tau' = \gamma(\tau)$ for some $\gamma \in SL_2(\mathbb{Z})$

as $\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \frac{1}{(c\tau+d)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} \Rightarrow \Lambda_{\tau'} = \frac{1}{(c\tau+d)} \Lambda_\tau$.

hence the map $z + \Lambda_\tau \longmapsto \frac{z}{c\tau+d} + \Lambda_{\tau'}$ is an isomorphism of $\phi: \mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\tau'}$. whit with $m = \left(\frac{1}{c\tau+d}\right)$. Further. we have

if $\left(\langle 1/N + \Lambda_\tau\rangle, \mathbb{C}/\Lambda_\tau\right) \cong \left(\langle 1/N + \Lambda_{\tau'}\rangle, \mathbb{C}/\Lambda_{\tau'}\right)$

then. $\phi\left(\frac{1}{N} + \Lambda_\tau\right) = \frac{1}{N(c\tau+d)} + \Lambda_{\tau'}$ or equivalently

we can also take $\phi' : \mathbb{C}/\Lambda_{\tau'} \to \mathbb{C}/\Lambda_\tau$ by sending ~~$z+\Lambda_{\tau'} \to z + \Lambda$~~

$z + \Lambda_{\tau'} \longrightarrow (c\tau+d)z + \Lambda_\tau$. hence.

$\phi(1/N + \Lambda_{\tau'}) \longrightarrow \frac{c\tau+d}{N} + \Lambda_\tau$ but any generator of $\langle \frac{1}{N} + \Lambda_\tau\rangle$ is of the form $\frac{k}{N} + \Lambda_\tau$ ~~hence~~ where $(k, N) = 1$. Hence

$\frac{c\tau+d}{N} + \Lambda_\tau = \frac{k}{N} + \Lambda_\tau \Rightarrow \frac{c\tau+d}{N} - \frac{k}{N} \in \Lambda_\tau \Rightarrow \frac{c\tau}{N} + \frac{d-k}{N} \in \Lambda_\tau$

$\Rightarrow$ $N \mid c$ and $d \equiv k \bmod N$ where k is a unit in $\frac{\mathbb{Z}}{N\mathbb{Z}}$

In other words we can look at $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ having special condition that $a, d \in \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*$ s.t. $ad \equiv 1 \bmod N$. and also $c \equiv 0 \bmod N$. $\Rightarrow$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \bmod N$. Hence this illustrates that we have special kind of γ's doing this. Further if we want our ϕ to take $\frac{1}{N} + \Lambda_{\tau'}$ to $\frac{1}{N} + \Lambda_\tau$ then it becomes very special.. as this implies. that our k previously as above $= 1$. $\Rightarrow$ $d \equiv 1 \bmod N \Rightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N$.

Hence something is interesting is happening with these restriction further if we take and γ satisfying the above property i.e $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \bmod N$ ~~or~~ or $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N$. then we can show that we exactly at the above equivalent relation or we exactly get the ~~#~~ $\phi: \mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\tau'}$ satisfying the condition for equivalent relations on $\mathbb{C} \times T(N)$. We make these things more precise. We claim that every cyclic ~~group of~~ subgroup of $E[N]$ of order N. and $E = \mathbb{C}/\Lambda$. where $\Lambda = \{\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. can be isomorphically sent to Λ_τ and $C \to \frac{1}{N} + \Lambda_\tau$. To see this we have an obvious isomorphism from ~~Λ~~ $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda_\tau$ when $\tau = \omega_1/\omega_2$. hence clearly an order N subgroup of $E[N]$ has to go to $\left\langle \frac{c\tau+d}{N} + \Lambda_\tau \right\rangle$ which is the generator of any order N subgroup of $\left\langle \frac{1}{N} + \Lambda_\tau \right\rangle \oplus \left\langle \frac{\tau}{N} + \Lambda_\tau \right\rangle$. Obviously with some constraints on c and d that are c, d are coprime. to N. hence we have

$a, b, k \in \mathbb{Z}$. s.t. ~~$ad - bc + kN = 1$~~ $ad - bc + kN = 1$

$\Rightarrow$ mod N $ad - bc \equiv 1$. Hence as we have surjection of $SL_2(\mathbb{Z})$ to $SL_2\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)$. Hence we can lift a, b to $\mathbb{Z}$ (abusing the notation slightly by saying lift of a, b as a, b, so that. $ad - bc = 1$.

$\Rightarrow$ now take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and look at lattice.

$\Lambda = \mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)$. we know that $\Lambda = \Lambda_\tau$ hence. dividing by $(c\tau+d)$ take the lattice $\Lambda_\tau \to \Lambda_{\tau'}$ where $\tau' = \gamma(\tau)$. Hence an isomorphism taking $\frac{c\tau+d}{N} + \Lambda_\tau \to \frac{1}{N} + \Lambda_{\tau'}$. $\Rightarrow$. $\exists$ τ and isomorphism of $\Lambda(\omega_1, \omega_2) \to \Lambda_\tau$ s.t. $C \to \left\langle \frac{1}{N} + \Lambda_\tau \right\rangle$.

Now we directly state the result which we indirectly have showed or before lets define some more important subgroups of $SL_2(\mathbb{Z})$. and then look at their quotient space of their action on $\mathbb{H}$.

□ Define. $\Gamma(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$ to be congruence subgroup of level N.

Further ~~¥~~ $\Gamma_0(N) := \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \}$

$$\Gamma_1(N) := \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \}$$

We first saw the importance of $\Gamma_1(N)$ and $\Gamma_0(N)$ in some sense we take on belief for now that $\Gamma(N)$ also comes from slighty changing the equivalence relation on $\mathcal{E} \times T(N)$. ~~by some~~.

We call a subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ <u>congruence subgroup</u> ~~of level N~~ if for some $N \in \mathbb{N}$ $\boxed{\Gamma(N) \subseteq \Gamma}$

Clearly $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N)$ and equality holds for $N=1$. (trivially). Let $Y_0 = \mathbb{H}/\Gamma_0(N)$

$$Y_1 = \mathbb{H}/\Gamma_1(N)$$

$$Y = \mathbb{H}/\Gamma(N)$$

where the $\mathbb{H}/\Gamma$ represents the orbit space or quotient space in ~~topological and~~ group theoretic and topological sense. We now state the ~~b~~ first "equivalence" result (kind of) dont go literal things. We are taking of an intuitionistic relation which we can ~~think~~ hope to exist after so much theory.

<u>Theorem</u>: Given $N \in \mathbb{N}$. and $S_0 = \{ [E, C] \}$; $S_1 = \{ [E, P] \}$ ~~and S =~~ where $[E, C]$ is ~~an equivalence~~ representation of the (E, C) equivalent under $\phi : (E_1, C_1) \to (E_2, C_2)$ s.t $\phi : E_1 \to E_2$ is isomorphism & $\phi(C_1) = C_2$.

and (E_1, P) is equivalent of (E_2, Q) if $\exists$ an isomorphism from $E_1 \to E_2$ taking the given generator of order N in E_1 to given generator of order N in E_2 i.e. Q

~~Equivalence class~~ S_1 is the quotient space of ~~E~~ $\mathcal{E} \times T[N]$ quotiented with the later relation. Then. $\exists$ a bijection ~~from $Y_0 : S_0 \to E[N]$~~

$\Psi_0 : S_0(N) \to Y_0(N)$ and $\Psi_1 : S_1(N) \to Y_1(N)$

$[E, C] \mapsto \Gamma_0(N)\tau$ $\qquad$ $[E, P] \mapsto \Gamma_1(N)\tau$

where $\Gamma_i(N)\tau$ is the orbit of τ under $\Gamma_i(N)$. for $i \in \{0, 1\}$

One can easily show that $\Gamma(N)$, $\Gamma_0(N)$, $\Gamma_1(N)$ are subgroups of $SL_2(\mathbb{Z})$.

" Wiles' Modularity Thm : Let E be a complex elliptic curve with $j(E) \in \mathbb{Q}$. Then for some positive integer N there exists a surjective holomorphic function of compact Reimann surfaces from the modular curve $X_0(N)$ to the elliptic curve E. "

$X_0(N)$ is ~~essentially~~ (approximately) $Y_0(N)$ but for few points. One can give $Y_0(N)$ a topology induced from the usual topology over ~~$\mathbb{C}$~~ $\mathbb{H}$ (quotient topology). Using this we have ~~that~~ a Reimann surface structure over $Y_0(N)$. One can show that only finitely many points are needed to compactify ~~$\mathbb{C}$~~ $Y_0(N)$ (because $[SL_2(\mathbb{Z}) : \Gamma_0(N)] < \infty$). This compactification of $Y_0(N)$ is $X_0(N)$. Wiles' Modularity thm. which is a remarkable result in Number theory in last few decades. essentially connexts compactified Reimann surfaces $X_0(N)$ and E via a surjective holomorphic map.

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