Equilibrium shapes of ellipsoidal soil asteroids

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Recent research has suggested that asteroids might be particle aggregates held together by self-gravity alone. This has important implications on the possible equilibrium shapes of spinning asteroids. As in the case of spinning fluid masses, not all shapes and spins may be compatible with a granular rheology. We take the asteroid to be an ellipsoid with an interior modelled as a rigid-plastic, cohesion-less soil. Using an approximate volume-averaged procedure, we are able to derive regions in spin-shape parameter space that allow equilibrium solutions to exist. Our results are the same as those reported by Holsapple (2000), but are obtained with much less effort. We also investigate the dynamics of such spinning asteroids and attempt to recover the results of Richardson et al. (2004), who obtained equilibrium shapes of smooth spherical aggregates by numerically studying their passage into equilibrium.

1 INTRODUCTION

Investigations of spinning fluid masses go back to the time of Newton who, assuming the asphericity to be small, determined the flattening of the Earth by modelling it as such. Later, Maclaurin determined the equilibrium shapes of oblate fluid rotators, the so-called Maclaurin ellipsoids. He further showed that prolate equilibrium shapes cannot be obtained. Truly triaxial ellipsoidal shapes of equilibrium for spinning fluids were not supposed to be realisable until Jacobi put forth an argument in support of their existence. These ellipsoids branch off from the Maclaurin sequence and are called the Jacobi ellipsoids. The equilibrium shapes of spinning fluid ellipsoids are comprehensively covered by Chandrasekhar (1969) in a unified manner, using the volume-averaged method introduced below.

Recent research (see Richardson et al. 2002) has suggested that asteroids might be incoherent structures held together by self-gravity and best modelled as granular aggregates. Observations also show that a majority of asteroids are in a state of pure spin about their axis of maximum inertia. This motivates a study of equilibrium shapes of spinning aggregates taken to be ellipsoids as a first approximation. Like fluids, granular materials place restrictions, though not as severe, on the allowable shapes of a spinning ellipsoid by limiting the amount of stress that it can tolerate. Such restrictions may help constrain their interiors and thus, constitute a first step towards solving the inverse problem of inferring the asteroids' interiors from a knowledge of their shapes and spins. This is especially important now that the shape and spin states of many asteroids are known to a high degree of accuracy, either from radar observations (see Ostro et al. 2002) or by inverting light curve data (see Pravec et al. 2002).

Granular materials display a wide range of behaviour, from nearly rigid structures to loose fluid like flows. It seems appropriate to consider asteroids as dense frictional aggregates modelled as a rigid-plastic soil obeying a Mohr-Coulomb failure law. Holsapple (2001), in his analysis of equilibrium shapes, used this rheology for asteroid interiors. His analysis relied on techniques of limit analysis from plasticity theory, in particular, the lower limit theorem (see, e.g., Chen and Han 1988). He was able to map out regions in spin-shape space where ellipsoids could exist in equilibrium. In his sudy, the body fails at a point whenever the inhomogeneous distribution of stress satisfies the yield condition at that point. His analysis was a local one in contrast to the volume-averaged approach presented below.

Richardson et al. (2004) studied the equilibrium shapes of spinning dense granular aggregates modelling them as collections of smooth spheres held together by their gravity. From an initial (non-equilibrium) configuration they followed the evolution of each sphere using a smooth particle hydrodynamic (SPH) code until equilibrium was attained.

Here we investigate asteroids modelled as rigid-plastic soil ellipsoids in pure spin using a volume-averaged method. We will first obtain regions in parameter space describing the shape where, for a given spin, an ellipsoidal asteroid can exist. Our results match Holsapple's (2001) exactly, but are obtained in a very transparent manner using a minimum of computation. We then study the dynamical evolution of a non-equilibrium ellipsoid, which is very complicated if Holsapple's (2001) approach is followed. This will help evaluate the appropriateness of comparing our, and Holsapple's (2001), continuum approach to the discrete model used by Richardson et al. (2004).

2 VOLUME-AVERAGING

We begin with a kinematic assumption, viz. the ellipsoid's deformation is homgeneous, so that the deformation gradient $\boldsymbol{F}$ is constant. Thus, ellipsoids can deform only into ellipsoids. This is a first approximation, but was shown by Chandrasekhar (1969) to yield physically meaningful results in the case of spinning fluid masses, and is further motivated by the fact that spinning elastic ellipsoids deform into ellipsoids (see Love 1946). Equations governing the evolution of $\boldsymbol{F}$ are obtained by taking the first moment[1] of the linear-momentum-balance equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathsf{b} = \rho \ddot{\mathsf{x}}, \tag{1}$$

and assuming a traction-free surface, to yield

$$\ddot{\boldsymbol{F}} \boldsymbol{F}^{-1} \boldsymbol{I} = \boldsymbol{M}^T - \boldsymbol{\sigma} V, \tag{2}$$

where x is the position vector, $\boldsymbol{\sigma}$ the average stress,

$$\boldsymbol{I} = \int_V \rho \mathsf{x} \otimes \mathsf{x} dV, \tag{3}$$

the inertia dyad, and

$$\boldsymbol{M} = \int_V \mathsf{x} \otimes \rho \mathsf{b} dV, \tag{4}$$

the moment tensor due to the body forces b. In our case, the body force is due only to internal gravity, and we can compute the above integral to find

$$\boldsymbol{M} = -2\pi\rho G \boldsymbol{I}\boldsymbol{A}, \tag{5}$$

where G is the gravitational constant and the tensor $\boldsymbol{A}$ captures the effect of the ellipsoidal shape on its internal gravity (see Chandrasekhar 1969). It depends only on the axes ratio $\alpha = a_2/a_3$ and $\beta = a_3/a_1$.

Introducing the velocity gradient

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}, \tag{6}$$

and using (5) for $\boldsymbol{M}$, reduces (2) to

$$\left(\dot{\boldsymbol{L}} + \boldsymbol{L}^2\right) \boldsymbol{I} = \boldsymbol{M}^T - \boldsymbol{\sigma} V. \tag{7}$$

The inertia dyad's evolution is governed by

$$\dot{\boldsymbol{I}} = \boldsymbol{L}\boldsymbol{I} + \boldsymbol{I}\boldsymbol{L}^T, \tag{8}$$

which is obtained by taking the first moment of the mass conservation equation.

In the case of an ellipsoid in pure spin and in equilibrium, $\boldsymbol{L}$'s symmetric part, the *strain rate* (or *stretching rate*) tensor $\boldsymbol{D} = 0$, so that (7) simplifies to

$$\boldsymbol{\sigma} V = \left(2\pi\rho G \boldsymbol{A} - \boldsymbol{W}^2\right) \boldsymbol{I}, \tag{9}$$

where $\boldsymbol{W}$, the *angular velocity* (or *spin*) tensor, is the anti-symmetric part of $\boldsymbol{L}$. Note that the above equation is a balance between "centrifugal" and gravitational stresses in a volume-averaged sense.

In general, Eqns. (6), (7) and (8) govern the motion of a homogeneously deforming gravitating ellipsoid in free space, once a constitutive equation for $\boldsymbol{\sigma}$ is specified.

2.1 *Rheology*

It is possible to explore different material models by specifying an appropriate constitutive relation. We restrict attention to a rigid-plastic frictional soil with an appropriate failure criterion. The material remains rigid until a failure criterion is violated, whereafter plastic flow begins. For statics, we use the Mohr-Coulomb (MC) criterion:

$$\sigma_{max} - k_{MC}\sigma_{min} \leq 0, \tag{10}$$

in terms of the extremum principal stresses and k_{MC} is related to the internal friction angle ϕ_F by

$$k_{MC} = \frac{1 + \sin\phi_F}{1 - \sin\phi_F}. \tag{11}$$

For dynamics, we use a smoothed MC criterion, the Drucker-Prager (DP) failure criterion

$$|\boldsymbol{\sigma}'|^2 \leq k^2 p^2, \tag{12}$$

where p is the pressure, $\boldsymbol{\sigma}'$ the deviatoric stress and

$$k = \frac{2\sqrt{6}\sin\phi_F}{3 - \sin\phi_F}, \tag{13}$$

chosen so that the DP-yield surface is the inner envelope of the MC-yield surface. Subsequent plastic flow, assumed incompressible, is governed by the flow rule

$$\boldsymbol{D} = \dot{q}\boldsymbol{\sigma}', \tag{14}$$

with $\dot{q}$ a constant. The stress during plastic flow is

$$\boldsymbol{\sigma} = p\left(\boldsymbol{1} - k\frac{\boldsymbol{D}}{|\boldsymbol{D}|}\right). \tag{15}$$

[1] I.e. integrating the dyadic product ($\otimes$) of each quantity in the equation with the position vector, over the body's volume V.

3 EXAMPLES

We give two examples of the above volume-averaging approach in the context of ellipsoids spinning about the 3-axis with $1 \geqslant \alpha \geqslant \beta$.

3.1 *Statics*

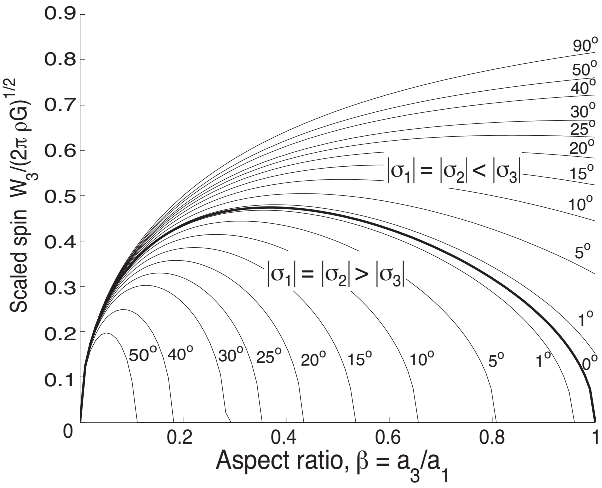
By using stresses from (9) in the MC-failure criterion (10) with an equality sign, we obtain critical curves for the spin W_3 in terms of the axis ratio β. We use $\alpha = (1+\beta)/2$ for triaxial ellipsoids. In each case, we have to choose σ_{max} and σ_{min}, thus yielding regions separated by heavy curves on which two of the principal stresses are equal, as shown in Fig. 1. We see that, for any friction angle, there is an upper and a lower curve bounding a region within which a stable spinning ellipsoid is possible. The failure can be understood by identifying the maximum and minimum principal stresses at failure. For example, in the oblate and prolate cases, the upper region(s) corresponds to a rotational disruption, the centrifugal stresses dominating gravitational stresses. In the oblate case, the lower region corresponds to gravitational collapse. This is not true for prolate objects, where the lower-most region corresponds to disruption due to a competition of rotational stresses in the equatorial plane. Inviscid fluids are indicated by $\phi_F = 0^o$. The heavy curve in Fig. 1(a) corresponds to the Maclaurin ellipsoids, while Fig. 1(b) shows that static prolate inviscid ellipsoids are not possible. A Jacobi ellipsoid is indicated by the intersection point in Fig. 1(c).

The constraints on spin and shape for rigid-plastic soils obtained using volume-averaged methods are in complete agreement with Holsapple's (2001) results. In some respects, this exact match is very surprising, because the homogeneous method deals with volume-averaged stresses and, thus, looks for failure on the average. In contrast, Holsapple's (2001) exact analysis seeks failure point-wise. Thus, it might be expected that a point-wise theory would be more sensitive than an average theory, which should be more conservative. The reason for the match lies in the fact that Holsapple's (2001) analysis predicts simultaneous failure at every point in the body. This means that failure at a point coincides with failure on the average.

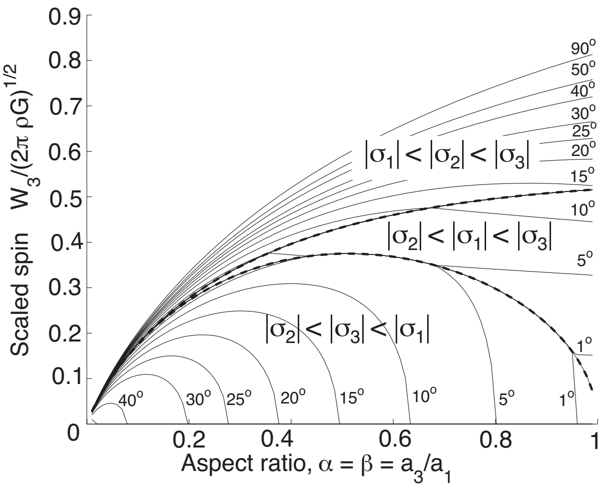
3.2 *Dynamics*

We study the passage into equilibrium of prolate ($\alpha = \beta$) ellipsoids in pure spin with friction angle $\phi_F = 40^o$. When thought of as rigid-plastic objects, the equilibrium shapes obtained by Richardson et al. (2004) were shown by them to lie in a region bounded by $\phi_F = 40^o$ curves (as in Fig. 1(b)).

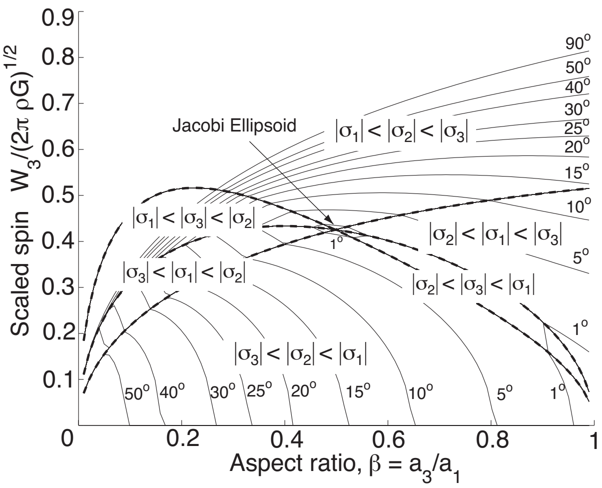
Requiring a compressive average pressure, as is often done for granular materials, yields the disruption curve C_3 in Fig. 2. We use C_2, the failure curve corresponding to the smooth DP criterion, instead of the



(a) Oblate ellipsoids



(b) Prolate ellipsoids



(c) Triaxial ellipsoids

Figure 1: Regions in spin-shape space where equilibrium ellipsoids are possible. Numbers next to the curves indicate the corresponding friction angle ϕ_F.

less conservative curve C_1 obtained from the MC criterion (see Fig 1(b)). Ellipsoids with initial conditions between C_3 and C_2 deform dynamically until they reach an equilibrium shape. Fig. 2 shows the results with the initial and final states connected. The average final α is indicated by numbers next to each group.

Comparing the results of Fig. 2 with those of Richardson et al. (2004), we find that there is qualitative, and in some respects, even quantitative match. We predict the presence of a disruption zone that corresponds to those of Richardson et al. (2004). Like them, we obtain a rather narrow "deformable" zone above the upper failure curve, with ellipsoids elongating before equilibrating. There are two principal differences. First, unlike Richardson et al. (2004), we can predict a disruption, but not follow it through. This may be addressed by using higher-order moments. Second, we observe less deformation than Richardson et al. (2004). This could be attributed to our use of the curve C_2 (DP criterion), rather than C_1 (MC criterion), as the upper boundary of the equilibrium region. The latter delays transition to rigidity, thus enhancing deformation. However, we believe this is because in the model of Richardson et al. (2004), the internal friction's origin is geometric alone, i.e., due to the aggregate's packing, as the constituent spheres are smooth. Thus, in their case, the frictional resistance contributes at higher packing fractions. In contrast, our model preserves volume, and does not differentiate between geometric and surface friction, leading to a greater frictional effect, so that the ellipsoid equilibrates faster. This could be amended by using a flow rule that is not volume preserving, and by allowing the friction to depend on the packing fraction to simulate a geometric dependency. Alternatively, we could model the initial aggregate as a dense gas of smooth spheres, whose rheology transitions to that of a rigid-plastic soil as the packing fraction increases.

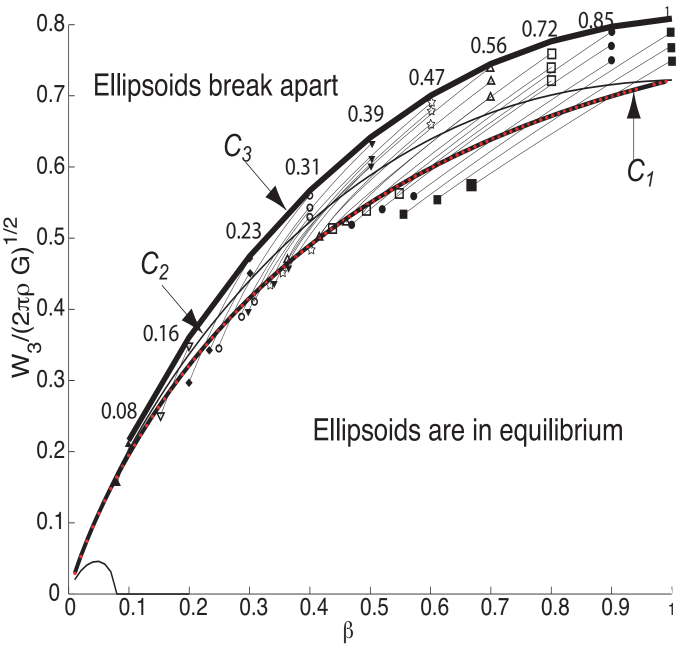


Figure 2: Passage into equilibrium of initially prolate ellipsoids.

4 CONCLUSIONS

We employ a simple and transparent method to characterise the equilibrium shapes, and passage into such states, of asteroids with interiors modelled by cohesion-less rigid-plastic soils. This approach can also be used to study disruption of asteroids during a planetary fly-by. The fact that we recover Holsapple's (2001) results exactly and those of Richardson et al. (2004) approximately has a number of implications. First, it may be used to determine the stability of the equilibrium shapes, as carried out by Chandrasekhar (1969) for fluid ellipsoids. Next, non-equilibrium dynamics of rigid-perfectly-plastic ellipsoids can be explored in a rather simple manner with the added ability of quantifying the results of Richardson et al. (2004). Finally, the volume-averaged approach is amenable to systematic improvements. Different internal rheologies may be explored. It may also be possible to extend the analysis to more general shapes by using higher-order moments. Such an analysis will be fruitful considering the wide variety of shapes seen amongst asteroids.

5 ACKNOWLEDGEMENTS

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