IDC:608: *Computability theory*
Homework 2023/24 II

(1) Let $d(x, \mathbb{Z})$ denote the nearest distance of $x \in \mathbb{R}$ from integers. For example, $d(3.4, \mathbb{Z}) = 0.4$ and $d(1.7, \mathbb{Z}) = 0.3$. Let $A = \{x > 1 : \lim_{n\to\infty} d(x^n, \mathbb{Z}) = 0\}$.

(a) Show that $\frac{1+\sqrt{5}}{2} \in A$.

(b) Call a real number x computable iff there is a computable function $f : \omega \to \mathbb{Q}$ such that for every $n < \omega$, $|f(n) - x| \leq 2^{-n}$. Show that every member of A is computable.

(c) Conclude that A is countable.

(2) Show that there are infinitely many pairs $e < e' < \omega$ such that $\mathrm{dom}(\varphi_e) = \{e'\}$ and $\mathrm{dom}(\varphi_{e'}) = \{e\}$.

(3) Show that every infinite c.e. $X \subseteq \omega$ has an infinite computable subset.

(4) Let $f : \omega \to \omega$. For $n \geq 1$, define f^n by $f^1 = f$, $f^{n+1} = f \circ f^n$.

(a) Show that if f is computable, then $\{f^n(0) : n \geq 1\}$ is c.e.

(b) Show that there is a computable $f : \omega \to \omega$ such that $\{f^n(0) : n \geq 1\}$ is not computable.

(5) Show that there is a partial computable function that cannot be extended to any total computable function. Hint: Use a pair of computably inseparable c.e. sets.

(6) Show that there is a simple set $X \subseteq \omega$ such that $\lim_{n\to\infty} \frac{|X \cap n|}{n} = 0$.

(7) Let $A, B \subseteq \omega$. Show that $A' \leq_m B \implies A' \leq_1 B$. Here A' is the Turing jump of A.

(8) Assume $x, y \in 2^\omega$ and x is not computable from y. Show that $\{z \in 2^\omega : x \leq_T y \oplus z\}$ is meager. Use this to show that every maximal Turing independent set is uncountable.

(9) Generalize Kleene-Post theorem as follows. For every $\mathbf{a} \in \mathcal{D} \setminus \{\mathbf{0}\}$, there exists $\mathbf{b} < \mathbf{a}'$ such that $\mathbf{a}$ and $\mathbf{b}$ are Turing incomparable.

(10) Construct a sequence $\langle x_n : n < \omega \rangle$ of reals such that for every n, $x_{n+1} <_T x_n$. Hint: Build $\langle y_n : n < \omega \rangle$ by finite approximation such that for every $m < n$, y_m is not computable from $\langle y_k : k \geq n \rangle$.

(11) Show that for every $x \in 2^\omega$, there exist $y, z \in 2^\omega$ such that $x <_T y$, $x <_T z$ and for every $w \in 2^\omega$, $(w \leq_T x \iff (w \leq_T y \text{ and } w \leq_T z))$.

(12) For $\mathbf{a}, \mathbf{b} \in \mathcal{D}$, the meet of $\mathbf{a}$ and $\mathbf{b}$ is the $\leq_T$-greatest lower bound of $\{\mathbf{a}, \mathbf{b}\}$ (if it exists). Show that for every $\mathbf{e} \in \mathcal{D}$, there exist $\mathbf{a}, \mathbf{b} \in \mathcal{D}$ such that $\mathbf{e} \leq_T \mathbf{a}$, $\mathbf{e} \leq_T \mathbf{b}$ and the meet of $\mathbf{a}, \mathbf{b}$ does not exist.

(13) Suppose $\langle p_n : n < \omega \rangle$ satisfies the following.

(a) Each $p_n \in \mathbb{S}$ and $(\forall n)(p_{n+1} \subseteq p_n)$.

(b) For every $n < \omega$, $\mathsf{splitnode}_n(p_n) = \mathsf{splitnode}_n(p_{n+1})$.

Put $p = \bigcap\limits_{n<\omega} p_n$. Show that $p \in \mathbb{S}$.

(14) Let $p \in \mathbb{S}_{rp}$. Prove the following.

(a) For every $y \in 2^\omega$, $p \leq_T y$ iff $(\exists x \in [p])(x \equiv_T y)$.

(b) For every $y \in 2^\omega$, if $p \leq_T y$, then there exists $q \in \mathbb{S}_{rp}$ such that $q \subseteq p$ and $q \equiv_T y$.

(c) If $q \in \mathbb{S}$, $q \subseteq p$ and $q \leq_T p$, then $q \in \mathbb{S}_{rp}$ and $q \equiv_T p$.

(15) Let $X \subseteq 2^\omega$ be a $\subseteq$-maximal set of pairwise Turing incomparable reals. Show that $|X| = \mathfrak{c}$. Hint: Use the fact that there is a perfect set of reals of minimal Turing degrees.

(16) Show that there is an infinite Turing independent family of c.e. sets.

(17) Prove the converse of Lemma 14.20. If x_0 is 1-generic and x_1 is 1-generic relative to x_0, then $x_0 \oplus x_1$ is 1-generic.

(18) Let $x \in 2^\omega$ be 1-generic. Show that x does not compute any non-computable c.e. set. Conclude that there is a degree below $0'$ that is not c.e.

(19) Let $A \subseteq \omega$ be hyperimmune. Show that A is immune and $A \oplus A$ is hyperimmune. Recall that $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$.

(20) (J. Miller) Show that there are $A \subseteq B \subseteq C \subseteq \omega$ such that A, C are 1-generic and B isn't.

(21) Show that $\mathsf{Rec} = \{e : W_e \text{ is computable}\}$ is Σ^0_3-complete.

(22) Show that for every $A \in \Delta^0_2$, there exists $B \in \Delta^0_2$ such that $B \not\leq_m A$. Hence there is no Δ^0_2-complete set.

(23) Prove the relativized limit lemma (Exercise 15.12 in notes).

(24) For $A, B \subseteq \omega$, we write $A \subseteq^\star B$ iff $A \setminus B$ is finite and $A =^\star B$ iff $A \subseteq^\star B$ and $B \subseteq^\star A$. Define A to be a maximal set iff A is c.e. and coinfinite subset of ω and for every c.e. $B \subseteq \omega$, if $A \subseteq^\star B$, then either $A =^\star B$ or $B =^\star \omega$. Show that every maximal set is high.

(25) Call $d : \omega \to \omega$ a DNC (diagonally non-computable) function iff

$$(\forall n)(\varphi_n(n)\downarrow \implies d(n) \neq \varphi_n(n)).$$

Show that $\mu(\{x \in 2^\omega : x \text{ computes a DNC function}\}) = 1$.