

EH GRAPHS ON RVMS

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1. ERDOS-HAJNAL GRAPHS IDEALIZED

Definition 1.1. Let $\mathcal{I}$ be an ideal on κ . $EH_{\mathcal{I}}(\kappa, \theta)$ is the graph whose vertices are functions $f : \kappa \rightarrow \theta$ modulo $\mathcal{I}$ -almost equality. The edge relationship is fEg iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{I}$.

Fact 1.2. Let κ be regular uncountable and $\omega \leq \theta < \kappa$.

- (1) If $\mathcal{I} \subseteq \mathcal{J}$, then there is a graph homomorphism from $EH_{\mathcal{I}}(\kappa, \theta)$ to $EH_{\mathcal{J}}(\kappa, \theta)$. Hence $\text{Chr}(EH_{\mathcal{I}}(\kappa, \theta)) \leq \text{Chr}(EH_{\mathcal{J}}(\kappa, \theta))$.
- (2) If $\mathcal{I} = [\kappa]^{<\kappa}$, then $EH_{\mathcal{I}}(\kappa, \theta)$ and $EH(\kappa, \theta)$ are homomorphic to each other and hence have the same chromatic number.
- (3) If $\mathcal{I}$ is a normal ideal on κ such that $\mathcal{P}(\kappa)/\mathcal{I}$ is σ -linked, then $EH_{\mathcal{I}}(\kappa, \omega)$ is countably chromatic.

Definition 1.3. Let $\mathcal{I}$ be an ideal on X . Define the transversal number $\text{Tr}_{\mathcal{I}}(X, Y)$ to be the supremum of $|\mathcal{F}|$ where $\mathcal{F} \subseteq {}^XY$ is an $\mathcal{I}$ -almost disjoint family of functions – for every distinct $f, g \in \mathcal{F}$, $\{x \in X : f(x) = g(x)\} \in \mathcal{I}$.

Lemma 1.4. Let $\mathcal{I}$ be a uniform ideal on κ . Then $\text{Chr}(EH_{\mathcal{I}}(\kappa, \theta)) \geq \log(\text{Tr}_{\mathcal{I}}(\kappa, 2^\theta))$.

Proof. Fix an independent family $\{A_\alpha : \alpha < 2^\theta\} \subseteq [\theta]^\theta$. Put $\lambda = \text{Chr}(EH_{\mathcal{I}}(\kappa, \theta)) \geq \theta$ and towards a contradiction, suppose $\mathcal{F}$ is a family of $\mathcal{I}$ -almost disjoint functions from κ to 2^θ such that $|\mathcal{F}| = (2^\lambda)^+$. Let $\langle f_i : i < (2^\lambda)^+ \rangle$ be an injective enumeration of $\mathcal{F}$. For each $i < j < (2^\lambda)^+$, define $g_{i,j} : \kappa \rightarrow \theta$ by

$$g_{i,j}(\xi) = \begin{cases} \min(A_{f_i(\xi)} \setminus A_{f_j(\xi)}) & \text{if } A_{f_i(\xi)} \setminus A_{f_j(\xi)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Fix a vertex coloring $c : EH_{\mathcal{I}}(\kappa, \theta) \rightarrow \lambda$ and define $d : [(2^\lambda)^+]^2 \rightarrow \lambda$ by $d(\{i < j\}) = c([g_{i,j}]_{\mathcal{I}})$. By Erdős-Rado theorem, there exist $i < j < k$ such that $d(\{i, j\}) = c([g_{i,j}]_{\mathcal{I}}) = c([g_{j,k}]_{\mathcal{I}}) = d(\{j, k\})$. Now observe that if $\alpha, \beta, \gamma < 2^\theta$ are distinct, then $\min(A_\alpha \setminus A_\beta) \neq \min(A_\beta \setminus A_\gamma)$. It follows that $g_{i,j}$ and $g_{j,k}$ are $\mathcal{I}$ -almost disjoint. A contradiction to the fact that c is a vertex coloring of $EH_{\mathcal{I}}(\kappa, \theta)$. $\square$

Corollary 1.5. For every uniform ideal $\mathcal{I}$ on 2^κ , $\text{Chr}(EH_{\mathcal{I}}(2^\kappa, \kappa)) \geq \kappa^+$. In particular, $\text{Chr}(EH(\mathfrak{c}, \omega)) \geq \omega_1$.

Proof. Use Lemma 1.4 and the fact that for every infinite λ , there is a family $\mathcal{F} \subseteq {}^\lambda\lambda$ such that $|\mathcal{F}| = \lambda^+$ and for every distinct $f, g \in \mathcal{F}$, $|f \cap g| < \lambda$. $\square$

Furthermore, if $\mathfrak{c} = 2^{<\mathfrak{c}}$, then $\text{Chr}(EH(\mathfrak{c}, \omega)) \geq \log(\text{Tr}_{[\mathfrak{c}]^{<\mathfrak{c}}}(\mathfrak{c}, \mathfrak{c})) = \mathfrak{c}$.

Definition 1.6. κ is *rvm* (resp. *rvc*) iff there is a normal ideal $\mathcal{I}$ on κ such that $\mathcal{P}(\kappa)/\mathcal{I} \cong \text{Random}_\lambda$ (resp. Cohen_λ) for some λ .

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Lemma 1.7. *If $\mathfrak{c}$ is rvm/rvc, then $\text{Chr}(EH(\mathfrak{c}, \omega)) \geq \mathfrak{c}$.*

Proof. Put $\mathcal{I} = [\mathfrak{c}]^{<\mathfrak{c}}$. By Prikry, $2^{<\mathfrak{c}} = \mathfrak{c}$ so there is an $\mathcal{I}$ -almost disjoint family $\mathcal{F} \subseteq {}^{\mathfrak{c}}\mathfrak{c}$ such that $|\mathcal{F}| = 2^{\mathfrak{c}}$. Hence by Lemma 1.4, we have

$$\text{Chr}(EH(\mathfrak{c}, \omega)) = \text{Chr}(EH_{\mathcal{I}}(\mathfrak{c}, \omega)) \geq \log(\text{Tr}_{\mathcal{I}}(\mathfrak{c}, \mathfrak{c})) \geq \log(|\mathcal{F}|) = \log(2^{\mathfrak{c}}) = \mathfrak{c}$$

where the last equality uses $2^{<\mathfrak{c}} = \mathfrak{c}$. $\square$

Lemma 1.8. *Suppose $\kappa \leq \mathfrak{c}$ is rvm/rvc.*

- (1) *If $2^{\kappa} \leq \mathfrak{c}$, then $\text{Chr}(EH(\kappa, \omega)) = \omega$.*
- (2) *If $2^{\kappa} \geq \mathfrak{c}^+$ and $2^{<\kappa} = \mathfrak{c}$, then $\text{Chr}(EH(\kappa, \omega)) \geq \omega_1$.*
- (3) *It is consistent that $2^{\kappa} = 2^{\omega_1} = \mathfrak{c}^+$ and $\text{Chr}(EH(\kappa, \omega)) = \omega$.*

Proof. (1) Fix a normal ideal $\mathcal{I}$ on κ such that $\mathbb{B} = \mathcal{P}(\kappa)/\mathcal{I} \cong \text{Random}_{\lambda}$ (resp. Cohen_{λ}). In the first case, $\lambda \leq \mathfrak{c}$ so $\mathbb{B}$ is σ -linked and thus by Fact 1.2, $\text{Chr}(EH(\kappa, \omega)) \leq \text{Chr}(EH_{\mathcal{I}}(\kappa, \omega)) = \omega$.

(2) In this case, there is an almost disjoint family $\mathcal{F}$ of size $\mathfrak{c}^+$ consisting of functions from κ to $\mathfrak{c}$ (for each $x : \kappa \rightarrow 2$, consider the map $\alpha \mapsto x \upharpoonright \alpha$). Now apply Lemma 1.4.

(3) Start with a model satisfying κ measurable, $\mu = \beth_{\omega_1}(\kappa)$, $2^{\mu} = \mu^{+\omega+2}$, $(\mu^{+\omega})^{\aleph_0} = \mu^{+\omega+1}$ and add $\mu^{+\omega}$ random reals. Then, in the generic extension κ carries a normal measure of Maharam type $= \mu^{+\omega} < \mathfrak{c} = \mu^{+\omega+1} < 2^{\omega_1} = 2^{\kappa} = \mu^{+\omega+2}$. $\square$

Corollary 1.9. *If $\mathfrak{c}$ is rvm/rvc, then for each rvm/rvc $\kappa < \mathfrak{c}$, $\text{Chr}(EH(\kappa, \omega)) = \omega$. So it is consistent that $\mathfrak{c}$ is weakly inaccessible and $\{\kappa < \mathfrak{c} : \text{Chr}(EH(\kappa, \omega)) = \omega\}$ is unbounded below $\mathfrak{c}$.*

Proof. Use Lemma 1.8(1) and the fact that if $\mathfrak{c}$ is rvm/rvc, then $2^{<\mathfrak{c}} = \mathfrak{c}$. $\square$

Lemma 1.10. *It is consistent that there are rvm/rvc $\kappa < \lambda < \mathfrak{c}$ such that $\text{Chr}(EH(\kappa, \omega)) = \omega$ and $\text{Chr}(EH(\lambda, \omega)) \geq \omega_1$.*

Proof. In V start with $\kappa < \lambda$ measurable such that $2^{\lambda} \geq \lambda^{++}$ (Silver's model) and force with $\mathbb{P} = \text{Random}_{\lambda^+}$. Check that $V^{\mathbb{P}} \models \lambda < \mathfrak{c} = 2^{\kappa} = 2^{<\lambda} = \lambda^+ < \lambda^{++} \leq 2^{\lambda}$. Now apply Lemma 1.8. $\square$

REFERENCES

- [1] S. Todorcevic, Combinatorial dichotomies in set theory, Bulletin of Symbolic Logic (2011), <https://www.math.toronto.edu/~stevo/dichotomies4.pdf>