Mon. Not. R. Astron. Soc. **359,** 79–92 (2005) doi:10.1111/j.1365-2966.2005.08864.x

Nutational damping times in solids of revolution

Ishan Sharma,[1⋆] Joseph A. Burns[1,2] and C.-Y. Hui[1]

[1]*Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, USA*
[2]*Department of Astronomy, Cornell University, Ithaca, NY 14853, USA*

Accepted 2005 January 25. Received 2004 August 25; in original form 2003 October 17

ABSTRACT

We derive the characteristic nutational damping time $T_{\rm d}$ for a linear, anelastic ellipsoid of revolution. Our calculation is based on the well-known idea that energy loss within an isolated spinning body causes the axis of maximum inertia of the body to align with its angular momentum vector, leading to pure spin. Energy loss occurs within an anelastic material whenever internal stresses are time variable; thus even freely rotating bodies in space, if they are wobbling, lose energy because internal stresses are associated with the accelerations caused by nutation. We find that $T_{\rm d} = D(h)(\frac{\mu Q}{\rho a^2 \Omega^3})$, where $D(h)$ is a constant of the order of a few times 10^2 that depends on the shape of the body with h being the (aspect) ratio of the lengths of axes to one another, μ is the elastic modulus, Q is a quality factor that describes the anelasticity of the material, ρ is the density of the body, a is its radius and Ω is an angular velocity. This functional form of the damping time is consistent with previous results but is more soundly based. Coefficients in past expressions vary between various authors, leading to predicted damping times that can differ by factors of the order of 10. To estimate damping times for typical asteroids, we choose values for the various parameters in this expression. We conclude that the extent of energy dissipation was *over*, rather than *under*estimated, in previous treatments. None the less, we argue that asteroids will generally be found in pure rotation, unless objects are small, spinning slowly and recently excited.

Key words: techniques: radar astronomy – comets: general – minor planets, asteroids.

1 INTRODUCTION

In celestial rotation, regularity rules: most spinning objects are found in nearly pure spin (i.e. they rotate at a constant rate around a single axis, which is the axis of maximum moment of inertia). Such a rotation state is surprising because, over long times, bodies are being continually jostled about, either through collisions or during close flybys. The pure spins that are so prevalent must therefore result from an ongoing process.

Energy and angular momentum are both conserved for an isolated rigid body because no external forces act nor are internal motions permitted. In general, rotation (described for biaxial bodies in Section 5), such a body 'wobbles' or nutates (i.e. executes free Eulerian motion); this motion, which is specified once the angular momentum $\boldsymbol{J}$, kinetic energy $E_{\rm k}$ and moments of inertia of the body are specified, can be quite complicated. The motions are usually described in combinations of elliptic functions (Landau & Lifshitz 1960).

Internal dissipation, which occurs once internal motions are possible, however, leads to energy loss. This moves the body closer to a state of pure rotation about its axis of maximum moment of inertia as this state has the least energy for a fixed angular momentum (Landau & Lifshitz 1960). In this state, the angular velocity and the angular momentum vectors are aligned with a body axis. The characteristic interval over which this alignment occurs is the nutational damping time $T_{\rm d}$; it will depend obviously on the energy loss rate which has to be estimated.

Time-varying loads (stresses) lead to energy loss in isolated, anelastic solid bodies; the former are a natural companion to the nutational motion of the body, because this requires elements of the body to undergo accelerations. Thus, in order to estimate the energy loss, we must first calculate the induced variable stresses, and then we must model the energy loss associated with the time-varying stresses. In each of these steps we must assume a model of material behaviour.

In the past, the nutational damping of asteroids as caused by internal energy dissipation was analysed approximately by Prendergast (1958), Burns & Safronov (1973), McAdoo & Burns (1974), Efroimsky & Lazarian (2000) and Efroimsky (2000, 2001). Many other similar dynamical studies concern planetary wobble decay (e.g. Peale 1973; Yoder & Ward 1979), including damping of the Chandler wobble of Earth (e.g. Munk & MacDonald 1960; Lambeck 1980, 1987), interstellar dust grain alignment (e.g. Purcell 1979; Lazarian & Efroimsky 1999) and comets (e.g. Molina, Moreno & Martinez-Lopez 2003).

In most of these cases, the familiar Q model (Knopoff 1964; Burns 1977; Lambeck 1980) was assumed to estimate energy loss. Internal stresses were computed by either invoking Love

⋆E-mail: is42@cornell.edu

numbers[1] (Munk & MacDonald 1960; Burns & Safronov 1973; Peale 1973; Yoder & Ward 1979; Lambeck 1980) or by directly attempting to solve the full three-dimensional (3D) elasticity problem for the body (Prendergast 1958; Purcell 1979; Denisov & Novikov 1987; Lazarian & Efroimsky 1999; Efroimsky & Lazarian 2000; Efroimsky 2000; Washabaugh & Scheeres 2002; Molina et al. 2003). When computing stresses, the body was usually taken to be elastic or at best viscoelastic (McAdoo & Burns 1974; Lambeck 1980, 1987); with the latter model, only nearly spherical bodies were considered. Lastly, we should mention that Denisov & Novikov (1987) managed to solve for the stresses inside a tumbling, symmetric, linear-elastic ellipsoid. However, we came to know of this reference only after we had completed the calculations for the stresses.

All the above authors derive the same functional form for the damping time (as could have been anticipated from dimensional analysis alone); see Section 9. However, the numerical coefficients in these published results are claimed to differ by factors of the order of 10. This disagreement could be attributed to: (a) the different body shapes that were used when stresses were estimated or, more fundamentally, (b) the alternate approaches that were employed to estimate stresses or, more prosaically, (c) errors in analysis or incorrect assumptions. To complicate matters further for astronomers or geophysicists who might wish to use these results when estimating actual damping times, the various authors often chose different values for the physical constants that appeared in their expressions.

The calculation to be presented below should be useful because, as an *exact* elasticity solution, it provides the best estimate of the internal stresses employed in this and earlier works, and because the character of the spheroid can be varied (from flattened discs to narrow cigars) in order to explore how shape influences damping.

2 OUTLINE OF THE CALCULATION

Before proceeding further, we outline the broad steps involved in our calculation. Questions as to why the problem is tackled in this manner and/or why a possible effect is neglected will be addressed in the following sections.

The general philosophy is that nutational motion of the body introduces time-varying internal stresses which cause energy loss. This dissipation then tends to align the angular velocity vector, the angular momentum vector and the axis of maximum moment of inertia. With this in mind, we proceed as follows.

(1) The acceleration field generated by any nutational motion is calculated in order to provide the body forces that act on internal elements.

(2) Using these body forces, internal stresses are calculated by solving the corresponding elasticity problem.

(3) The stress field thus calculated furnishes the strain energy necessary to estimate the energy dissipation using the energy dissipation mechanism at our disposal.

(4) Knowledge of the energy dissipation rate then allows the nutational damping time to be calculated.

3 MODEL AND SIMPLIFYING ASSUMPTIONS

In general, because the rotational dynamics are captured in the specification of *three* moments of inertia, one could consider modelling the asteroid as an anelastic, *tri*axial ellipsoid. Here the anelasticity accounts for energy dissipation while a suitable choice of the axes of the ellipsoid would set the correct dynamics and would approximate most physical bodies reasonably well. As a first step toward the general solution for an *anelastic* triaxial ellipsoid, we have derived the full 3D solution for any *elastic* solid of revolution (i.e. for a prolate or an oblate spheroid). The energy dissipated is calculated using a Q model (described below) as the loss mechanism.

The first step given in the outline (Section 3) does not pose serious difficulties because the acceleration field for a general triaxial ellipsoid undergoing free Eulerian motion is available (Landau & Lifshitz 1960). However, step 2 requires that the full 3D elasticity problem be solved for a given body. This is a formidable task due to: (a) the (likely) anelasticity of the material, (b) the absence of body forces derivable from a potential and (c) the likely complex shape of the body.

When deriving his eponymous numbers, Love (1946) assumed that the body forces are derivable from a potential and further that the body is linear-elastic spherical (or near-spherical) in shape. Neither of these assumptions is necessarily appropriate for our situation. First, we will demonstrate below the non-potential nature of body forces in our problem. Secondly, we often wish to consider aspherical objects and any restrictions on the shape of the body limits the usefulness of the result. These two failings have encouraged us to *not* employ Love numbers when estimating internal stresses.

An anelastic triaxial ellipsoid (with suitable choices for the anelasticity model and the axes of the body) is a very versatile model that can approximate many real objects fairly well. Furthermore, this is probably the most general model permitting an analytical solution. Unfortunately, this general model remains algebraically cumbersome and hence we have decided to simplify it. To start with, in order to ease the algebra while still preserving our ability to vary the shape of the body significantly, we restrict ourselves to ellipsoids of revolution. The construction of the elasticity solution for the general triaxial case is fairly straightforward but involves much more algebra.[2]

Next we take the material to be homogeneous, linear and elastic. Energy dissipation can be broadly classified into three categories: (a) frictional (rate-independent) dissipation; (b) viscous (rate-dependent) dissipation (as in fluids); and (c) a combination of rate-independent and dependent mechanisms, an example of which would be furnished by granular aggregates (which most asteroids are now thought to be). Energy dissipation in these mechanisms can be analysed rather simply by introducing a 'quality factor' Q (described below). In cases where we have rate-dependent dissipation mechanisms, Q varies with the loading frequency. Such a broad rate-based categorization of energy dissipation simplifies the problem considerably. Moreover, given the unknown internal structure of the body, there is little reason to favour a particular anelastic model (say Maxwell) over one with dissipation depending only on Q. Instead we ignore specific constitutive models and merely lump the dissipative effects into Q which we restrict to be constant. As we will see, this characterizes energy dissipation to be inherently frictional in nature.

3.1 The energy dissipation model

Owing to the inherent inhomogeneity of actual physical bodies and perhaps even the lack of a continuous structure (e.g. 'rubble-pile'

[1] Love numbers, which give the response of homogeneous elastic spheres to potential loads, are derived from the solution to an elasticity problem. See Munk & MacDonald (1960) or Lambeck (1980, 1987) for further discussion.

[2] Henceforth any reference to an ellipsoid should be understood to imply an ellipsoid of revolution unless stated otherwise.

models of asteroids), their internal energy dissipation is poorly understood. Introducing these processes into our model would make it intractable unless we resorted to numerical computation (where, even then, it might not prove solvable). Even in the latter case, we would need to know the nature of the dissipation; in various circumstances it might be due to grain-boundary sliding, creation and movement of dislocations, stress-induced flow of molten material and other effects (Lambeck 1980). Accounting for each of these mechanisms is obviously very complicated, indicating that a simpler model is preferable.

The Q model. Commonly, one accounts for energy dissipation in bodies under cyclic loads by assuming that the amount of energy lost per cycle (ΔE) is given by

$$\Delta E = \frac{2\pi \text{ (amount of 'fluctuating' strain energy)}}{Q} = \frac{2\pi \tilde{E}_\mathrm{s}}{Q}, \tag{1}$$

where Q is the (in)famous quality factor (Knopoff 1964; Lambeck 1980); it is also called the anelasticity. Although without much justification in terms of a simple, single physical mechanism, such an energy-loss model across many decades of frequency has been employed by a generation of geophysicists (e.g. Knopoff 1964; Lambeck 1980) to represent decay of seismic waves, for example. Typical values for Q in terrestrial materials at standard conditions range from hundreds to tens of thousands.

In order to use (1) we have to define a measure for the 'amount of 'fluctuating' energy'. The standard definition is that given by Lambeck (1980). He defines

$$\tilde{E}_\mathrm{s} = \int_V \langle e_\mathrm{s} \rangle \, \mathrm{d}V = \langle E_\mathrm{s} \rangle, \tag{2}$$

where V is the volume of the body, $\langle \cdot \rangle$ denotes the average over a cycle (of the load), e_s is the specific strain energy defined (in indical notation) as

$$e_\mathrm{s} = \frac{1}{2} \sigma_{ij} \epsilon_{ij}, \tag{3}$$

and E_s is the total strain energy ($= \int_V e_\mathrm{s} \, \mathrm{d}V$). Here, σ_{ij} is the stress on face i in the direction j, ϵ_{ij} is the corresponding strain and sums are taken on repeated indices.

The above measure of energy fluctuation is valid for the case when the stresses are purely sinusoidal which is not so for a tumbling body. In this case, the stresses fluctuate about a constant, non-zero value; because of this, the standard definition above cannot be used. Definitions of measures of energy fluctuations corresponding to the type of loadings encountered with tumbling bodies are not readily available. We discuss an alternative definition in Appendix A.

For the moment we use

$$\tilde{E}_\mathrm{s} = \langle E_\mathrm{s} \rangle - \int_V \langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle \, \mathrm{d}V. \tag{4}$$

The above is the same as computing the average of e_s *after* dropping all the time-independent terms in its expression and then integrating the resultant expression over the volume of the body. That is, if we decompose e_s into a constant and a time-varying part,

$$e_\mathrm{s} = e_\mathrm{s}^{\mathrm{constant}} + \tilde{e}_\mathrm{s},$$

then $\tilde{E}_\mathrm{s}$ defined above is the same as

$$\tilde{E}_\mathrm{s} = \int_V \langle \tilde{e}_\mathrm{s} \rangle \, \mathrm{d}V. \tag{5}$$

The amount of energy lost per cycle is computed by using the above equation in (1).

The measure of the amount of fluctuating energy described above is the same as that used by Efroimsky & Lazarian (2000). We use it here to ease comparisons with earlier work. This will also indicate that differences in our results are because we solve the accompanying elasticity problem exactly, and not because of a different definition of energy dissipation. However, the definition above is open to criticism as we discuss in Appendix A. Calculations based on the alternative (and perhaps more precise) formulation described in Appendix A are available from the first author.

3.2 Other assumptions

Next we list some assumptions that help to simplify our subsequent calculations.

Small deformations. We take the deformations induced in the body due to its nutational motion to be small compared with the flattening or elongation of the body. This requires that the body have adequate strength and/or the motion is not 'too fast' (so that the induced 'inertial forces' are small). This assumption implies that the moments of inertia of the body remain effectively unchanged throughout its motion. This in turn means that the 'feedback effect', i.e. any change in the dynamics of the body due to its deformations, is negligible. Thus, rigid-body equations (applied to the original undeformed body) describe the rotational motion of the body.

Small-strain elasticity. We consider that the small-strain theory of elasticity provides a reasonable estimate of the stress–strain field. The presence of internal gravity is an obvious problem especially in large enough bodies. To overcome this difficulty, following Love (1946), we think of the body as being already gravitationally pre-stressed. All subsequent stresses resulting from the motion of the body are considered to be superimposed on top of this pre-stress. The idea is that, if the deformations induced by the motion of the body are small enough, then one can apply the small-strain theory to the gravitationally pre-stressed body rather than taking an unstressed body and applying gravitational forces along with the motion-induced body forces. Obviously this assumption relies on the first assumption above in the sense that, if the shape changes are small compared with the flattening, then the gravitational field will not change much.

Material response. Finally, we assume that the body is isotropic. The primary reason for this is, of course, simplicity. However, on the other hand because so little is known about the internal properties of asteroids it may not be a good idea to consider more complicated models with many free parameters. The idea is to keep free parameters at a minimum so that the correct physics can be better captured.

The previous three assumptions reduce the problem to an elasticity problem for a linear, elastic and isotropic body. The stresses and the strains can now be related by Hooke's law.

Slow rotational dynamics. The frequency of loading is considered to be far slower than the lowest vibrational frequency of the body. This ensures that elastodynamic effects can be neglected in any

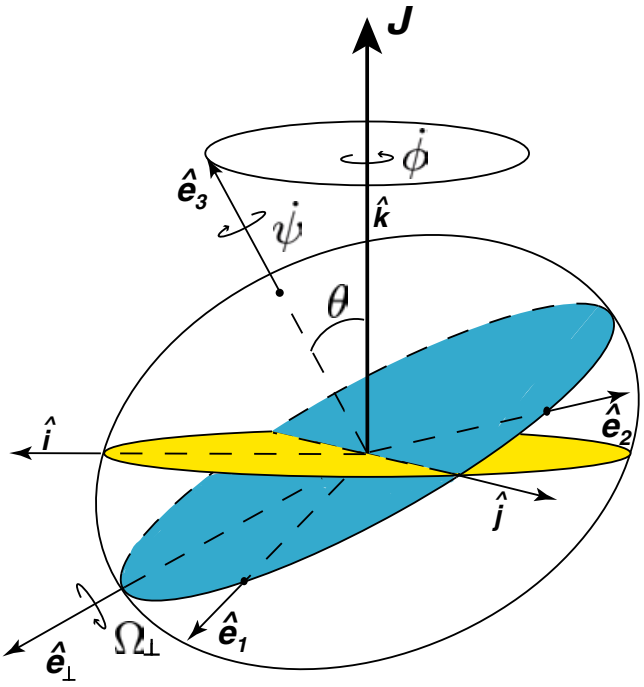


Figure 1. The rigid-body motion of an oblate ellipsoid. The unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ define an inertial coordinate system with the angular momentum $\boldsymbol{J}$ setting the $\hat{\boldsymbol{k}}$ direction. The unit vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ lie along the principal axes fixed in the body. The angle θ is the nutation angle, $\dot{\psi}$ is the spin and $\dot{\phi}$ is the inertial precession rate. The unit vector $\hat{\boldsymbol{e}}_\perp$ defines an intermediate axis along the intersection of the $\hat{\boldsymbol{e}}_1$–$\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{i}}$–$\hat{\boldsymbol{j}}$ planes and $\Omega_\perp$ is the associated component of angular velocity.

elasticity calculation. More concretely,

$$\frac{\sqrt{\mu/\rho}}{a}/\Omega_0 \gg 1.$$

This reduces our elasticity problem to a quasi-static one.

Most asteroid spin rates are about 10^{-4} rad s^{-1}. On the other hand, the rigidity of rock is of the order of 10^7 Pa and the density of asteroids is about 3 g cm^{-3} which leads to a speed of sound of 60 m s^{-1}. Thus, because of this large separation in time-scales, assuming quasi-statics is not a bad assumption to make.

Small energy loss. Energy loss over a loading cycle is small; in other words, the quality factor Q is large. This ensures that the rate of nutational damping as measured by $\dot{\theta}$ (Fig. 1) is much less than the loading frequency. With this assumption, two time-scales can be considered: the slow time-scale over which θ changes and the fast time-scale over which the body is loaded. Thus, θ is nearly constant over a load cycle, and the loading is (almost) periodic, justifying our use of energy-dissipation models that consider periodic loading.

4 ROTATIONAL DYNAMICS OF AN ELLIPSOID OF REVOLUTION

Before solving the elasticity problem, we first describe the rigid-body rotation of an isolated ellipsoid of revolution (see Fig. 1), i.e. the classical problem of free Eulerian motion (Landau & Lifshitz 1960; Greenwood 1988). $\boldsymbol{J}$ is the total angular momentum of the body, which is fixed in inertial space because no external torques act. An inertial coordinate system is partly aligned by specifying that the unit vector $\hat{\boldsymbol{k}}$ be parallel to $\boldsymbol{J}$. Our body-fixed coordinate system has unit vectors $\hat{\boldsymbol{e}}_i (i = 1, 2, 3)$, aligned with the principal axes of the body. The moments of inertia of the body are A, A and C about the axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$, respectively. With these assumptions, and as derived in Appendix B, the total angular velocity $\boldsymbol{\Omega}$ can be written as the sum of two terms

$$\Omega = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\Phi}}, \tag{6}$$

where the axial spin

$$\dot{\boldsymbol{\Psi}} = \dot{\psi}\hat{\boldsymbol{e}}_3 = \frac{(1-H)J\cos\theta}{C}\hat{\boldsymbol{e}}_3, \tag{7}$$

with $H = C/A$ and θ the nutation angle. The inertial precession rate

$$\dot{\boldsymbol{\Phi}} = \dot{\phi}\hat{\boldsymbol{k}} = \frac{HJ}{C}\hat{\boldsymbol{k}}. \tag{8}$$

Because the body is precessing uniformly,

$$\dot{\hat{\boldsymbol{e}}}_3 = \dot{\boldsymbol{\Phi}} \times \hat{\boldsymbol{e}}_3. \tag{9}$$

These equations merely say that an axially symmetric ellipsoid can be described to spin about $\hat{\boldsymbol{e}}_3$ at $\dot{\boldsymbol{\Psi}}$, carving out the body cone, while $\hat{\boldsymbol{e}}_3$ itself sweeps out a cone (the space cone) about $\hat{\boldsymbol{k}}$ at $\dot{\boldsymbol{\Phi}}$. Note that for a given nutational angle θ, $\dot{\boldsymbol{\Psi}}$ is constant while $\dot{\boldsymbol{\Phi}}$ is always constant regardless of θ.

Once the angular velocities are known, one can immediately write down the acceleration of any point in the body as

$$\boldsymbol{a}(\boldsymbol{r}) = \widehat{\dot{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}}\boldsymbol{r}} + 2\widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}}, \tag{10}$$

where $\widehat{\boldsymbol{abc}} \equiv \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ and $\boldsymbol{r} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3$ is the position vector written in the body-fixed system.

We write the kinetic energy E_k stored in the body as $E_\mathrm{k} = \frac{1}{2}(A\Omega_1^2 + A\Omega_2^2 + C\Omega_3^2)$. Lastly, defining $\Omega_0 = J/C$, we obtain

$$E_\mathrm{k} = \frac{C\Omega_0^2}{2}(\cos^2\theta + H\sin^2\theta). \tag{11}$$

5 INTERNAL STRESSES

When an ellipsoid executes its nutational motion, stresses are generated in its interior in conjunction with the acceleration field (10) derived in the previous section. Accordingly, we will solve an elasticity problem in which the body is loaded by the body force $-\rho\boldsymbol{a}$. Here ρ is the uniform density of the body. The body is taken to be homogeneous and elastic, allowing its material to be characterized by only two parameters, namely the Lamé constants λ and μ (Fung 1965). Furthermore, we assume that $\lambda \sim \mu$ for simplicity.

In order to estimate the strain–energy fluctuations, we must first solve for the stresses. For this we need to write down the body force $-\rho\boldsymbol{a}$ (associated with the presence of the acceleration field). Toward this end, we define

$$\omega_\perp = \dot{\phi}\sin\theta, \quad \omega_3 = \dot{\phi}\cos\theta, \tag{12}$$

permitting us, from (10), to write the body force at a point (x, y, z) in the body-fixed coordinate system as

$$\begin{aligned}\boldsymbol{F} = \rho\{&(a_1 z + (b_1 + b_4)x + c_1 y)\,\hat{\boldsymbol{e}}_1 \\ &+ (a_2 z + b_2 x + (c_2 + c_4)y)\,\hat{\boldsymbol{e}}_2 \\ &+ (a_4 z + b_3 x + c_3 y)\,\hat{\boldsymbol{e}}_3\}\end{aligned} \tag{13}$$

with

$$
\begin{aligned}
a_1 &= -\omega_3\omega_\perp \sin nt,\\
a_2 &= -\omega_3\omega_\perp \cos nt,\\
a_4 &= \omega_\perp^2,\\
b_1 &= \omega_\perp^2 \cos^2 nt,\\
b_2 &= c_1 = -\omega_\perp^2 \sin nt \cos nt,\\
c_2 &= \omega_\perp^2 \sin^2 nt,\\
b_3 &= -(\omega_3\omega_\perp + 2\omega_\perp\dot{\psi}) \sin nt,\\
c_3 &= -(\omega_3\omega_\perp + 2\omega_\perp\dot{\psi}) \cos nt,\\
b_4 &= c_4 = (\omega_3 + \dot{\psi})^2,
\end{aligned}
\tag{14}
$$

$$
n = |\dot{\psi}| = \frac{|(1-H)|\, J\cos\theta}{C}.
$$

Thus, the frequency of the load (or of the body forces) is simply the magnitude of the axial spin. Using the above constants in the expression for the force given by equation (13), we can see that the body force is not derivable from a potential. This is due to terms involving the angular acceleration of material points inside the ellipsoid.

Our second assumption (see Section 4.2) allows us to consider the elasticity problem to be *quasi-static*, which roughly means that at each instant during the nutational motion we can consider the body to be in static equilibrium. However, recall that during each moment the body force is different so that, in reality, the stress field is time-varying.

We are now in a position to solve for the stresses σ_{ij} and the accompanying strains ϵ_{ij} with $i, j = 1, 2, 3$. The procedure employed for the solution of the stress and the accompanying strains is outlined in Appendix B. For simplicity we report results obtained by letting $\lambda \sim \mu$. Because these expressions for the stresses are quite complicated, we list them in Appendix B. It can be verified that the calculated stresses do indeed satisfy the boundary condition that the surface of the ellipsoid is traction-free (i.e. force-free). From the stresses, the (specific) strain energy can be easily computed using (3). The total strain energy E_s, calculated by integrating e_s over the volume of the ellipsoid, can be arranged as

$$
E_\mathrm{s} = \frac{\rho^2 a^7 \Omega_0^4}{\mu} h(E_0 + E_2\cos 2\theta + E_4 \cos 4\theta), \tag{15}
$$

where E_0, E_2 and E_4 are complicated functions of the geometric parameter h.

We also estimate the fluctuating strain energy via (5):

$$
\tilde{E}_\mathrm{s} = \frac{\rho^2 a^7 \Omega_0^4}{\mu} h \sin^2\theta(\tilde{E}_0 + \tilde{E}_2 \cos 2\theta), \tag{16}
$$

where again $\tilde{E}_0$ and $\tilde{E}_2$ are functions of h alone.

A technical comment is in order with regard to the above formula. We expect the strain–energy fluctuations to be zero whenever the body rotates about a principal axis, i.e. when $\theta = 0$ or $\pi/2$. That is true. What is not necessarily true is that these fluctuations should *approach* zero as the body approaches a principal-axes rotation. In fact, from (16) we see that $\tilde{E}_\mathrm{s} \to 0$ *only* when $\theta \to 0$. The reason for it not vanishing (when $\theta \to \pi/2$) is not hard to see; this purely mathematical feature is probably best understood by considering the strain (potential) energy of a simple harmonic oscillator (SHO):

$$
E_\mathrm{SHO} = \frac{kx^2}{2} \sim \frac{k\sin^2\omega t}{2}.
$$

However,

$$
\langle E_\mathrm{SHO}\rangle = \frac{k}{4} = \text{constant},
$$

no matter what the frequency ω is!

Thus, in the case of the SHO, there will always be a finite (in fact the same) amount of fluctuating strain energy as the frequency approaches zero. This is exactly what happens in our case. On the other hand, $\tilde{E}_\mathrm{s} \to 0$ when $\theta \to 0$. This is understood by considering the specific strain energy e_s. This is a complicated function containing terms in $\sin nt$, $\cos nt$ and their powers. It is assumed to be constant (at every point in the body) when the body is in principal-axis rotation. When $\theta = \pi/2$, the loading frequency $n = 0$. Thus, e_s is automatically a constant. However, at $\theta = 0, n \neq 0$ but e_s is still a constant. This can only happen if the coefficients of the trigonometric terms (which are functions of the spin [3] $\dot{\psi}$) vanish. In other words, $\tilde{e}_\mathrm{s} = 0$ when $\theta = 0$. Thus, in this case, $\langle e_\mathrm{s}\rangle \to e_\mathrm{s}$, as $\theta \to 0$ so that (5) yields $\tilde{E}_\mathrm{s} \to 0$.

6 EXPRESSION FOR THE NUTATIONAL DAMPING TIME

Finally, we bring together the results of the previous sections in order to calculate a characteristic nutational damping time. For this, we use our solution of the elasticity problem involving time-varying stresses along with the Q model to estimate ΔE (energy dissipated in one cycle of the load).

Now, if there were no energy loss in this system, then the total energy $E = E_\mathrm{s} + E_\mathrm{k}$ would be constant. Thus, when energy is lost in the system, we can write

$$
\dot{E} = -\frac{\Delta E}{(2\pi/n)} = \dot{E}_\mathrm{s} + \dot{E}_\mathrm{k} = \dot{\theta}\left(\frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}\theta} + \frac{\mathrm{d}E_\mathrm{s}}{\mathrm{d}\theta}\right). \tag{17}
$$

From (11) we obtain

$$
\frac{\mathrm{d}E_\mathrm{k}}{\mathrm{d}\theta} = C\Omega_0^2(H-1)\sin\theta\cos\theta. \tag{18}
$$

Combining the derivative of (15) with this, we can express

$$
\left|\frac{\dot{E}_\mathrm{k}}{\dot{E}_\mathrm{s}}\right| = \left|\frac{\mathrm{d}E_\mathrm{k}/\mathrm{d}\theta}{\mathrm{d}E_\mathrm{s}/\mathrm{d}\theta}\right| = \frac{2\pi}{15}\left(\frac{\sqrt{\mu/\rho}}{a\Omega_0}\right)^2 \frac{(1-h^2)/(1+h^2)}{E_2 + 4E_4\cos 2\theta}.
$$

By the second assumption of Section 3.2, the term $(\sqrt{\mu/\rho})/a\Omega_0$ is large so that $\dot{E}_\mathrm{k}/\dot{E}_\mathrm{s} \gg 1$. All the remaining terms are of $o(1)$ except possibly when $h \sim 1$; thus $\dot{E}_\mathrm{s} \ll \dot{E}_\mathrm{k}$ and can generally be neglected for simplicity. For the moment, we will avoid the regime $h \sim 1$.

Then, rewriting (17) gives

$$
\dot{\theta} \approx \frac{\dot{E}}{\mathrm{d}E_\mathrm{k}/\mathrm{d}\theta} = -\frac{n}{2\pi}\frac{\Delta E}{\mathrm{d}E_\mathrm{k}/\mathrm{d}\theta}, \tag{19}
$$

which then describes the time evolution of the nutation angle θ. The damping time can be obtained by integrating (19):

$$
T_\mathrm{d} = \int_0^{t_\mathrm{d}} \mathrm{d}t = -\int_{\theta_\mathrm{i}}^{\theta_\mathrm{f}} \frac{2\pi(\mathrm{d}E_\mathrm{k}/\mathrm{d}\theta)}{n\Delta E}\mathrm{d}\theta, \tag{20}
$$

with $\theta_\mathrm{f} = \delta_\mathrm{c}$ for oblate bodies and $\theta_\mathrm{f} = \pi/2 - \delta_\mathrm{c}$ for prolate ones. In the above equation, ΔE is obtained from (1) using (16), while $\mathrm{d}E_\mathrm{k}/\mathrm{d}\theta$ is given by (18). The parameter δ_c is a small angle that is chosen as the smallest angular resolution of present-day

[3] Recall that $n = |\dot{\psi}|$.

instrumentation. Following Efroimsky & Lazarian (2000) we choose $\delta_c = 6^\circ$. There are two reasons for using δ_c. First, because the damping time oblate bodies is seen to diverge as $\theta_f \to 0$. Secondly, though the time for prolate bodies to damp to $\theta_f = \pi/2$ does not diverge, it does become quite large, and thus, it seems appropriate to compare the damping times of oblate and prolate bodies when they are assumed to damp to within an angle of δ_c of their final states.

We now proceed to calculate the damping times using the Q model.

6.1 Damping with the Q model

We first compute ΔE by using (1) and (16) to obtain

$$\Delta E = 2\pi \frac{\rho^2 a^7 \Omega_0^4}{\mu Q} h \sin^2\theta (\tilde{E}_0 + \tilde{E}_2 \cos 2\theta). \tag{21}$$

Now, we use this expression for ΔE, (18) and the last equation of (14) for n in (20) which, after substituting for C, H and J in terms of ρ, a and h, is

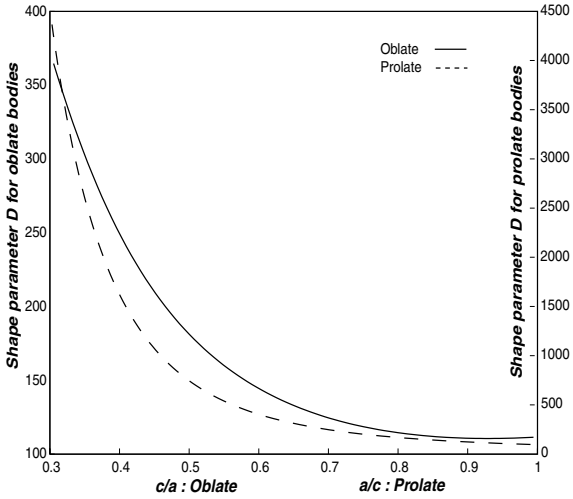
$$T_d = \text{sign}(1-h)\left(\frac{\mu Q}{\rho a^2 \Omega_0^3}\right)\frac{8\pi}{15}\int_{\theta_i}^{\theta_f} \frac{1}{\sin\theta(\tilde{E}_0 + \tilde{E}_2\cos 2\theta)}\,\mathrm{d}\theta$$
$$= D(h)\left(\frac{\mu Q}{\rho a^2 \Omega_0^3}\right), \tag{22}$$

where $D(h)$ is the shape parameter and θ_f is as defined before. We have suppressed the dependence of D on θ_i. Note that, for fixed initial angles, D is a purely geometric parameter, depending on the aspect ratio of the body alone, that is computed numerically. In Figs 2(a) and (b) we plot D as a function of $h = c/a$ (and $1/h = a/c$). Recall that $h < 1$ signifies oblate spheroids that damp to $\theta = 0$ while $h > 1$ corresponds to prolate ones that damp to $\theta = \pi/2$. Note that we have plotted both the prolate and the oblate cases on the same plot by using $1/h$ for prolate ellipsoids.

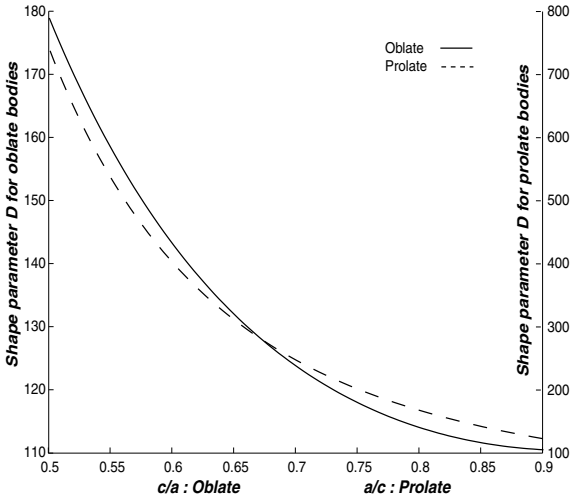
We see in (22) that the damping time depends on the initial nutational angle as well as on the parameter h which characterizes the shape of the body. These dependences are absent from most previous works except Efroimsky & Lazarian (2000); the shape dependence is also hinted at in Prendergast (1958), Burns & Safronov (1973) and McAdoo & Burns (1974). However, because the initial angle, once the final angle is chosen, does not affect the result by orders of magnitude, we only plot the results for $\theta_i = \pi/4$. The reason for a perceived discontinuity at $h = 1$ is the fact that while oblate bodies (with $h < 1$) damp to $\theta_f = 0$, prolate ones relax to $\theta_f = \pi/2$. Note that a spherical object ($h = 1$) does not have a damping time as every axis is a principal axis of inertia.

6.2 Discussion

We have to exercise some caution before interpreting the results of the previous section. Figs 2(a) and (b) plot the shape parameter versus the aspect ratio. However, when using the shape parameters to predict damping times we have to be careful that we do not compare objects with different physical properties and, as a consequence, vastly different dynamical properties. For example, if a is of unit length an object of oblateness $h = 0.5$ and an object of prolateness $h = 2$ will have a volume ratio of 1/4. Furthermore, if we assume that the density is the same for both, then the masses and the maximum inertia scale by 1/4. Consequentially, if Ω_0 is also taken to be the same, the prolate object has four times the angular momentum and kinetic energy of the oblate one. Thus, these objects have very different initial conditions and it would be inappropriate



(a) Shape parameter D for $0.3 \leqslant h < 1$



(b) Shape parameter D for $0.5 \leqslant h \leqslant 0.9$

Figure 2. The function $D(h)$ is plotted versus h and $1/h$. The two figures show the shape parameter $D(h)$ for oblate bodies that damp to $\theta_f = \delta_c$ and prolate bodies that damp to $\theta_f = \pi/2 - \delta_c$.

to compare them directly. Another way of saying this would be that using the shape parameter defined through (22) as a measure of the damping time-scale would be appropriate only if the objects of interest have the same a, J and ρ. However, frequently damping time comparisons are made between objects with similar densities, volumes and angular momenta but different shapes. Because, as pointed out above, the volumes and angular momenta of these will be quite different for different shapes, we need to rephrase (22) in terms of the magnitude of the angular momentum J, the volume V and the density ρ. For this we use the relations

$$V = \frac{4\pi}{3} h a^3, \quad J = C\Omega_0 = \frac{8\pi}{15}\rho h a^5 \Omega_0.$$

Inverting, we obtain

$$\rho a^2 \Omega_0^3 = \frac{125}{8}\left(\frac{4\pi}{3}\right)^{4/3}\left(\frac{J^3}{V^{13/3}\rho^2}\right) h^{4/3}.$$

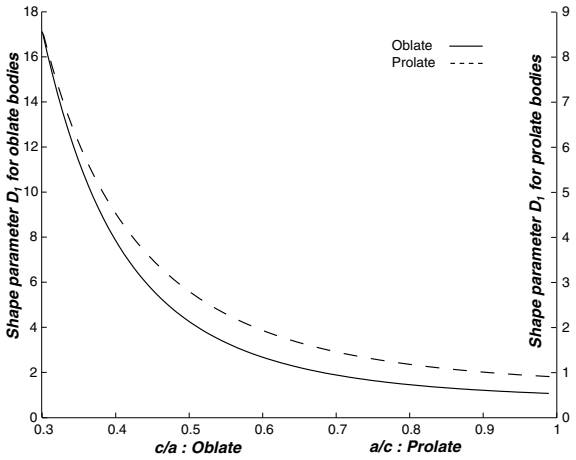


Figure 3. The function $D_1(h)$ plotted against h and $1/h$. The shape parameter D_1 is shown for oblate bodies that damp to $\theta_f = \delta_c$ and for prolate bodies that damp to $\theta_f = \pi/2 - \delta_c$.

Using the above in (22), we obtain

$$T_d = \frac{8}{125}\left(\frac{3}{4\pi}\right)^{4/3} D(h)h^{-4/3}\left(\frac{\mu Q V^{13/3}\rho^2}{J^3}\right) \equiv D_1(h)\left(\frac{\mu Q V^{13/3}\rho^2}{J^3}\right). \quad (23)$$

In this form, the 'new' shape parameter D_1 is a measure of the damping time of objects with similar densities, volumes and angular momenta. The behaviour of D_1 is plotted in Fig. 3.

From Figs 2(a), (b) and 3 we see qualitatively the following.

(a) Damping times are predicted to be infinite whenever h reaches infinity or becomes zero. Thus, we expect long damping episodes for flat plate-like or long cylinder-like objects.

(b) Note that the shape parameter using (22) and that from (23) display qualitatively different results. While in the latter, prolate ellipsoids always damp faster than oblate ones, this is not the case when (22) is used. This reinforces the cautionary note above about comparison between two objects. We should know whether we are comparing objects with similar semimajor axes a and representative angular velocity Ω_0, in which case we will employ (22) and use Fig. 2, or we are comparing objects with similar angular momentum J and volume V, whence we should employ (23) and Fig. 3.

(c) The model predicts (relatively) rapid damping when $h \sim 1$. This might seem counter-intuitive because as $h \to 1$ the frequency n of the body forces approaches zero; hence any 'viscous-like' dissipation would decay to zero. On the other hand, we recall that the Q model does not tax the loading frequency and, in this sense, behaves much like a frictional dissipation model.

(d) For $h > 1$, the Q model displays a finite time decay to a principal rotation state. The reason for this again lies in the frictional nature of the Q model. This is not surprising because a spring–mass system with a frictional damper stops in finite time as opposed to one with a viscous damper which takes infinitely long. However, as we mention above, finite-time damping is not observed in the case of oblate bodies, where the damping time diverges logarithmically. We believe the reason for this lies in the way we estimate the 'amount of 'fluctuating' energy'. As we state in Appendix A, this method does not measure all the fluctuations. Indeed, when we use the alternative estimation method outlined in Appendix A, we observe finite-time damping, reinforcing the frictional nature of the Q model.

In closing this section, we note that there is nothing sacrosanct in using ρ, V and J as 'comparison' parameters and one can employ other parameters such as the kinetic energy T, the maximum inertia C, the density ρ, etc., depending upon the type of comparison one wishes to make.

7 COMMENTS ON NEARLY SPHERICAL BODIES

An indicator that things could be complicated when $h \sim 1$ is the fact (noted in Section 6) that the rate of change of strain energy could be comparable to the rate of change of kinetic energy as the nutational angle varies (see equation 17).

Therefore, let us consider cases when $h \sim 1$ but not exactly 1. A fundamental assumption throughout this calculation is the first of Section 3.2, namely, that one does not need to include the altered shape in order to predict the dynamics even though the shape itself is influenced by dynamical effects. Thus we have a one-way decoupling of dynamics and elasticity. However, this assumption fails whenever $h \sim 1$, because then the body is nearly spherical, in which case small shape changes may profoundly alter the dynamics. For example, small deformations may change the ordering among the axes of moments of inertia which in turn will affect the stability of rotation about them. This can be further understood by recognizing that a sphere is singular in so far as its free Eulerian motion goes, which is perhaps best illustrated by the property that, for a given angular momentum, just one nutational angle ($\theta = 0$) is possible, whereas for a general body all nutational angles between 0 and $\pi/2$ are allowed. That is to say, 'nutational damping of a sphere' is an oxymoron because a rotating sphere – whether rigid, elastic or viscoelastic – never nutates; it always rotates uniformly about the specified angular momentum vector.

With nearly spherical bodies at small wobble angles, the effect of deformations can be more easily incorporated into the dynamics. This is accomplished using perturbation techniques and Liouville's equation as outlined by Munk and MacDonald (1960) or Lambeck (1980). For slightly flattened objects, the deformation forces (i.e. the body forces resulting from the acceleration field) are derivable from a potential function at least up to the first order in the flattening. [4] Then as previous authors have argued, one can drastically simplify the elasticity analysis by directly invoking Love numbers to estimate internal deformations and stresses (neglecting the asphericity of the body). If the flattening is not small, this simplification is not possible. Then one has to solve to the complete elasticity problem and this immensely complicates the application of the perturbation technique. However, if the shape of the body is not nearly spherical, the dynamical effects of slight shape changes will be small and thus we can ignore the 'feedback' effect, at least up to the first order.

8 ESTIMATES OF NUTATIONAL DAMPING TIMES

All researchers to date *agree* on the functional form of the characteristic damping time. For example, Prendergast (1958), Burns & Safronov (1973), Peale (1973), Purcell (1979), Yoder & Ward (1979), Efroimsky & Lazarian (2000) and Efroimsky (2000, 2001) report the characteristic damping time T_d has the form:

$$T_d = D\left(\frac{\mu Q}{\rho a^2 \Omega_0^3}\right). \quad (24)$$

[4] Flattening $f \sim \delta r/r$ characterizes the amount of oblateness.

The results of the various authors differ solely through the constant D, the values of which will be listed below. The form above indicates that comparisons should be made with the damping time give by (22), so that we have to use the shape parameter D plotted in Fig. 2, rather than D_1 given in Fig. 3. Such a functional dependence could have been anticipated, of course, directly from dimensional analysis.

Let us now broadly categorize the aforementioned works. Prendergast (1958) considers general symmetric ellipsoids and calculates the strain energy using only the 'centrifugal' part of the body force. Though he does not indicate how he calculates the stresses, we surmise that the calculations follow Chree (1888). It should be pointed out though that the body force will be nearly 'centrifugal' in nature only when the nutational angle θ and/or the flattening is small. Peale (1973), as well as Yoder & Ward (1979), investigated nearly spherical objects at small nutational angles (θ) in their calculations. In contrast, as just discussed in Section 8, our calculation and a few others are *not* valid for nearly spherical objects. Both of these 1970s papers call upon the Love numbers to estimate the strain energy of the body (as is entirely appropriate for nearly spherical objects) and they also carefully include 'feedback effects'. However, as described at the beginning of Section 4, Love numbers apply merely to spherical objects under loads that are derivable from a potential, and neither of these conditions holds for large flattening (when h is far from 1). In such circumstances, the complete elasticity problem must be solved. Furthermore, the perturbation technique that they employ to incorporate 'feedback effects' works solely for small θ. Burns and Safronov (1973) estimated the strain energy that is periodically stored in the polar bulge (employing a Love number to determine the rotational flattening) and in bending stresses. Because their result is given as an order-of-magnitude estimate, any difference (e.g. by a factor of several) that it has from another author's expression should not be a cause for concern.

Moreover, only Prendergast (1958), McAdoo & Burns (1974), Efroimsky & Lazarian (2000) and the present paper state explicitly how D varies with the shape parameter. Burns & Safronov (1973) did note that D would differ with h.

We note again that the expressions for damping time derived above and by Efroimsky & Lazarian (2000) depend on the final angle to which the body damps. Their damping times grew exponentially as the body damped into a principal rotation state. Thus, they had to define a *characteristic* damping time, which was the time taken by the body to damp to within an angle δ_c of a principal rotation state. This is seen to occur in our calculation too in the case of oblate bodies that damp to $\theta_f = 0^\circ$, except that the growth in the damping time is logarithmic. Thus, we too report the times for damping to an angle of δ_c for oblate bodies, and to an angle $\pi/2 - \delta_c$ for prolate bodies. We take $\delta_c = 6^\circ$. It should be pointed out that the damping time is sensitive to the final angle, i.e. if we choose δ_c to be smaller, say less than a degree, the damping times can increase by an order of magnitude. Earlier estimates were merely e-folding times and did not consider the final angle explicitly (and so can be thought of as a measure of a 'characteristic' damping time).

In some sense, this logarithmic growth is unexpected because, as we noted above, the Q model is frictional in nature and does not tax the frequency of external loading. Thus, we should expect damping in finite time. This is indeed seen to be true if we use the alternate method of Appendix A. In this case, the sensitivity of the damping time on the final angle is reduced significantly, which is another reason, we believe, that the approach in Appendix A is preferable.

The numerical coefficients previously obtained are, in chronological order:

(i) Prendergast (1958):

$D \sim 10$ for $h \sim 0.5$
$D \sim 30$ for $h \sim 1$
$D \sim 5$ for $h \sim 2.5$

(ii) Burns & Safronov (1973):

$D \sim 100$ for nearly spherical objects
but $D \sim 20$ for elongated objects;

(iii) Peale (1973):

$D \sim 7$ for nearly spherical objects;

(iv) McAdoo & Burns (1974):

$D \sim 50$ for viscoelastic spheres;
$D \sim 4h^{4/3}$ for cylinders;

(v) Yoder & Ward (1979):

$D \sim 5$ for nearly spherical objects;

(vi) Efroimsky & Lazarian (2000):

$D \sim 4$ for non-spherical objects that damp to within 6° of a principal rotation state.

(vii) Molina et al. (2003):

$D \sim 25$ for symmetric ellipsoids with aspect ratios int he range 0.5–0.9.

We emphasize that the closeness in the D values for some of these solutions is somewhat misleading. The results of Peale and of Yoder and Ward pertain to nearly spherical bodies, whereas those of Efroimsky and Lazarian, and Molina et al. apply to non-spherical objects and are, in fact, reported to break down when the shape of the body approaches a sphere (i.e. when $h \to 1$).

(viii) This paper: the functional dependence of D on h, for damping to within 6° of the final state, is plotted in Fig. 2. It shows that (see Fig. 2 for other h values)

$D \sim 200$ for $h \sim 0.5$
$D \sim 100$ for $h \sim 1$
$D \sim 200$ for $h \sim 2$.

Despite the functional similarity of these results, recently Efroimsky & Lazarian (2000) and Efroimsky (2000, 2001) have sharply criticized all earlier works as missing important dynamics, failing to perfectly satisfy boundary conditions and applying unphysical choices for the parameters in (24). Later, Molina et al. (2003) claim to put forward a solution better than that due Efroimsky & Lazarian (2000). The procedures in these papers are very similar to those that we have presented here. Furthermore, these calculations apply to a cuboid (of dimensions $a \times a \times c$), whereas we study symmetric ellipsoids. Efroimsky & Lazarian (2000) claim to have solved the full elasticity problem for the cuboid in order to estimate the stresses. However, their solution satisfies only some, not all, of the boundary conditions; in particular, shear stresses do not vanish on the faces of the cuboid, as they should. This is also pointed out by Molina et al. (2003), who themselves claim to have a better 'approximation' to the elasticity solution. However, we do not find any justification as to why their solution is a better approximation. The expression for the damping time in these papers, like ours, is reported to be valid when the flattening (or elongation) is large. Thus, we make the following detailed comparison between our solution and theirs.

(a) The stresses that Efroimsky & Lazarian (2000) obtain are stated to reach a maximum in the interior of the body. In contradistinction, our *exact* solution does not share this property. One might

hope to attribute this discrepancy to the difference in the object shapes that were considered. However, with suitable choices for the axes, we can crudely approximate a cuboid; thus we maintain that such a drastic change in the stress behaviour with depth is improbable.

(b) As both Efroimsky & Lazarian (2000) and Molina et al. (2003) note, their formula suffers from the problem of a 'fast start', i.e. if an inertia axis is parallel to $\boldsymbol{J}$, nutational damping continues to be predicted to take place when in fact it should be absent. This is equivalent to saying that in equation (19), $\dot{\theta}$ is non-zero when $\theta \rightarrow 0$ or $\pi/2$. This 'problem' persists in our formulae. However, the truth is that this is not really a problem: $\dot{\theta}$ does in fact turn out to be zero when $\theta = 0$ or $\pi/2$ *provided* we use these values of θ right from the beginning, i.e. when we estimate the amount of fluctuating energy $\tilde{E}$. Analogous to what we saw in the technical comment at the end of Section 6, there is no reason to expect that the value of the limit as $\theta \rightarrow 0$ or $\pi/2$ will be equal to the value at these points due to the averaging integrals involved. This in fact is a classic case of confusing the limit of a function with its value at a point. We note that this has nothing to do with the validity of the averaging approximation in equation (19) near the poles, i.e. when $\theta_{\rm f} = 0^\circ$ or 90°.

(c) Both Efroimsky & Lazarian (2000) and Molina et al. (2003) report that damping times are insensitive to the aspect ratio h. This is somewhat true in the range 0.5–2 of the aspect ratio. However, away from this regime, our results (see Fig. 2a) indicate a strong dependence on h.

(d) Comparing our values of D with those of Efroimsky & Lazarian (2000) and Molina et al. (2003), we see that the extent of the energy dissipation does *not* seem to have been previously *under*estimated. If we consider damping to within a degree of the final state, then calculations show that, in fact, if anything, it appears to have been *over*estimated by as much as a factor of 100.

Calculations show that D can be much larger than this if one uses an alternate rate-dependent (viscous) energy dissipation model. This slower energy loss in the latter case happens because a viscous model only penalizes the *rate* of change of the strain energy. On the other hand, calculations also show that if we use the alternate estimate of the energy fluctuations outlined in Appendix A, the shape parameter is reduced. This occurs because the alternative formulation penalizes energy fluctuations that are neglected in the present formulation.

(e) Efroimsky & Lazarian (2000) and Efroimsky (2000, 2001), maintain that their solution differs fundamentally from all earlier ones, except for Peale's (1973), because they had recognized a double-frequency term, and this accounted for order-of-magnitude discrepancies in damping times. (For the historical record, we note that, much earlier, Prendergast 1958, uncovered this term.) Later, Molina et al. (2003) also spend considerable space on this feature. This claim of substantial energy loss associated with the double-frequency term seems implausible for three reasons. First, we have used the same acceleration field as they, but our solution is distinct from theirs. Secondly, the functional dependence on n is not very strong. Lastly, when using the Q model (with a constant Q), in any Fourier decomposition of the strain energy each Fourier term dissipates exactly the same amount of energy. In other words, the fractional energy dissipation associated with each loading frequency is exactly the same. This is easily seen by noting that the (absolute) area underneath $\sin kt$ over a period of 2π is the same for *each* k so that the amount of dissipation predicted by the Q model (being directly proportional to this area) is the same. Thus, one does not expect Fourier terms generated by a missed double-frequency term in the acceleration to alter the amount of dissipation (and so the dissipation time) by orders of magnitude.

More likely, discrepancies between ourselves and Efroimsky & Lazarian (2000) and Molina et al. (2003) result from the fact that they both do *not* solve the elasticity problem for a homogeneous, isotropic, linear elastic body *exactly* or even *approximately*. In the case of the former, their solution does not satisfy the force-free boundary conditions nor the compatibility equations. On the other hand, Molina et al. (2003), while satisfying the force-free boundary conditions, bypass the compatibility conditions by placing the (ad hoc) restriction on the shear stresses to be zero. Though, while this manoeuver may guarantee uniqueness of the stress solution, we still do not know the rheology for which these solutions have been derived. The rheology may be something completely unphysical. In both these papers, the authors have chosen not to satisfy the compatibility equations. As noted in the remark in Appendix C, not satisfying the compatibility equations is a rather serious drawback. For an example that illustrates this point, see Appendix C.

9 DAMPING TIMES

The actual times over which the nutation disappears can be computed from (24) once values for the parameters μ, Q, etc., are substituted in that expression. However, when estimating $T_{\rm d}$ for asteroids and comets, neither μ nor Q is known, nor can these parameters be measured. Even for the proximate Earth, their values are not found from experiment but rather were originally inferred by the gross properties of our planet (e.g. the elastic modulus of the Earth is derived from seismic wave speeds, whereas its Q value comes from the decay of seismic wave amplitudes, see Knopoff 1964 or Lambeck 1980, 1987).

Our understanding of the interior make-up of small, distant solar-system bodies (asteroids and comets) has improved dramatically over the last decade (see, e.g., Asphaug, Ryan & Zuber 2002; Binzel et al. 2003). Many researchers now accept that such bodies are 'rubble piles', innumerable particles (whether collisional debris or primordial building blocks) held together principally by self-gravity augmented perhaps by feeble intermolecular forces. Puzzles remain as to how asteroids can look so solid (support large craters and irregular shapes, rotate as units) but apparently possess so little strength as indicated by their rotational characteristics, high porosity and apparent susceptibility to fragmentation under weak tidal forces (Binzel et al. 2003).

Our estimate of $\mu\, Q$ follows Harris (1994), who selects $\mu\, Q = 5 \times 10^{12}$ and $\rho = 2.5$, each in cgs units. These choices differ from those of Burns & Safronov (1973), who had assumed a much stronger and harder material, as well as those of Efroimsky & Lazarian (2000), who took a weaker material. With these choices and selecting D as 200 (roughly corresponding to oblate or prolate bodies with a/c either 2 or $1/2$), we find that the nutational damping time, given in millions of years, is

$$T_{\rm d} = 0.24 \frac{P^3}{a^2}, \tag{25}$$

where the spin period P is expressed in hours and the radius of the asteroid is stated in kilometres. Almost all of the more than a thousand asteroids with published light-curves are considerably larger than a kilometre in size and have spins that are on average 10 h long (summary plots are given in fig. 1 in Pravec et al. 2003 or fig. 2 in Paolicchi, Burns & Weidenschilling 2002). Thus almost all asteroids have nutational damping times that are considerably shorter than the age of the solar system. Only about a dozen asteroids (or one in a hundred) are known to be wobbling (see the tabulation in Paolicchi et al. 2002 and the plots in Harris 1994;

Paolicchi et al. 2002). Most of the tumblers known today are small and/or slowly rotating. All but one of the wobblers have nutational damping times that are longer than the age of the solar system as computed according (25) above. An approximately equal number of asteroids that are not apparently wobbling fall into the region where damping times exceed 4.5 billion years. The tumbling of some fraction of these may, of course, have simply gone undetected; others may, for one reason or another, suffer more severe internal energy dissipation.

About two-thirds of the tumblers are Earth-crossing objects and another is a Mars-crosser. However, it is uncertain whether this result occurs because most tiny objects with measured light-curves are predominantly Earth-crossers or because planet-crossing events themselves may engender the wobble (Black et al. 1999). Among the very fastest-spinning asteroids, only 2000 WL107, a 20-m radius near-Earth asteroid turning in 20 min, is tumbling. From (25), its damping time is 20 Myr, a time similar to the estimated collisional age of 10 Myr of such a minor planet in the main belt. Thus, its nutation is unremarkable.

Only a few cometary spin rates are known and, in the mean, they are longer than those of asteroids. The inferred sizes of cometary nuclei are generally much smaller than those of typical asteroids. Hence, cometary wobble damping times, according to (25), are considerably longer than corresponding values for asteroids. It is not surprising, then, that – despite a small sample – several comets, most notably Halley, exhibit complex light-curves.

Because the sizes and spin rates of the few bodies that are known to be wobbling appear to match well the predictions of our theory, we take this as prima facie evidence that the above treatment may indeed be correct. It also suggests that collision rates in the asteroid belt are, and have been for some time, quite modest.

10 CONCLUSION

In this work we have presented a reasonably complete calculation of the nutational damping times for prolate and oblate spheroids in free space. Any reader who wishes further details on the calculation should contact the lead author. Our approach has relied on several assumptions, some of which (e.g. small distortions and the neglect of their effect on the subsequent dynamics) may leave our methodology open to criticism. The next step toward creating a better model must necessarily involve tracking the change in the shape of the body and incorporating its influence on the dynamics of the body. At the same time one could consider moving away from the regime of small strains to allow for large deformations of the body, which in turn would require accounting for possible changes in the internal gravitational field. Finally, one should introduce more realistic constitutive laws (i.e. better descriptions of how stresses link to strains).

The completeness of our computation is perhaps its principal merit. Unfortunately, this came at a considerable cost: many phenomena were overlooked. Yet we are confident that it would be complicated to include most of the 'improvements' mentioned above. Thus it may be time to attack this problem from an entirely novel analytical philosophy (namely to sacrifice some rigour so as to permit a more complex model) or from a numerical viewpoint (cf. Richardson et al. 2002). The benefits may be substantial now as landings on comets (Rosetta) and asteroids (Muses C) are being planned, and as we attempt to characterize potentially hazardous asteroids in order to develop strategies that will mitigate their possible threat to our planet.

ACKNOWLEDGMENTS

We thank A. Ruina, S. J. Peale, J. T. Jenkins and H. D. Conway for helpful discussions. We also thank A. Dobrovolskis, M. Efroimsky and an anonymous reviewer for helpful comments. This research was supported by NASA's Planetary Geology and Geophysics programme.

REFERENCES

Asphaug E., Ryan E. V., Zuber M. T., 2002, in Bottke W. F., Jr, Cellino A., Paolicchi P., Binzel R. P., eds, Asteroids III. Univ. Arizona Press, Tucson, p. 463
Binzel R. P. et al., 2003, Planet. Space Sci. 51, 443
Black G. J., Nicholson P. D., Bottke W. F., Jr, Burns J. A., 1999, Icarus, 140, 239
Burns J. A., 1977, in Burns J. A., ed., Planetary Satellites. Univ. Arizona Press, Tucson, p. 113
Burns J. A., Safronov V. S., 1973, MNRAS, 165, 403
Chree C., 1886, Quart. J. Math. 22, 89
Chree C., 1888, Quart. J. Math. 23, 11
Denisov G. G., Novikov V. V., 1987, Mechanics of Solids (English translation of Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela) 22, 69
Efroimsky M., 2000, J. Math. Phys. 41, 1854
Efroimsky M., 2001, Planet. Space Sci. 49, 937
Efroimsky M., Lazarian A., 2000, MNRAS, 311, 269
Fung Y. C., 1965, Foundations of Solid Mechanics. Prentice-Hall, Englewood Cliffs
Greenwood D. T., 1988, Principles of Dynamics. 2nd edn, Prentice-Hall, Englewood Cliffs
Harris A. W., 1994, Icarus, 107, 209
Knopoff L., 1964, Rev. Geophys., 2, 625
Lambeck K., 1980, The Earth's Variable Rotation: Geophysical Causes and Consequences. Cambridge Univ. Press, Cambridge
Lambeck K., 1987, Geophysical Geodesy: the Slow Deformation of the Earth. Cambridge Univ. Press, Cambridge
Landau L. D., Lifshitz E. M., 1960 Principles of Dynamics, 2nd edn. Pergamon, Oxford
Lazarian A., Efroimsky M., 1999, MNRAS, 303, 673
Love A. E. H., 1946, A Treatise on the Mathematical Theory of Elasticity, 4th edn. Dover, New York
McAdoo D. C., Burns J. A., 1974, Icarus, 21, 86
Molina A., Moreno F., Martinez-Lopez F., 2003, A&A, 398, 809
Munk W. H., MacDonald G. J. F., 1960, The Rotation of the Earth. Cambridge Univ. Press, Cambridge
Paolicchi P., Burns J. A., Weidenschilling S. J., 2002, in Bottke W. F., Jr, Cellino A., Paolicchi P., Binzel R. P., eds, Asteroids III. Univ. Arizona Press, Tucson, p. 517
Peale S. J., 1973, Rev. Geophys. Space Phys., 11, 767
Pravec P., Harris A. W., Michalowski T., 2002, in Bottke W. F., Jr, Cellino A., Paolicchi P., Binzel R. P., eds, Asteroids III. Univ. Arizona Press, Tucson, p. 113
Prendergast K. H., 1958, AJ, 63, 412
Purcell E. M., 1979, ApJ, 231, 404
Richardson D. C., Leinhardt Z. M., Melosh H. J., Bottke W. F., Asphaug E., 2002, in Bottke W. F., Jr, Cellino A., Paolicchi P., Binzel R. P., eds, Asteroids III. Univ. Arizona Press, Tucson, p. 501
Timoshenko S., Young D. H., 1955, Vibration Problems in Engineering, 3rd edn. Van Nostrand, New York
Timoshenko S., Goodier J. N., 1970, Theory of Elasticity, 3rd edn. McGraw-Hill, New York
Washabaugh P. D., Scheeres D. J., 2002, Icarus, 159, 314
Yoder C. F., Ward W. R., 1979, ApJ, 233, L33

APPENDIX A: REMARKS ON THE *Q* MODEL

As mentioned in Section 4.1, the definition of a measure of the 'amount of fluctuating energy' is not entirely obvious due to the

type of body forces acting on a wobbling body. Consider a simple one-dimensional example (e.g. a beam in tension). We consider a case where the stress in this object is $\sigma = \sqrt{2}(a + b \sin nt)$, and thus is not purely sinusoidal but has a constant offset as well. This is similar to the stress field in a nutating body. Proceeding to calculate the (specific) strain energy, we obtain for a linear elastic material,

$$e = \frac{1}{2}\sigma\epsilon = a^2 + b^2 \sin^2 nt + 2ab \sin nt.$$

In the above we assumed that the elastic constants are normalized to unity so that the strain $\epsilon = \sigma$. Now blindly using the standard definition (2) will pick up the offset a^2. This seems inappropriate because one does not expect constant stresses to lead to energy dissipation. A possible alternative could be to, as Efroimsky & Lazarian (2000) and Molina et al. (2003) do and similar to the definition outlined in the previous section, simply drop the offset term a from the strains at the outset and calculate the average of the resulting energy expression ($b^2 \sin^2 nt$ in the above example). Unfortunately, there is no easily identifiable reason why this procedure should logically yield a good measure of the amount of fluctuation in the strain energy. After all, in the above expression some of fluctuations are carried by the $2ab \sin nt$ term also. Thus, it would seem that by dropping the offset from the stresses we are not accounting for these fluctuations.

Alternatively, if energy dissipation is due to frictional sliding between grain boundaries or at crack faces, it would make sense for the dissipation to be affected by the presence of an offset.[5] This is because an offset in the stress functions as a pressure, the presence (or absence) of which increases or decreases the frictional force, in turn affecting the amount of dissipation. The energy dissipation model obtained by simply throwing out the offset in the stresses will miss this mechanism of dissipation.

To the best of our knowledge, definitions of Q corresponding to stresses of the kind '$a + b \sin nt$' are not readily available; thus one needs to define an appropriate measure of the fluctuating strain energy. One way to do this would be to think of fluctuations of the strain energy as the amount of area under the strain energy graph that actually changes sign over a loading cycle. So we would define the amount of 'fluctuating' strain energy given by the specific strain energy (e_s) via the definition

$$E_s^{\text{fluc}} = \int_V \langle |e_s - \langle e_s \rangle| \rangle \, \mathrm{d}V. \tag{A1}$$

In this estimate one taxes the (absolute) amount of time-varying strain energy. Unfortunately this expression suffers from the (obvious) problem as to how to handle the absolute value in the integrand. Thus, rather than trying to evaluate E_s^{fluc} as suggested by equation (A1), we introduce a new measure of the 'fluctuating' strain energy $\tilde{E}_s'$ which dominates the previous estimate in the sense that

$$E_s^{\text{fluc}} \leqslant \tilde{E}_s'.$$

Towards this end, we define

$$\tilde{E}_s' = \left[V \int_V \langle (e_s - \langle e_s \rangle)^2 \rangle \, \mathrm{d}V \right]^{1/2}, \tag{A2}$$

which can be seen to have the aforementioned property by employing the following two inequalities:[6]

$$\int_0^T |f| \, \mathrm{d}t \leqslant \left(T \int_0^T f^2 \, \mathrm{d}t \right)^{1/2},$$

and

$$\int_V \sqrt{g} \, \mathrm{d}V \leqslant \left(V \int_V g \, \mathrm{d}V \right)^{1/2}.$$

We can now write (1) as

$$\Delta E = \frac{2\pi \tilde{E}_s'}{Q'}, \tag{A3}$$

with $\tilde{E}_s'$ given by (A2). The reason for using Q' instead of Q will be clearer in the next paragraph. This formula manages to capture the fluctuation in the strain energy when the stresses are not purely sinusoidal and also gives estimates comparable to Lambeck's definition when the stresses are pure sinusoids.

A possible criticism is that it is not correct to compare damping time-scales using this definition with those obtained by using the previous one. The reason being that the Q' used here might be very different from the Q used in Lambeck's definition of Section 3.1. However, then this criticism will be true too for the (ad hoc) extension of Lambeck's model postulated in that section.

However, consider the following experiment. Take a bar of unit volume and apply stresses of the kind $b \sin nt$. In such loadings we know that Lambeck's definition holds true. It predicts the dissipation to be

$$\Delta E = \frac{2\pi}{Q} \frac{b^2}{2}.$$

Meanwhile the dissipation as mentioned by this model would yield

$$\Delta E = \frac{2\pi}{Q'} \frac{b^2}{2\sqrt{2}}.$$

Because the energy dissipated is a physical quantity, comparing the two equations we see that

$$Q = Q'\sqrt{2}.$$

This indicates that perhaps Q and Q' are comparable and not *vastly* different as might be suggested. Moreover, we prefer to think of the quality factor as a parameter, and do not think that the Qs should be differentiated, especially because they do not seem to differ by orders of magnitude.

The above definition is supposed to capture fluctuations in the strain energy that were missed by the earlier formulation. Thus, one expects greater energy dissipation when using this definition which in turn would imply shorter damping times. This indeed turns out to be the case. The corresponding shape parameter, is calculated to be at least a factor of 10 lower. This should not be attributed to a difference in Qs, because as mentioned in the previous paragraph, we do not expect the corresponding Qs of the two definitions to differ by an order of magnitude.

APPENDIX B: DYNAMICS OF A RIGID ELLIPSOID OF REVOLUTION

For a free rigid body in space, angular momentum and energy are conserved quantities. In addition to the angular velocity components

[5] It was pointed out by the referee that the standard definition of Q, which is the one employed in the main text, when used for rocks, is insensitive to pre-stressing over a broad range of pressures. Whether this holds true for asteroid interiors that might be granular aggregates is not, as yet, known.

[6] These are really two instances of the Cauchy–Schwarz inequality.

Ω_i ($i = 1, 2, 3$) along the principal axes, we introduce an intermediate component measured along $\hat{\boldsymbol{e}}_\perp$ (see Fig. 1), $\Omega_\perp$. Then, because of axial symmetry, we can write the angular momentum vector as $A\Omega_\perp\hat{\boldsymbol{e}}_\perp + C\Omega_3\hat{\boldsymbol{e}}_3$. Conservation of angular momentum would imply that this sum is a constant vector $\boldsymbol{J}$. We orient the body as shown in Fig. 1 with θ measuring the tilt of the axis of symmetry $\hat{\boldsymbol{e}}_3$.

From straightforward geometry, we have

$$C\Omega_3 = J\cos\theta, \tag{B1}$$

$$A\Omega_\perp = J\sin\theta. \tag{B2}$$

Energy conservation states that the kinetic energy[7] E_k is given by

$$A\Omega_\perp^2 + C\Omega_3^2 = 2E_\mathrm{k} = \text{constant}.$$

Substituting from the previous two equations, we obtain

$$\frac{J^2\sin^2\theta}{A} + \frac{J^2\cos^2\theta}{C} = 2E_\mathrm{k}.$$

Then angular-momentum conservation implies that the angle θ is a constant (derivable from the above expression).

Let us write $\boldsymbol{\Omega} = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\Phi}}$ in terms of a spin $\dot{\boldsymbol{\Psi}}(= \dot{\psi}\hat{\boldsymbol{e}}_3)$ and an inertial precession $\dot{\boldsymbol{\Phi}}(= \dot{\phi}\hat{\boldsymbol{k}})$; see Fig. 1. Their magnitudes can be solved by first noting that

$$\dot{\phi}\sin\theta = \Omega_\perp \quad \dot{\psi} = \Omega_3 - \dot{\phi}\cos\theta.$$

From these expressions and (B1) we obtain the formulae for $\dot{\boldsymbol{\Psi}}$ and $\dot{\boldsymbol{\Phi}}$ given by equations (7) and (8), respectively. Referring to (7) and (8), one should note the constancy of the spin and the precession because H, J, C and θ are fixed for a rigid body.

Finally, we calculate the acceleration at a body point $\boldsymbol{r}$:

$$\boldsymbol{a}(\boldsymbol{r}) = \dot{\boldsymbol{v}} = \dot{\overline{\boldsymbol{\Omega}\times\boldsymbol{r}}} = \dot{\boldsymbol{\Omega}}\times\boldsymbol{r} + \boldsymbol{\Omega}\times\dot{\boldsymbol{r}}, \tag{B3}$$

but

$$\begin{aligned}\dot{\boldsymbol{\Omega}} &= \ddot{\boldsymbol{\Psi}} + \ddot{\boldsymbol{\Phi}} = \ddot{\psi}\hat{\boldsymbol{e}}_3 = \dot{\psi}\boldsymbol{\Omega}\times\hat{\boldsymbol{e}}_3\\ &= \boldsymbol{\Omega}\times\dot{\boldsymbol{\Psi}},\end{aligned}$$

while

$$\dot{\boldsymbol{r}} = \boldsymbol{\Omega}\times\boldsymbol{r}.$$

Using these in (B3), with the notation that $\widehat{\boldsymbol{abc}} \equiv \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$, we obtain

$$\begin{aligned}\boldsymbol{a}(\boldsymbol{r}) &= (\boldsymbol{\Omega}\times\dot{\boldsymbol{\Psi}})\times\boldsymbol{r} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})\\ &= -\boldsymbol{r}\times(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Psi}}) + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})\\ &= -\boldsymbol{r}\widehat{\boldsymbol{\Omega}\dot{\boldsymbol{\Psi}}} + \widehat{\boldsymbol{\Omega}\boldsymbol{\Omega}\boldsymbol{r}}\\ &= -\boldsymbol{r}\widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Psi}}} + \widehat{\dot{\boldsymbol{\Psi}}\dot{\boldsymbol{\Phi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}}\\ &= 2\widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}}\boldsymbol{r}} + \widehat{\dot{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}}\boldsymbol{r}},\end{aligned} \tag{B4}$$

where in the fourth line we have used the vector identity $\widehat{\boldsymbol{abc}} + \widehat{\boldsymbol{bca}} + \widehat{\boldsymbol{cab}} = 0$ in the first two terms.

APPENDIX C: ELASTICITY SOLUTIONS

In equation (13) the body force is

$$\begin{aligned}\boldsymbol{F} = \rho\{&[a_1 z + (b_1 + b_4)x + c_1 y]\,\hat{\boldsymbol{e}}_1\\ &+ [a_2 z + b_2 x + (c_2 + c_4)y]\,\hat{\boldsymbol{e}}_2\\ &+ (a_4 z + b_3 x + c_3 y)\,\hat{\boldsymbol{e}}_3\}\end{aligned} \tag{C1}$$

[7] Because we are neglecting self-gravity and outside effects, there is no potential energy and all energy is kinetic.

with the constants given by equation (14). The equations to be solved are the well-known Navier equations

$$(\lambda + \mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F} = 0, \tag{C2}$$

where λ and μ are the Lamé constants and $\boldsymbol{u}$ is the displacement of a material point. The boundary condition that the surface of the body (B) is free from applied forces can be stated in terms of stresses as

$$\sigma_{ij}n_j|_{\partial B} = 0. \tag{C3}$$

Because the body is taken to be isotropic and linear elastic the stresses are related to the strains (ϵ_{ij}) by Hooke's law

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}. \tag{C4}$$

Finally, the strains can be obtained from the displacements by the equations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{C5}$$

Equations (C2) and (C3) represent a closed system of partial differential equations that (given the geometry of a body) can be solved for the displacements $\boldsymbol{u}$. Once the displacements are known, the strains and the stresses can be calculated from equations (C5) and (C4), respectively.

In order to solve (C2) with the boundary conditions (C3), we exploit the linearity of (C2). We seek solutions of the form $\boldsymbol{u} = \boldsymbol{u}_\mathrm{H} + \boldsymbol{u}_\mathrm{P}$ such that $\boldsymbol{u}_\mathrm{H}$ satisfies (C2) with $\boldsymbol{F} = 0$ (the so-called homogeneous solution) while $\boldsymbol{u}_\mathrm{P}$ is any solution to (C2) (this is the particular solution). As Love (1946) noted, $\boldsymbol{u}_\mathrm{H}$ is a bi-harmonic function[8] and each of its components (u_H, v_H, w_H) can thus be represented (Fung 1965; Chree 1886) most generally in terms of infinite sums of spherical harmonics (H_n) of order n as

$$u_\mathrm{H} = \sum H_n + r^2\sum H_{n-2},$$

with $r = (x^2 + y^2 + z^2)^{1/2}$ being the distance from the origin. Similar representations apply to v_H and w_H. Note that neither u_H nor u_P will necessarily satisfy the boundary conditions (C3), but the hope is that by appropriately choosing the various constants (which have been suppressed in the expansion above but are incorporated in H_n) one can satisfy (C3). Once these constants are found, we can calculate the total strain energy (E_s) stored in the body from $E_\mathrm{s} = \frac{1}{2}\int_V \sigma_{ij}\epsilon_{ij}\,\mathrm{d}V$.

REMARK

An elasticity problem can be formulated either in terms of displacements (the Navier equation above) or directly in terms of stresses. A displacement formulation incorporates the specific rheological model of the body (a linear elastic solid in our case) under consideration. However, a stress formulation is simply a statement concerning the static equilibrium of the body. It makes no connection with the rheology of the solid. Thus, there exists infinitely many sets of six independent stresses that satisfy (static) equilibrium as well as the prescribed boundary conditions. Each of these infinitely many solutions are the solutions of a solid with *some* rheological model. However, the nature of this rheology is unknown, and it is very likely that the rheology would be completely unphysical. In order to help us choose those stresses which are a solution to a specified rheology, there exist compatibility equations. Thus, if we need stresses that are solutions to a linear elastic solid, we must make sure that the

[8] That is, it satisfies $\nabla^4\boldsymbol{u}_\mathrm{H} = 0$.

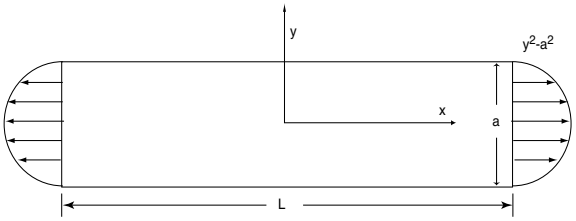


Figure C1. An example problem in which a beam of length L and cross-section $a \times a$ is subjected to end loads given by $y^2 - a^2$. The coordinate system is as shown. The length L is assumed to be much larger than the thickness a.

stresses not only satisfy equilibrium and the boundary conditions but also the compatibility equations that accompany the linear elastic rheology. In summary, a set of stresses, satisfying equilibrium and boundary conditions but not the compatibility equations, run the risk of being solutions to a solid for which the rheology, if it was known, might not be very useful. See Love (1946) or Fung (1965) for further details.

Efroimsky & Lazarian (2000) and Molina et al. (2003) have solutions that satisfy equilibrium and the boundary conditions (approximately in the former case and exactly in the latter). However, because their solutions do not satisfy compatibility conditions, it is very probable that their solutions will be far from being the solutions of a linear elastic solid. In fact, one has no knowledge of the elastic behaviour of the solid for which they have obtained solutions. It should be mentioned that though Molina et al. (2003) do not satisfy the compatibility conditions, they make each individual stress component vanish at the boundary. This is more than what is required at the boundary because, according to (C3), the net force should vanish at the boundary, which is not the same as all the stresses vanishing. Thus, Molina et al. (2003) enforce extra conditions on their stresses which they claim substitute for their having to satisfy the compatibility conditions. However, there is no reason to believe that enforcing such artificial constraints on their stresses will make their solution any more or less physical than Efroimsky & Lazarian (2000). Nor does such an imposition help us in identifying the rheology of their body.

Finally, consider a simple example that illustrates that it is impossible to find (compatible) displacement fields for solutions of a linear elastic material that do not satisfy the compatibility conditions. Consider the long beam of Fig. C1 with $a \ll L$ and subjected to the forces shown. A solution that satisfies equilibrium and boundary conditions but *not* the compatibility conditions is

$$\sigma_{xx} = y^2 - a^2,$$
$$\sigma_{yy} = \sigma_{xy} = 0.$$

Immediately we know that there is something wrong. Consider the stresses on a cross-section far away from the ends. The stresses on it will be the given by σ_{xx} above. However, according the Saint-Venant's principle[9] the stresses on a cross-section far away from the ends should be uniform as the elasticity solution is not sensitive to the distribution of the loads at sufficient distances from the point of application of the loads. This is an indication that everything may not be correct concerning the solution above. However, there is a more direct reason. Consider solving for the displacement field using the above solution assuming the body to be isotropic, linear elastic. Using (C4), we obtain the strains

$$\epsilon_{xx} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}(y^2 - a^2) \equiv \frac{y^2 - a^2}{E},$$
$$\epsilon_{yy} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}(y^2 - a^2) \equiv \frac{-\nu(y^2 - a^2)}{E},$$
$$\epsilon_{xy} = 0,$$

where E is the Young's modulus. Using these in (C5) to solve for the displacements u_x and u_y, we see that the following equations must be satisfied:

$$\frac{\partial u_x}{\partial x} = \frac{y^2 - a^2}{E},$$
$$\frac{\partial u_y}{\partial y} = \frac{-\nu(y^2 - a^2)}{E},$$
$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0.$$

In the above, ν is the Poisson ratio. It is easily seen that it is impossible to find any u_x and u_y that are compatible with the above three equations simultaneously. The purpose of the compatibility equations is precisely to reject such solutions that are unable to yield compatible displacement fields. If a solution does satisfy the compatibility equations, we are assured that the displacement equations, analogous to the three above, will have a solution.

APPENDIX D: THE STRESS FIELD

In this section we give the stresses arising at any point (x, y, z) of an elastic body with $\lambda \sim \mu$ due to its tumbling motion in free space. For cosmetic purposes we introduce A_i and B_i as functions of the aspect ratio. Let $[m, n, p] \equiv m + nh^2 + ph^4$. Then these functions can be enumerated as follows:

$$A_1 = \frac{5[0, 1, 0]}{[13, 20, 0]} \quad A_2 = \frac{[-39, 20, 0]}{30[9, 8, 16]}$$

$$A_3 = \frac{[0, 13, -60]}{15[9, 8, 16]} \quad A_4 = \frac{5[1, 0, 0]}{2[15, 10, 8]}$$

$$B_1(x, y, z) = \frac{[5, 5, 4](a^2 - x^2 - y^2) - [5, 4, 0]z^2}{2[15, 10, 8]}$$

$$B_2(x, y, z) = \frac{[39, 52, 100](a^2 - x^2) - [21, 16, 60]y^2 - 4[18, 25, 0]z^2}{30[9, 8, 16]}$$

$$B_3(x, y, z) = \frac{19[0, 1, -2](a^2 - x^2) - [0, 31, 6]y^2 - [45, -158, 0]z^2}{30[9, 8, 16]}$$

$$B_4(x, y, z) = \frac{-[0, 13, 15](a^2 - x^2) + [0, 13, 5]y^2 + [13, 15, 0]z^2}{2[13, 20, 0]}$$

$$B_5(x, y, z) = \frac{[135, 146, 120](h^2a^2 - z^2) - [0, 135, 172](x^2 + y^2)}{30[9, 8, 16]}.$$

[9] See Timoshenko & Goodier (1970) for more details.

Now we can write the stresses as

$$\frac{1}{\rho}\sigma_{11} = -2A_1\omega_3\omega_\perp yz\cos nt + a_4B_1(x,y,z)\cos 2nt + (2b_4+a_4)B_2(x,y,z) + a_4B_3(x,y,z)$$

$$\frac{1}{\rho}\sigma_{22} = -2A_1\omega_3\omega_\perp xz\sin nt - a_4B_1(x,y,z)\cos 2nt + (2b_4+a_4)B_2(y,x,z) + a_4B_3(y,x,z)$$

$$\frac{1}{\rho}\sigma_{12} = A_1\omega_3\omega_\perp(x\cos nt + y\sin nt)z - a_4B_1(x,y,z)\sin 2nt + (2b_4+a_4)\left[B_2(x,y,z) - B_2(y,x,z)\right]\frac{xy}{x^2-y^2} + a_4\left[B_3(x,y,z) - B_3(y,x,z)\right]\frac{xy}{x^2-y^2}$$

$$\frac{1}{\rho}\sigma_{13} = A_1h^2\omega_3\omega_\perp xy\cos nt + A_2(2b_4+a_4)xz + A_3a_4xz + A_4a_4(y\sin 2nt - x\cos 2nt)z + \omega_3\omega_\perp B_4(x,y,z)\sin nt$$

$$\frac{1}{\rho}\sigma_{23} = A_1h^2\omega_3\omega_\perp xy\sin nt + A_2(2b_4+a_4)yz + A_3a_4yz + A_4a_4(y\cos 2nt + x\sin 2nt)z + \omega_3\omega_\perp B_4(y,x,z)\cos nt$$

$$\frac{1}{\rho}\sigma_{33} = A_2(2b_4+a_4)\left[h^2(a^2-2x^2-2y^2) - z^2\right] + A_4h^2a_4\left[(x^2-y^2)\cos 2nt - 2xy\sin 2nt\right] + a_4B_5(x,y,z).$$

The angular velocities $\omega_\perp$, ω_3, a_4, b_4 are defined in (12) and (14) and n is the magnitude of the spin angular velocity (see the last of equation 14).

This paper has been typeset from a TEX/LATEX file prepared by the author.