MTH302: *Set Theory and Mathematical Logic*
Homework: These will be periodically updated. Odd Semester, 2021/22

(1) Show that there is no set V such that every set is a member of V.

(2) Show that $(x, y) = (a, b)$ iff $x = a$ and $y = b$.

(3) Suppose R is an equivalence relation on A. For each $a \in A$, define the R-equivalence class of a by $[a] = \{b \in A : aRb\}$. Show that $\{[a] : a \in A\}$ is a partition of A. Furthermore, show that for every partition $\mathcal{F}$ of A, there is an equivalence relation S on A such that $\mathcal{F}$ is the set of all S-equivalence classes.

(4) Let $(L, \prec)$ be a linear ordering. Prove the following.

(a) $(L, \prec)$ is a well-ordering iff there is no sequence $\langle x_n : n < \omega \rangle$ in L such that $(\forall n < \omega)(x_{n+1} \prec x_n)$.

(b) $(L, \prec)$ is a well-ordering iff for every $A \subseteq L$, $(A, \prec)$ is isomorphic to an initial segment of $(L, \prec)$.

(5) Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well-orderings. Then exactly one of the following holds.

(a) $(X, \prec_1) \cong (Y, \prec_2)$.

(b) For some $x \in X$, $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.

(c) For some $y \in Y$, $(\mathsf{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

(6) Let $f : \mathcal{P}(\omega) \setminus \{\emptyset\} \to \omega$ be defined by $f(X) = \min(X)$. Call a well-ordering $(A, \prec)$ f-directed iff $A \subseteq \omega$ and for every $x \in A$,

$$f(\omega \setminus \mathsf{pred}(A, \prec, x)) = x$$

Describe all f-directed well-orderings.

(7) Show that if $\alpha < \beta$ are ordinals, then there is a unique ordinal γ such that $\alpha + \gamma = \beta$. (**Hint**: $\gamma = \mathsf{type}(\beta \setminus \alpha, \in)$).

(8) Suppose α, β, γ are ordinals and $\alpha + \beta = \alpha + \gamma$. Show that $\beta = \gamma$.

(9) Suppose $\alpha \cdot \alpha = \beta \cdot \beta$. Show that $\alpha = \beta$.

(10) Show that there is an uncountable chain in $(\mathcal{P}(\omega), \subseteq)$. (**Hint**: Identify ω with the set of rationals $\mathbb{Q}$ and for each real number x, consider $\{r \in \mathbb{Q} : r \leq x\}$).

(11) Call an ordinal α good iff there exists $X \subseteq \mathbb{R}$ such that $(X, <)$ is order isomorphic to α. Show that α is good iff $\alpha < \omega_1$.

(12) Let $(P, \preceq_1)$ be a partial ordering. Show that there exists $\preceq_2$ such that $(P, \preceq_2)$ is a linear ordering and $\preceq_2$ extends $\preceq_1$ which means the following:

$$(\forall a, b \in P)(a \preceq_1 b \implies a \preceq_2 b)$$

(13) Suppose $f : \mathbb{R} \to \mathbb{R}$ is additive and $a = f(1)$.

(a) Show that $f(0) = 0$.
(b) Show that for every $x \in \mathbb{R}$, $f(-x) = -f(x)$.
(c) Show that for every $x \in \mathbb{Q}$, $f(x) = ax$.

(14) Let $H \subseteq \mathbb{R}$ be a Hamel basis.

(a) Show that every nonzero $x \in \mathbb{R}$ can be uniquely written as

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $x_1 < x_2 < \cdots < x_n$ are in H and $a_1, a_2, \ldots a_n$ are nonzero rational numbers. Uniqueness means the following: Suppose

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1y_1 + b_2y_2 + \cdots + a_my_m$$

where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ are in H and $a_1, \ldots a_n, b_1, \ldots, b_m$ are nonzero rationals. Show that $m = n$ and for every $1 \leq k \leq n$, $x_k = y_k$ and $a_k = b_k$.

(b) Let $f : H \to \mathbb{R}$. Show that there is a unique additive function $g : \mathbb{R} \to \mathbb{R}$ such that $f \subseteq g$.

(15) Show that for every $f : \mathbb{R} \to \mathbb{R}$ there are injective functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.

(16) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

(a) Show that either f is identically zero or range$(f) \subseteq \mathbb{R}^+$.

(b) Suppose f is continuous and not identically zero. Show that $f(x) = a^x$ for some $a > 0$.

(17) Show that there is a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

(18) Prove the following.

(a) For every ordinal α, $|\alpha| \leq \alpha$.
(b) If κ is a cardinal and $\alpha < \kappa$, then $|\alpha| < \kappa$.
(c) There is an injection from X to Y iff $|X| \leq |Y|$.
(d) There is a surjection from X to Y iff $|Y| \leq |X|$.
(e) There is a bijection from X to Y iff $|X| = |Y|$.

(19) Prove the following.

(a) $|\mathbb{R}^\omega| = \mathfrak{c}$.

(b) $|C(\mathbb{R})| = \mathfrak{c}$ where $C(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(c) Let A be the set of all real numbers which are roots of some polynomial equation with rational coefficients. Show that $|A| = \omega$.

(20) Show that $\mathbb{R}^2$ cannot be partitioned into circles of positive radii.

(21) Show that $\mathbb{R}^3$ can be partitioned into circles of positive radii.

(22) Suppose $A \subseteq \mathbb{R}^2$ and every vertical section of A is finite. Show that some horizontal section of $\mathbb{R}^2 \setminus A$ is uncountable.

(23) Let ϕ be a propositional formula in which $\neg$ doesn't occur. Show that ϕ is satisfiable.

(24) Suppose the set of propositional variables $\mathcal{V}ar$ is uncountable. Use Zorn's lemma to show the following: Let S be a set of propositional formulas such that every finite subset of S is satisfiable. Then S is satisfiable. **Hint**: Apply Zorn's lemma to $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is the set of all functions h such that $\text{dom}(h) \subseteq \mathcal{V}ar$, $\text{range}(h) \subseteq \{0, 1\}$ and for every finite $F \subseteq S$, there exists a valuation $val : \mathcal{V}ar \to \{0, 1\}$ such that $h \subseteq val$ and every formula in F is true under val.

(25) Let $\mathcal{L}, \mathcal{L}'$ be two first order languages where $\mathcal{L}'$ is obtained from $\mathcal{L}$ by adding a new constant symbol c to $\mathcal{L}$. Suppose T is an $\mathcal{L}$-theory, ϕ is an $\mathcal{L}$-formula with only free variable x, ψ is an $\mathcal{L}$-sentence and t is an $\mathcal{L}$-term with no variables. Show that the following hold.

(**UG**) If $T \vdash_{\mathcal{L}'} \phi(c/x)$, then $T \vdash_{\mathcal{L}} (\forall x)(\phi)$.

(**EI**) If $T \cup \{\phi(c/x)\} \vdash_{\mathcal{L}'} \psi$, then $T \cup \{(\exists x)(\phi)\} \vdash_{\mathcal{L}} \psi$.

(26) Suppose $\mathcal{L}, \mathcal{L}'$ are first order languages and $\mathcal{L}'$ extends $\mathcal{L}$. Let T be an $\mathcal{L}$-theory and ϕ be an $\mathcal{L}$-sentence. Then $T \vdash_{\mathcal{L}} \phi$ iff $T \vdash_{\mathcal{L}'} \phi$.

(27) Suppose T is a maximally consistent $\mathcal{L}$-theory and ϕ, ψ are $\mathcal{L}$-sentences. Show the following.

(a) $T \vdash \phi$ iff $\phi \in T$.

(b) $\neg\phi \in T$ iff $\phi \notin T$.

(c) $(\phi \wedge \psi) \in T$ iff $\phi \in T$ and $\psi \in T$.

(d) $(\phi \vee \psi) \in T$ iff either $\phi \in T$ or $\psi \in T$.

(e) $(\phi \implies \psi) \in T$ iff either $\psi \in T$ or $\phi \notin T$.

(f) $(\phi \iff \psi) \in T$ iff "$\phi \in T$ iff $\psi \in T$".

(28) Suppose T is a consistent complete $\mathcal{L}$-theory. Let S be the set all $\mathcal{L}$-sentences ϕ such that $T \vdash \phi$. Show that S is a maximally consistent $\mathcal{L}$-theory.

(29) Let $\mathcal{L} = \mathcal{L}_{PA} \cup \{c\}$ where c is a new constant symbol. Let $\mathsf{Primes} = \{2, 3, 5, 7, \dots\}$ be the set of all primes numbers. For each $p \in \mathsf{Primes}$, let "p $\mathsf{divides}$ c" denote the $\mathcal{L}$-sentence $(\exists y)(S^p(0) \cdot y = c)$. For each $X \subseteq \mathsf{Primes}$, let T_X be the $\mathcal{L}$-theory

$$T_X = TA \cup \{(p \text{ divides } c) : p \in X\} \cup \{\neg(p \text{ divides } c) : p \in \mathsf{Primes} \setminus X\}$$

where $TA = Th(\omega, 0, S, +, \cdot)$ denotes true arithmetic.

(a) Show that T_X is consistent for every $X \subseteq \mathsf{Primes}$.

(b) Show that TA has continuum many pairwise non-isomorphic countable models.

(30) Show that every countable linear ordering is isomorphic to a subordering of the rationals $(\mathbb{Q}, <)$.

(31) Let $W \subseteq \omega$. Show that W is c.e. iff there exists a computable function $f : \omega \to \omega$ such that range$(f) = W$.

(32) Suppose $r_1, r_2, \dots, r_n, d_1, d_2, \dots, d_n$ are natural numbers and for every $1 \leq i \leq n$, $0 \leq r_i < d_i$. Assume that for every $1 \leq i < j \leq n$, d_i and d_j are relatively prime. Show that there exists a positive integer N such that for every $1 \leq i \leq n$, $rem(N, d_i) = r_i$.

(33) Let $W \subseteq \omega$ be nonempty. Show that W is c.e. iff there exists a computable $A \subseteq \omega^2$ such that $W = \{n \in \omega : (\exists m)((n, m) \in A)\}$.

(34) Suppose $X \subseteq \omega$ is numeralwise representable in PA. Show that X is computable.

(35) Let $H \subseteq \omega$ be a non-computable c.e. set. Show that H is definable in $\mathcal{N} = (\omega, 0, S, +, \cdot)$ but not numeralwise representable in PA.

(36) Do the Exercise on Lecture slide 202.