MTH202: *Set Theory and Discrete Mathematics*
Homework: These will be periodically updated. Odd Semester, 2020/21

(1) Show that there is no set V such that every set is a member of V.

(2) Show that $(x, y) = (a, b)$ iff $x = a$ and $y = b$.

(3) Suppose R is an equivalence relation on A. For each $a \in A$, define the R-equivalence class of a by $[a] = \{b \in A : aRb\}$. Show that $\{[a] : a \in A\}$ is a partition of A. Furthermore, show that for every partition $\mathcal{F}$ of A, there is an equivalence relation S on A such that $\mathcal{F}$ is the set of all S-equivalence classes.

(4) Let $(L, \prec)$ be a linear ordering. Prove the following.

(a) $(L, \prec)$ is a well-ordering iff there is no sequence $\langle x_n : n < \omega \rangle$ in L such that $(\forall n < \omega)(x_{n+1} \prec x_n)$.

(b) $(L, \prec)$ is a well-ordering iff for every $A \subseteq L$, $(A, \prec)$ is isomorphic to an initial segment of $(L, \prec)$.

(5) Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well-orderings. Then exactly one of the following holds.

(a) $(X, \prec_1) \cong (Y, \prec_2)$.

(b) For some $x \in X$, $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.

(c) For some $y \in Y$, $(\mathsf{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

(6) Let $f : \mathcal{P}(\omega) \setminus \{\emptyset\} \to \omega$ be defined by $f(X) = \min(X)$. Call a well-orderings $(A, \prec)$ f-directed iff $A \subseteq \omega$ and for every $x \in A$,

$$f(\omega \setminus \mathsf{pred}(A, \prec, x)) = x$$

Describe all f-directed well-orderings.

(7) Prove the following.

(a) If x is an ordinal and $y \in x$, then y is an ordinal and $y = \mathsf{pred}(x, \in, y)$.

(b) If x, y are ordinals and $(x, \in) \cong (y, \in)$, then $x = y$.

(c) If x is an ordinal, then $x \notin x$.

(d) If x, y are ordinals, then exactly one of the following holds: $x = y$, $x \in y$, $y \in x$.

(e) If C is a nonempty set of ordinals, then there exists $x \in C$ such that $(\forall y \in C)(y = x \text{ or } x \in y)$.

(f) If A is a set of ordinals, then $(A, \in)$ is a well-ordering. Hence if A is a transitive set of ordinals, then A is an ordinal.

(8) Show that if $\alpha < \beta$ are ordinals, then there is a unique ordinal γ such that $\alpha + \gamma = \beta$. (**Hint**: $\gamma = \mathsf{type}(\beta \setminus \alpha, \in)$).

(9) Suppose α, β, γ are ordinals and $\alpha + \beta = \alpha + \gamma$. Show that $\beta = \gamma$.

(10) Suppose $\alpha \cdot \alpha = \beta \cdot \beta$. Show that $\alpha = \beta$.

(11) Show that there is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. (**Hint**: Identify ω with the set of rationals $\mathbb{Q}$ and for each real number x, consider $\{r \in \mathbb{Q} : r \leq x\}$).

(12) Call an ordinal α good iff there exists $X \subseteq \mathbb{R}$ such that $(X, <)$ is order isomorphic to α. Show that α is good iff $\alpha < \omega_1$.

(13) Let $(P, \preceq_1)$ be a partial ordering. Show that there exists $\preceq_2$ such that $(P, \preceq_2)$ is a linear ordering and $\preceq_2$ extends $\preceq_1$ which means the following:

$$(\forall a, b \in P)(a \preceq_1 b \implies a \preceq_2 b)$$

(14) Let $H \subseteq \mathbb{R}$ be a Hamel basis.

(a) Show that every nonzero $x \in \mathbb{R}$ can be uniquely written as

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

where $x_1 < x_2 < \cdots < x_n$ are in H and $a_1, a_2, \ldots a_n$ are nonzero rational numbers.

(b) Let $f : H \to \mathbb{R}$. Show that there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f \subseteq g$.

(15) Show that for every $f : \mathbb{R} \to \mathbb{R}$ there are injective functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.

(16) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

(a) Show that either f is identically zero or $\text{range}(f) \subseteq \mathbb{R}^+$.

(b) Suppose f is continuous and not identically zero. Show that $f(x) = a^x$ for some $a > 0$.

(17) Show that there is a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

(18) Prove the following.

(a) For every ordinal α, $|\alpha| \leq \alpha$.

(b) If κ is a cardinal and $\alpha < \kappa$, then $|\alpha| < \kappa$.

(c) There is an injection from X to Y iff $|X| \leq |Y|$.

(d) There is a surjection from X to Y iff $|Y| \leq |X|$.

(e) There is a bijection from X to Y iff $|X| = |Y|$.

(19) Prove the following.

(a) $|\mathbb{R}^\omega| = \mathfrak{c}$.

(b) $|C(\mathbb{R})| = \mathfrak{c}$ where $C(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(c) Let A be the set of all real numbers which are roots of some polynomial equation with rational coefficients. Show that $|A| = \omega$.

(20) Show that $\mathbb{R}^2$ cannot be partitioned into circles of positive radii.

(21) Show that $\mathbb{R}^3$ can be partitioned into circles of positive radii.

(22) Suppose $A \subseteq \mathbb{R}^2$ and every vertical section of A is finite. Show that some horizontal section of $\mathbb{R}^2 \setminus A$ is uncountable.

(24) Let $\mathcal{U}$ be a non-principal ultrafilter over ω. Suppose $\langle x_n : n < \omega \rangle$ is a sequence of real numbers. We say that x is the $\mathcal{U}$-limit of $\langle x_n : n < \omega \rangle$ and write $\mathcal{U}\lim_n x_n = x$ iff for every $\varepsilon > 0$, $\{n < \omega : |x_n - x| < \varepsilon\} \in \mathcal{U}$.

Show the following.

(a) If $\lim_n x_n = x$, then $\mathcal{U}\lim_n x_n = x$.

(b) If $\mathcal{U}\lim_n x_n = a$ and $\mathcal{U}\lim_n x_n = b$, then $a = b$.

(c) If $\langle x_n : n < \omega \rangle$ is bounded in $\mathbb{R}$, then there exists a unique real x such that $\mathcal{U}\lim_n x_n = x$.

(25) Let $\langle A_n : n < \omega \rangle$ be a sequence of infinite subsets of ω such that for every $n < \omega$, $A_{n+1} \subseteq A_n$. Define $\mathcal{F} = \{X \subseteq \omega : (\exists n < \omega)(A_n \subseteq X)\}$.

(a) Show that $\mathcal{F}$ is a filter on ω.

(b) Show that $\mathcal{F}$ is not an ultrafilter on ω.

(26) Let $(P, \preceq)$ be a partial ordering where P is an infinite set. Show that there exists an infinite $X \subseteq P$ such that either X is a chain in $(P, \preceq)$ or X is an antichain in P – This means that for every distinct $a, b \in X$ neither $a \preceq b$ nor $b \preceq a$.

(27) Suppose $1 \leq N < \omega$ and $h : \omega \to N$. Show that there exist $a < b < c < d < e < \omega$ such that $h(a) = h(b) = h(c) = h(d) = h(e)$ and $a + b + c + d = e$.

(28) Suppose $1 \leq N < \omega$ and $h : \omega \to N$. Show that there exist $a < b < c < d < e < \omega$ such that $h(a) = h(b) = h(c) = h(d) = h(e)$ and $abcd = e$.

(29) Suppose $S \subseteq \mathbb{R}^2$ and $\mathcal{F}$ is a family of circles in $\mathbb{R}^2$. Assume $|S| < \mathfrak{c}$, $|\mathcal{F}| < \mathfrak{c}$, S does not contain 11 concyclic points and $|S \cap C| = 10$ for every $C \in \mathcal{F}$. Show that for every circle $T \subseteq \mathbb{R}^2$, there exists $F \subseteq T$ such that $|F| \leq 10$, $S \cup F$ does not contain 11 concyclic points and $|(S \cup F) \cap T| = 10$.

(30) Find $\gcd(1887, 1295)$.

(31) Show that for every integer $n \geq 1$, $6 \mid n(n+1)(n+2)$ and $24 \mid n(n+1)(n+2)(n+3)$.

(32) Show that there are no positive integers a, b such that $a^2 = 17b^2$.

(33) Show that the equation $2x^2 - 5y^2 = 1$ has no integer solution.

(34) Show that $\gcd(a, b, c) = \min(\{ax + by + cz : x, y, z \in \mathbb{Z} \text{ and } ax + by + cz > 0\})$.

(35) Show that the minimum positive value of $112x - 105y + 49z$ for integers x, y, z is 7.

(36) Find the remainder when 554^{455} is divided by 5.

(37) Show that for some integer n, every member of $\{n + k : 1 \leq k \leq 2020\}$ is composite.

(38) Let a, b be positive integers. Define $\mathsf{lcm}(a, b)$ to be the least common positive multiple of a, b. Show that $\mathsf{lcm}(a, b)\gcd(a, b) = ab$.

(39) Find all the units in $\mathbb{Z}/20\mathbb{Z}$ and for each one of them, compute its multiplicative inverse.

(40) Show the following.

(a) Let $E_n = \{(a, b) : a \equiv b \pmod{n}\}$. Then E_n is an equivalence relation on $\mathbb{Z}$ whose equivalence classes are $\{[r] : 0 \leq r < n\}$.

(b) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$ and $a - c \equiv b - d \pmod{n}$.

(c) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

(d) If $a \equiv b \pmod{n}$ and $k \geq 1$, then $a^k \equiv b^k \pmod{n}$.

(e) If $\gcd(c, n) = 1$ and $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}$.

(41) Let a, b, c be positive integers. Show that the equation $ax + by = c$ has integers solutions iff $\gcd(a, b) \mid c$.

(42) Prove or disprove: If $a^2 \equiv b^2 \pmod{n}$, then either $a \equiv b \pmod{n}$ or $a \equiv -b \pmod{n}$.

(43) Let ϕ be the Euler's totient function. Show the following.

(a) $\phi(1) = 1$ and for every prime p, $\phi(p) = p - 1$.

(b) If p is prime and $k \geq 1$, then $\phi(p^k) = p^k - p^{k-1}$.

(c) If $\gcd(a, b) = 1$, then $\phi(ab) = \phi(a)\phi(b)$.

(d) For each $n \geq 2$, $\phi(n)$ is the number of units in $\mathbb{Z}/n\mathbb{Z}$.

(44) Suppose p is a prime and a, b are integers. Assume that neither a nor b is a multiple of p. Show that $p \mid (a^{p-1} - b^{p-1})$.

(45) Let $p \geq 3$ be a prime. Show that $p \mid (1^p + 2^p + \cdots + (p-1)^p)$.

(46) Let G be a finite graph. Show that the sum of the degrees of all the vertices in G is twice the number of its edges. This is called the handshaking lemma.

(47) Recall that a tree is a connected acyclic graph.

(a) Show that every tree with n vertices has exactly $n-1$ edges.

(b) A vertex in T is called a leaf iff it has degree one. Let T be a tree on 101 vertices. Suppose T has a vertex of degree 5. Show that T has at least 5 leaves.

(48) The complement of a graph $G = (V, E)$ is defined by $\overline{G} = (V, \overline{E})$ where $\overline{E} = [V]^2 \setminus E$. Show that for every graph G, either G or $\overline{G}$ is connected.

(49) Let G be a finite graph with $n \geq 3$ vertices and strictly more than $\binom{n-1}{2}$ edges. Show that G is connected.

(50) Let G be a connected graph on 100 vertices such that each vertex has degree at least 5. Show that G contains a path of length 10.

(51) Let G be a 5-regular finite bipartite graph with bipartition (A, B).

(a) Show that $|A| = |B|$.

(b) Show that G has a perfect matching.

(52) Show that for any partition of a deck of playing cards into 13 stacks of 4 cards each, we can choose one card from each stack in a way such that the resulting set of 13 cards consists of one card of each one of the ranks: Ace, 2, 3, ..., 10, Jack, Queen, King.

(53) Complete the proof of De Bruijn-Erdős theorem: Suppose $G = (V, E)$ is an infinite graph and $1 \leq n < \omega$. Suppose for every finite $A \subseteq V$, the chromatic number of the induced subgraph of G on A is at most n. Then $\chi(G) \leq n$.

(54) Show that every countable linear ordering is order-isomorphic to a subset of the set of rational numbers.

(55) Let $G = (V, E)$ be a Rado graph. Let $H = (V_1, E_1)$ be a graph such that V_1 is countable. Show that there exists an injective function $f : V_1 \to V$ such that for every $x, y \in V_1$,

$$\{x, y\} \in E_1 \iff \{f(x), f(y)\} \in E$$