# MTH404: Analysis II

LECTURE NOTES

# $\sigma$-ideals

## Definition (Ideals)

*Let $X$ be a nonempty set. We say that $\mathcal{I}$ is an ideal on $X$ iff $\mathcal{I}$ is a family of subsets of $X$ satisfying the following.*

(i) $\emptyset \in \mathcal{I}$ *and* $X \notin \mathcal{I}$.

(ii) *If* $A \subseteq B \subseteq X$ *and* $B \in \mathcal{I}$, *then* $A \in \mathcal{I}$.

(iii) *If* $A, B \in \mathcal{I}$, *then* $A \cup B \in \mathcal{I}$. *So* $\mathcal{I}$ *is closed under finite unions.*

## Definition (Sigma-ideals)

*$\mathcal{I}$ is a $\sigma$-ideal on $X$ iff $\mathcal{I}$ is an ideal on $X$ and for every countable $\mathcal{A} \subseteq \mathcal{I}$, $\bigcup \mathcal{A} \in \mathcal{I}$.*

**Example**: Let $X$ be an uncountable set and $\mathcal{I}$ be the family of all countable subsets of $X$. Then $\mathcal{I}$ is a $\sigma$-ideal on $X$.

# Meager sets

Let $(X, d)$ be a metric space and $D \subseteq X$.

(a) $D$ is **dense in** $X$ iff for every nonempty open $U \subseteq X$, $D \cap U \neq \emptyset$.

(b) $D$ is **nowhere dense in** $X$ iff for every nonempty open $U \subseteq X$, there exists a nonempty open $V \subseteq X$ such that $V \subseteq U$ and $V \cap D = \emptyset$.

(c) $D$ is **open dense in** $X$ iff it is both open and dense in $X$.

(d) $D$ is **meager in** $X$ iff there exists a countable family $\{D_n : n \geq 1\}$ such that each $D_n$ is nowhere dense in $X$ and $D \subseteq \bigcup\{D_n : n \geq 1\}$.

(e) For historical reasons, some people also write "$D$ is of the **first category in** $X$" instead of "$D$ is meager in $X$" and "$D$ is of the **second category in** $X$" instead of "$D$ is non-meager in $X$".

**Exercise**: Let $(X, d)$ be a metric space and $D \subseteq X$. Show that the following are equivalent.

(1) $D$ is nowhere dense in $X$.

(2) $cl(D)$ (closure of $D$) is nowhere dense in $X$.

(3) $X \setminus D$ contains an open dense subset of $X$.

# Baire Category Theorem

Recall that a metric space $(X, d)$ is complete iff every Cauchy sequence in $X$ converges to some point in $X$.

## Theorem (Baire Category Theorem)

*Suppose $(X, d)$ is a complete metric space and $\{D_n : n \geq 1\}$ is a countable family of nowhere dense subsets of $X$. Then $X \setminus \bigcup\{D_n : n \geq 1\}$ is dense in $X$.*

**Proof**: Put $D = \bigcup\{D_n : n \geq 1\}$. Let $U$ be a nonempty open subset of $X$. We must show that $U \setminus D \neq \emptyset$. Indutively, choose a sequence $\langle B_n : n \geq 1\rangle$ of open balls in $X$ as follows. $B_1$ is an open ball of radius $\leq 1$ such that $cl(B_1) \subseteq U$ and $B_1 \cap D_1 = \emptyset$. This can be done because $D_1$ is nowhere dense in $X$. Having chosen $B_n$, let $B_{n+1}$ be an open ball of radius $\leq 2^{-n}$ such that $cl(B_{n+1}) \subseteq B_n$ and $B_{n+1} \cap D_{n+1} = \emptyset$. Once again, we are using the fact that $D_{n+1}$ is nowhere dense in $X$. Note that $B_{n+1} \subseteq B_n \subseteq U$ for every $n \geq 1$. Let $a_n$ be the center of $B_n$. Since the radii of $B_n$ converge to 0, it follows that $\langle a_n : n \geq 1\rangle$ is a Cauchy sequence in $X$. Since $X$ is complete, this sequence converges to say $a \in X$. It is easy to see that $a \in U \setminus D$. It follows that $X \setminus D$ is dense in $X$. □

# Meager ideal

## Corollary

*Let $(X, d)$ be a complete metric space and $\mathcal{M}$ be the family of all meager subsets of $X$. Then $\mathcal{M}$ is a $\sigma$-ideal on $X$.*

**Proof**: The only nontrivial thing to check is $X \notin \mathcal{M}$. But this follows from the Baire Category theorem. $\square$

## Lemma

*Let $\mathcal{M}$ be the meager ideal on $\mathbb{R}^n$. Let $\mathcal{I}$ be the $\sigma$-ideal of all countable subsets of $\mathbb{R}^n$. Then $\mathcal{I}$ is a proper subideal of $\mathcal{M}$.*

**Proof**: Note that $\{x\}$ is nowhere dense in $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$. So $\mathcal{I} \subseteq \mathcal{M}$.
Next suppose $n = 1$. Let $A$ be the set of all $x \in [0, 1]$ whose decimal expansion contains only two digits: $0, 1$. Then it is easy to see that $A$ is uncountable and nowhere dense in $\mathbb{R}$. So $A \in \mathcal{M} \setminus \mathcal{I}$.
Finally suppose $n \geq 2$. Let $A$ be a line in $\mathbb{R}^n$. Then $A$ is uncountable and nowhere dense in $\mathbb{R}^n$. So $A \in \mathcal{M} \setminus \mathcal{I}$. It follows that $\mathcal{I}$ is a proper subideal of $\mathcal{M}$. $\square$

# Null subsets of $\mathbb{R}$

## Definition (Null sets)

*Let $X \subseteq \mathbb{R}$. We say that $X$ is Lebesgue null (or just null) iff for every $\varepsilon > 0$, there exists a countable family $\{J_k : k \geq 1\}$ of open intervals in $\mathbb{R}$ such that*

(a) $X \subseteq \bigcup\{J_k : k \geq 1\}$ *and*

(b) $\sum_{k\geq 1} \mathit{length}(J_k) < \varepsilon$.

## Lemma

*Suppose $X_n \subseteq \mathbb{R}$ is null for each $n \geq 1$. Then $\bigcup\{X_n : n \geq 1\}$ is also null.*

**Proof**: Put $X = \bigcup\{X_n : n \geq 1\}$. Let $\varepsilon > 0$. Since each $X_n$ is null, we can find a countable family $\{J_{n,k} : k \geq 1\}$ of open intervals such that
$X_n \subseteq \bigcup\{J_{n,k} : k \geq 1\}$ and $\sum_{k\geq 1} \text{length}(J_{n,k}) < \varepsilon/2^n$.
Since the union of a countable family of countable sets is countable, the family $\mathcal{F} = \{J_{n,k} : n, k \geq 1\}$ is countable. Let $\{I_k : k \geq 1\}$ enumerate all members of $\mathcal{F}$. Then $X \subseteq \bigcup\{I_k : k \geq 1\}$ and $\sum_{k\geq 1} \text{length}(I_k) < \sum_{n\geq 1} \varepsilon/2^n = \varepsilon$. It follows that $X$ is null. $\square$

# Null ideal on $\mathbb{R}$

## Lemma

(i) *Every countable $X \subseteq \mathbb{R}$ is null.*

(ii) *For every $a < b$, the interval $(a, b)$ is not null.*

(iii) *The ternary Cantor set is an uncountable null set.*

**Proof**: See Homework. $\square$

## Corollary

*Let $\mathcal{N}$ be the family of Lebesgue null subsets of $\mathbb{R}$. Then $\mathcal{N}$ is a $\sigma$-ideal on $\mathbb{R}$ that properly extends the $\sigma$-ideal of countable subsets of $\mathbb{R}$.*

**Proof**: Follows from the previous two lemmas. $\square$

# Null vs Meager

The following theorem says that the null and the meager ideals on $\mathbb{R}$ are orthogonal in the following sense.

## Theorem

*There is a partition $A \sqcup B = \mathbb{R}$ such that $A$ is null and $B$ is meager.*

**Proof**: Let $\{a_k : k \geq 1\}$ be an enumeration of all rationals. For each $n \geq 1$, let

$$U_n = \bigcup_{k \geq 1} \left( a_k - 2^{-(n+k)}, a_k + 2^{-(n+k)} \right)$$

Define $A = \bigcap \{U_n : n \geq 1\}$ and $B = \mathbb{R} \setminus A$. The reader should check that $A, B$ are as required. $\square$

# What is a measure?

Our starting point is the following question. Are there any interesting generalizations of the notions of length/area/volume to arbitrary subsets of $\mathbb{R}/\mathbb{R}^2/\mathbb{R}^3$?

To simplify matters, let us try to extend the notion of "length" to arbitrary subsets of $\mathbb{R}$. So we are looking for a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ that satisfies some desirable properties. Here's a list of such properties.

(1) $m((a, b)) = b - a$ for every $a < b$ in $\mathbb{R}$.

(2) (Isometric invariant) If $A$ is congruent to $B$, then $m(A) = m(B)$.

(3) (Countably additive) For every countable family $\{A_n : n \geq 1\}$ of pairwise disjoint subsets of $\mathbb{R}$, $m\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} m(A_n)$.

Unfortunately, there is no such generalization.

## Theorem (Vitali, 1905)

*There is no* $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ *that satisfies* $(1) - (3)$ *above.*

# Vitali's obstruction

First note that any $m$ satisfying (3) is also monotone is the following sense: If $A \subseteq B \subseteq \mathbb{R}$, then $m(A) \leq m(B)$.

Towards a contradiction, suppose there is such an $m$. Define a binary relation $E$ on $\mathbb{R}$ by $aEb$ iff $a - b \in \mathbb{Q}$. It is easy to check that $E$ is an equivalence relation on $\mathbb{R}$. For each $a \in \mathbb{R}$, let $[a]_E$ denote the $E$-equivalence class of $a$. Then $[a]_E = \mathbb{Q} + a$ and $\mathcal{F} = \{[a]_E : a \in \mathbb{R}\}$ is a partition of $\mathbb{R}$. Since each $[a]_E = a + \mathbb{Q}$ is dense in $\mathbb{R}$, the sets $[a]_E \cap [0,1]$ are all nonempty. Therefore, using the axiom of choice, we can find $V \subseteq [0,1]$ such that $V$ intersects each member of $\mathcal{F}$ at exactly one point. Observe that if $a \neq b$ are rationals, then $V + a$ and $V + b$ are disjoint.

We first claim that $m(V) > 0$. Suppose not. Then by properties (2) and (3), $\infty = m(\mathbb{R}) = \sum_{r \in \mathbb{Q}} m(V + r) = 0$ which is a contradiction. So $m(V) > 0$.

Define $W = \bigcup\{V + r : r \in \mathbb{Q} \cap [0,1]\}$. Since $V \subseteq [0,1]$, we get $W \subseteq [0,2]$. Now $m(W) = m(\bigcup\{V + r : r \in \mathbb{Q} \cap [0,1]\}) = \sum_{r \in \mathbb{Q} \cap [0,1]} m(V + r) = \infty$. But $W \subseteq [0,2]$. So $2 = m([0,2]) \geq m(W) = \infty$ which is a contradiction. It follows that no such $m$ exists. $\square$

# Banach Measure Problem

Banach measure problem asks the following. Is there a function $m : \mathcal{P}([0,1]) \to [0,1]$ that satisfies the following?

(1) For every $0 \leq a < b \leq 1$, $m((a,b)) = b - a$.

(2) (Countably additive) For every countable family $\{A_n : n \geq 1\}$ of pairwise disjoint subsets of $[0,1]$,

$$m\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} m(A_n)$$

Banach and Kuratowski (1920) showed that under the continuum hypothesis (CH), there is no such $m$. Godel (1938) showed that CH cannot be disproved in ZFC. Solovay (1971) showed that it is consistent with ZFC that there is such an $m$. So Banach's measure problem is undecidable in ZFC.

# Lebesgue outer measure on $\mathbb{R}$

## Definition (Lebesgue outer measure on $\mathbb{R}$)

*Define* $\mu^\star : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ *as follows.*

$$\mu^\star(X) = \inf \left\{ \sum_{n \geq 1} \mathit{length}(J_n) : \langle J_n : n \geq 1 \rangle \text{ is a sequence of open intervals such that } X \subseteq \bigcup_{n \geq 1} J_n \right\}$$

## Lemma

(1) $X \subseteq \mathbb{R}$ *is Lebesgue null iff* $\mu^\star(X) = 0$.

(2) *(Translation invariant) For every* $X \subseteq \mathbb{R}$ *and* $t \in \mathbb{R}$, $\mu^\star(X + t) = \mu^\star(X)$.

(3) *(Monotone) If* $X \subseteq Y \subseteq \mathbb{R}$, *then* $\mu^\star(X) \leq \mu^\star(Y)$.

(4) *(Countably subadditive) If* $X_n \subseteq \mathbb{R}$ *for each* $n \geq 1$ *and* $X = \bigcup_{n \geq 1} X_n$, *then*

$$\mu^\star(X) \leq \sum_{n \geq 1} \mu^\star(X_n)$$

# Lebesgue outer measure on $\mathbb{R}$

**Proof**: Facts (1), (2) and (3) are immediate from the definition of $\mu^\star$. Let us check (4). Suppose $X_n \subseteq \mathbb{R}$ for each $n \geq 1$ and $X = \bigcup_{n\geq 1} X_n$. We can assume that $\mu^\star(X_n) < \infty$ for every $n \geq 1$, otherwise the inequality is trivial. Let $\varepsilon > 0$ be arbitrary. For each $n \geq 1$, choose a sequence $\langle J_{n,k} : k \geq 1\rangle$ of open intervals such that $X_n \subseteq \bigcup_{k\geq 1} J_{n,k}$ and

$$\sum_{k\geq 1} \text{length}(J_{n,k}) < \mu^\star(X_n) + \varepsilon/2^n$$

Let $\langle I_k : k \geq 1\rangle$ enumerate all intervals in the countable family $\{J_{n,k} : k, n \geq 1\}$. Then

$$\sum_{k\geq 1} \text{length}(I_k) \leq \sum_{n\geq 1}\sum_{k\geq 1} \text{length}(J_{n,k}) < \sum_{n\geq 1}(\mu^\star(X_n) + \varepsilon/2^n) = \varepsilon + \sum_{n\geq 1}\mu^\star(X_n)$$

Since $X \subseteq \bigcup_{k\geq 1} I_k$, it follows that $\mu^\star(X) \leq \varepsilon + \sum_{n\geq 1}\mu^\star(X_n)$. As this inequality holds for all $\varepsilon > 0$, we must have

$$\mu^\star(X) \leq \sum_{n\geq 1}\mu^\star(X_n)$$

$\square$

# Lebesgue outer measure on $\mathbb{R}$

## Lemma

*For every closed interval $J \subseteq \mathbb{R}$, $\mu^\star(J) = length(J)$.*

**Proof**: Let $J = [a, b]$ where $-\infty < a < b < \infty$. For each $\varepsilon > 0$, the open intervals $(a, b), (a - \varepsilon, a + \varepsilon)$ and $(b - \varepsilon, b + \varepsilon)$ cover $[a, b]$ and the sum of their lengths is $(b - a) + 2\varepsilon$. So $\mu^\star(J) \leq b - a$. For the other inequality, suppose $\langle J_n : n \geq 1 \rangle$ is a sequence of open intervals that cover $[a, b]$. Since $[a, b]$ is compact, finitely many of $J_n$'s already cover $[a, b]$. Fix $k \geq 1$ such that $[a, b] \subseteq \bigcup_{n \leq k} J_n$. Now use induction on $k$ to show that

$$\sum_{n \leq k} \text{length}(J_n) \geq b - a$$

Hence $\sum_{n \geq 1} \text{length}(J_n) \geq b - a$. It follows that $\mu^\star(J) \geq b - a$ and we are done. □

## Corollary

*For every open interval $J \subseteq \mathbb{R}$, $\mu^\star(J) = length(J)$.*

**Proof**: Exercise. □

# Lebesgue outer measure on $\mathbb{R}^n$

## Definition (Open boxes, Volumes)

*A subset $B \subseteq \mathbb{R}^n$ is an open n-box iff there are bounded open intervals $J_1, J_2, \ldots, J_n$ in $\mathbb{R}$ such that $B = J_1 \times J_2 \times \cdots \times J_n$. We define the n-volume of $B$ by*

$$vol_n(B) = \prod_{1 \leq k \leq n} length(J_k)$$

## Definition (Lebesgue outer measure on $\mathbb{R}^n$)

*Define $\mu_n^\star : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ as follows.*

$$\mu_n^\star(X) = \inf \Big\{ \sum_{n \geq 1} vol_n(B_n) : \langle B_n : n \geq 1 \rangle \text{ is a sequence of } n\text{-boxes such that } X \subseteq \bigcup_{n \geq 1} B_n \Big\}$$

# Lebesgue outer measure on $\mathbb{R}^n$

The following lemma can be proved exactly like the one for 1-dimensional Lebesgue outer measure.

## Lemma

(1) *(Translation invariant) For every $X \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$, $\mu_n^\star(X+t) = \mu_n^\star(X)$.*

(3) *(Monotone) If $X \subseteq Y \subseteq \mathbb{R}^n$, then $\mu_n^\star(X) \leq \mu_n^\star(Y)$.*

(4) *(Countably subadditive) If $X_m \subseteq \mathbb{R}^n$ for each $m \geq 1$ and $X = \bigcup_{m\geq 1} X_m$, then*

$$\mu_n^\star(X) \leq \sum_{m\geq 1} \mu_n^\star(X_m)$$

We will sometimes write $\mu^\star$ instead of $\mu_n^\star$ if the dimension $n$ is clear from the context.

$\mu^\star$ is a highly non-additive function: A result of Lusin says that for every $X \subseteq \mathbb{R}^n$, there exists a partition $X = A \sqcup B$ such that $\mu^\star(A) = \mu^\star(B) = \mu^\star(X)$. But Caratheodory showed that there is a reasonably "large family" $\mathcal{M}$ of subsets of $\mathbb{R}^n$ such that $\mu^\star \restriction \mathcal{M}$ is countably additive. His arguments for proving this work in a much more general setting that will now be described.

# Abstract outer measures and measurable sets

## Definition (Outer measure)

*Let $X$ be a nonempty set. An outer measure on $X$ is a function $m : \mathcal{P}(X) \to [0, \infty]$ satisfying the following.*

(1) $m(\emptyset) = 0$.

(2) *(Monotone) If $A \subseteq B \subseteq X$, then $m(A) \leq m(B)$.*

(3) *(Countably subadditive) If $A_n \subseteq X$ for every $n \geq 1$ and $A = \bigcup_{n\geq 1} A_n$, then $m(A) \leq \sum_{n\geq 1} m(A_n)$*

## Definition (Caratheodory's criterion)

*Suppose $m$ is an outer measure on $X$. We say that $E \subseteq X$ is $m$-measurable iff for every $A \subseteq X$, $m(A) = m(A \cap E) + m(A \setminus E)$.*

**Remark**: Note that $m(A) \leq m(A \cap E) + m(A \setminus E)$ follows from the fact that $m$ is countably subadditive. Therefore, **to show that $E \subseteq X$ is $m$-measurable it suffices to show that $m(A) \geq m(A \cap E) + m(A \setminus E)$ for every $A \subseteq X$.**

# Caratheodory's theorem

## Theorem (Caratheodory)

*Suppose $m$ is an outer measure on $X$. Let $\mathcal{M} = \{E \subseteq X : E \text{ is } m\text{-mesurable}\}$.*

(1) $\emptyset \in \mathcal{M}$ *and* $X \in \mathcal{M}$. *If* $m(E) = 0$, *then* $E \in \mathcal{M}$.

(2) *If* $E \in \mathcal{M}$, *then* $X \setminus E \in \mathcal{M}$.

(3) *If* $E_1, E_2 \in \mathcal{M}$, *then* $E_1 \cup E_2 \in \mathcal{M}$.

(4) *If* $E_1, E_2 \in \mathcal{M}$ *are disjoint, then* $m(E_1 \cup E_2) = m(E_1) + m(E_2)$.

(5) *If* $E_n \in \mathcal{M}$ *for every* $n \geq 1$, *then* $\bigcup_{n \geq 1} E_n \in \mathcal{M}$.

(6) *If* $\langle E_n : n \geq 1 \rangle$ *is a sequence of pairwise disjoint sets in* $\mathcal{M}$ *and* $E = \bigcup_{n \geq 1} E_n$, *then* $m(E) = \sum_{n \geq 1} m(E_n)$.

**Proof**: (1) and (2) are obvious from the definition of $m$-measurable.

(3) Assume $E_1, E_2 \in \mathcal{M}$. For $W \subseteq X$, we will write $W^c$ (complement of $W$) for $X \setminus W$. Let $A \subseteq X$ be arbitrary. As $E_1, E_2 \in \mathcal{M}$, we have
$m(A) = m(A \cap E_1) + m(A \cap E_1^c) =$
$= m(A \cap E_1 \cap E_2) + m(A \cap E_1 \cap E_2^c) + m(A \cap E_1^c \cap E_2) + m(A \cap E_1^c \cap E_2^c)$.

# Caratheodory's theorem

Since $E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c) \cup (E_1^c \cap E_2)$, after intersecting both sides with $A$ and applying countable subadditivity of $m$, we get
$m(A \cap (E_1 \cup E_2)) \leq m(A \cap (E_1 \cap E_2)) + m(A \cap (E_1 \cap E_2^c)) + m(A \cap (E_1^c \cap E_2))$.
Adding $m(A \cap (E_1 \cup E_2)^c)$ on both sides and using the fact that $(E_1 \cup E_2)^c = E_1^c \cap E_2^c$, we get $m(A \cap (E_1 \cup E_2)) + m(A \cap (E_1 \cup E_2)^c) \leq m(A)$.
As $A \subseteq X$ was arbitrary, it follows that $E_1 \cup E_2 \in \mathcal{M}$.

(4) Since $E_1 \cap E_2 = \emptyset$, we get $(E_1 \cup E_2) \cap E_1^c = E_2$. Using the fact that $E_1$ is $m$-measurable, we get
$m(E_1 \cup E_2) = m((E_1 \cup E_2) \cap E_1) + m((E_1 \cup E_2) \cap E_1^c) = m(E_1) + m(E_2)$.

(5) Suppose $E_n \in \mathcal{M}$ for every $n \geq 1$. Define $F_1 = E_1$ and for $n \geq 2$, $F_n = E_n \cap (E_1 \cup \cdots \cup E_{n-1})^c$. Then $F_n$'s are pairwise disjoint members of $\mathcal{M}$ (by clauses (2)-(3) above) and $\bigcup_{n \geq 1} E_n = \bigcup_{n \geq 1} F_n$. It follows that to prove (5), it suffices to show that $\mathcal{M}$ is closed under countable unions of pairwise disjoint sets.

# Caratheodory's theorem

So assume that $E_n$'s are pairwise disjoint members of $\mathcal{M}$. Put $E = \bigcup_{n \geq 1} E_n$, $G_0 = \emptyset$ and $G_k = E_1 \cup \cdots \cup E_k$ for each $k \geq 1$. Since $E_k \in \mathcal{M}$, for any $A \subseteq X$, we have $m(A \cap G_k) = m((A \cap G_k) \cap E_k) + m((A \cap G_k) \cap E_k^c) =$
$= m(A \cap E_k) + m(A \cap G_{k-1})$ for every $k \geq 1$. It follows that $m(A \cap G_k) = \sum_{n \leq k} m(A \cap E_n)$. Hence

$$m(A) = m(A \cap G_k) + m(A \cap G_k^c) \geq \sum_{n \leq k} m(A \cap E_n) + m(A \cap E^c)$$

Letting $k \to \infty$, we get $m(A) \geq \sum_{n \geq 1} m(A \cap E_n) + m(A \cap E^c)$. Using countable subadditivity of $m$, we have

$$m(A) \geq \sum_{n \geq 1} m(A \cap E_n) + m(A \cap E^c) \geq m(A \cap E) + m(A \cap E^c)$$

So $m(A) \geq m(A \cap E) + m(A \cap E^c)$ which implies that $E \in \mathcal{M}$ and that all inequalities above are equalities. Plugging $A = E$ gives $\sum_{n \geq 1} m(E_n) = m(E)$. This proves both (5) and (6). $\square$

# Sigma-algebras and measurable spaces

The family $\mathcal{M}$ of $m$-measurable sets in Caratheodory's theorem is an example of a $\sigma$-algebra.

## Definition (Sigma-algebra)

*Let $X$ be a nonempty set. A $\sigma$-algebra on $X$ is a family $\mathcal{F}$ of subsets of $X$ that satisfies the following.*

(1) $\emptyset \in \mathcal{F}$ *and* $X \in \mathcal{F}$.

(2) *(Closed under complements) If* $E \in \mathcal{F}$, *then* $X \setminus E \in \mathcal{F}$.

(3) *(Closed under countable unions) If* $E_n \in \mathcal{F}$ *for every* $n \geq 1$, *then* $\bigcup_{n\geq 1} E_n \in \mathcal{F}$.

## Definition (Measurable space)

*A measurable space is a pair* $(X, \mathcal{F})$ *where* $X$ *is a nonempty set and* $\mathcal{F}$ *is a* $\sigma$-*algebra on* $X$.

# Measures and measure spaces

Suppose $\mathcal{F}$ is a $\sigma$-algebra on a nonempty set $X$ and $m : \mathcal{F} \to [0, \infty]$. We say that $m$ is a **measure** iff the following hold.

(1) $m(\emptyset) = 0$.

(2) (Countably additive) For every sequence $\langle E_n : n \geq 1 \rangle$ of pairwise disjoint sets in $\mathcal{F}$,

$$m(\bigcup_{n\geq 1} E_n) = \sum_{n\geq 1} m(E_n)$$

If $m(X) < \infty$, we say that $m$ is a **finite measure**. If $m(X) = 1$, we say that $m$ is a **probability measure**.

## Definition (Measure space)

*A measure space is a triplet* $(X, \mathcal{F}, \nu)$ *where* $(X, \mathcal{F})$ *is a measurable space and* $\nu : \mathcal{F} \to [0, \infty]$ *is a measure.*

**Example**: Let $m : \mathcal{P}(X) \to [0, \infty]$ be an outer measure on $X$. Let $\mathcal{M}$ be the family of all $m$-measurable sets and $\nu = m \restriction \mathcal{M}$. Then $(X, \mathcal{M}, \nu)$ is a measure space.

# Lebesgue measure on $\mathbb{R}^n$

## Definition (Lebesgue measure)

*Recall that $\mu_n^\star$ is an outer measure on $\mathbb{R}^n$. We say that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable iff $E$ is $\mu_n^\star$-measurable (Caratheodory criterion). Let $\mathcal{M}_n$ denote that family of all Lebesgue measurable subsets of $\mathbb{R}^n$. By Caratheodory's theorem, $\mathcal{M}_n$ is a $\sigma$-algebra on $\mathbb{R}^n$ and $\mu_n^\star \restriction \mathcal{M}_n$ is a measure. We denote the restricted map $\mu_n^\star \restriction \mathcal{M}_n$ by $\mu_n$ and call it Lebesgue measure. So a set is Lebesgue measurable iff it belongs to $dom(\mu_n) = \mathcal{M}_n$. The triplet $(\mathbb{R}^n, dom(\mu_n), \mu_n)$ is called Lebesgue measure space.*

# Intervals are Lebesgue measurable

## Lemma

*Every open interval $J \subseteq \mathbb{R}$ is Lebesgue measurable.*

**Proof**: Let $J = (a, b)$. We need to show that for every $A \subseteq \mathbb{R}$, $\mu^\star(A) \geq \mu^\star(A \cap J) + \mu^\star(A \setminus J)$. If $\mu^\star(A \cap J) + \mu^\star(A \setminus J) = \infty$, this is clear as $\mu^\star$ is monotone. So assume both $\mu^\star(A \cap J)$ and $\mu^\star(A \setminus J)$ are finite. If $\mu^\star(A) = \infty$, we are done. So assume $\mu^\star(A)$ is also finite. Let $\varepsilon > 0$ be arbitrary. Choose a sequence of open intervals $\langle J_n : n \geq 1 \rangle$ whose union contains $A$ such that $\sum_{n \geq 1} \text{length}(J_n) < \mu^\star(A) + \varepsilon$. Let $I_n = J_n \cap (a, b)$, $K_{1,n} = J_n \cap (-\infty, a)$ and $K_{2,n} = (b, \infty)$. Then $A \cap J \subseteq \bigcup_{n \geq 1} I_n$ and $A \setminus J \subseteq \bigcup_{n \geq 1} (K_{1,n} \cup K_{2,n}) \cup \{a, b\}$. Therefore, $\mu^\star(A \cap J) \leq \sum_{n \geq 1} \text{length}(I_n)$ and $\mu^\star(A \setminus J) \leq \sum_{n \geq 1} (\text{length}(K_{1,n}) + \text{length}(K_{2,n}))$. Now adding these two inequalities and using $\text{length}(J_n) = \text{length}(I_n) + \text{length}(K_{1,n}) + \text{length}(K_{2,n})$, we get

$$\mu^\star(A \cap J) + \mu^\star(A \setminus J) \leq \sum_{n \geq 1} \text{length}(J_n) < \mu^\star(A) + \varepsilon$$

Letting $\varepsilon \to 0$, we get $\mu^\star(A) \geq \mu^\star(A \cap J) + \mu^\star(A \setminus J)$. It follows that $J$ is Lebesgue measurable. $\square$

# Open sets are Lebesgue measurable

The following can be proved just like the previous Lemma. We omit the tedious details.

## Lemma

*Every open n-box $B \subseteq \mathbb{R}^n$ is Lebesgue measurable.*

Let $U \subseteq \mathbb{R}^n$ be open. Let $\mathcal{F}$ be the family of all open $n$-boxes $B = J_1 \times \cdots \times J_n$ where $B \subseteq U$ and $J_1, \ldots, J_n$ are open intervals with **rationals end points**. Then each member of $\mathcal{F}$ is Lebesgue measurable and since $\mathcal{F}$ is countable, $\bigcup \mathcal{F} = U$ is also Lebesgue measurable. So we have the following.

## Corollary

*Every open $U \subseteq \mathbb{R}^n$ is Lebesgue measurable.*

# $\sigma$-algebra generated by a family of sets

Suppose $X$ is a nonempty set and $\mathcal{A}$ is a collection of subsets of $X$. The $\sigma$-**algebra generated by** $\mathcal{A}$ is defined to be the smallest (under inclusion) $\sigma$-algebra $\mathcal{F}$ on $X$ such that $\mathcal{A} \subseteq \mathcal{F}$.

**Exercise**: Suppose $X$ is a nonempty set and $\mathcal{A}$ is a collection of subsets of $X$. Show that the $\sigma$-algebra generated by $\mathcal{A}$ is the intersection of all $\sigma$-algebras on $X$ that contain $\mathcal{A}$.

# Borel subsets of $\mathbb{R}^n$

The Borel $\sigma$-algebra on $\mathbb{R}^n$ is the $\sigma$-algebra generated by the family of all open subsets of $\mathbb{R}^n$. We denote it by **Borel**$(\mathbb{R}^n)$. If the dimension $n$ is clear from the context, we drop $\mathbb{R}^n$ and just write **Borel**.

For each $k \geq 1$, define $\Sigma^0_k(\mathbb{R}^n)$ and $\Pi^0_k(\mathbb{R}^n)$ as follows.

(a) $\Sigma^0_1(\mathbb{R}^n)$ is the family of all open subsets of $\mathbb{R}^n$ and $\Pi^0_1(\mathbb{R}^n)$ is the family of all closed subsets of $\mathbb{R}^n$.

(b) $\Sigma^0_{k+1}(\mathbb{R}^n)$ is the family of all countable unions of members of $\Pi^0_k(\mathbb{R}^n)$ and $\Pi^0_{k+1}(\mathbb{R}^n)$ is the family of all countable intersections of members of $\Sigma^0_k(\mathbb{R}^n)$.

We sometimes drop $\mathbb{R}^n$ and just write $\Sigma^0_n$ and $\Pi^0_n$. Members of $\Sigma^0_2$ (resp. $\Pi^0_2$) are also called $F_\sigma$-sets (resp. $G_\delta$-sets). Members of $\Sigma^0_3$ (resp. $\Pi^0_3$) are also called $G_{\delta\sigma}$-sets (resp. $F_{\sigma\delta}$-sets) and so on. It can be shown that $\Sigma^0_k$ (resp. $\Pi^0_k$) is a proper subset of $\Sigma^0_{k+1}$ (resp. $\Pi^0_{k+1}$) and their union does not exhaust **Borel**

$$\bigcup_{k\geq 1} \Sigma^0_k \cup \Pi^0_k \subsetneq \textbf{Borel}$$

# Borel hierarchy

This section assumes some background in ordinals and cardinals. Recall that $\omega_1$ is the least uncountable cardinal. Using transfinite recursion, define for each $1 \leq \alpha < \omega_1$, the families $\Sigma^0_\alpha(\mathbb{R}^n)$ and $\Pi^0_\alpha(\mathbb{R}^n)$ as follows.

(a) $\Sigma^0_1(\mathbb{R}^n)$ is the family of all open subsets of $\mathbb{R}^n$ and $\Pi^0_1(\mathbb{R}^n)$ is the family of all closed subsets of $\mathbb{R}^n$.

(b) $\Sigma^0_{\alpha+1}(\mathbb{R}^n)$ is the family of all countable unions of members of $\Pi^0_\alpha(\mathbb{R}^n)$ and $\Pi^0_{\alpha+1}(\mathbb{R}^n)$ is the family of all countable intersections of members of $\Sigma^0_\alpha(\mathbb{R}^n)$.

(c) If $\alpha$ is a limit ordinal, then $\Sigma^0_\alpha(\mathbb{R}^n) = \bigcup\{\Sigma^0_\beta(\mathbb{R}^n) : 1 \leq \beta < \alpha\}$ and $\Pi^0_\alpha(\mathbb{R}^n) = \bigcup\{\Pi^0_\beta(\mathbb{R}^n) : 1 \leq \beta < \alpha\}$.

Since every countable subset of $\omega_1$ is bounded below $\omega_1$, it follows that $\mathbf{Borel} = \bigcup\{\Sigma^0_\alpha \cup \Pi^0_\alpha : 1 \leq \alpha < \omega_1\}$. Let $\mathfrak{c} = |\mathbb{R}|$ be the cardinality of $\mathbb{R}$. Using the fact that $|\mathfrak{c}^\omega| = \mathfrak{c}$, it is easy to check that $|\Sigma^0_\alpha| = |\Pi^0_\alpha| = \mathfrak{c}$ for each $1 \leq \alpha < \omega_1$. Hence $|\mathbf{Borel}| = |\mathfrak{c} \times \omega_1| = \mathfrak{c}$.

# Borel vs Lebesgue measurable

Since each open subset of $\mathbb{R}^n$ is Lebesgue measurable and the family $\text{dom}(\mu_n)$ of Lebesgue measurable sets is a $\sigma$-algebra, it follows that every Borel set is Lebesgue measurable.

## Lemma

$\text{Borel}(\mathbb{R}^n) \subsetneq \textit{dom}(\mu_n)$.

**Proof**: We only need to show that this inclusion is proper. Fix $A \subseteq \mathbb{R}^n$ such that $|A| = \mathfrak{c}$ and $\mu_n(A) = 0$. Such $A$ exists because when $n = 1$, we can take $A$ to be the ternary Cantor set and when $n \geq 2$, we can take $A$ to be a line in $\mathbb{R}^n$. Note that for every $B \subseteq A$, $\mu_n(B) = 0$. So every subset of $A$ is Lebesgue measurable. It follows that the cardinality of the set of all Lebesgue measurable subsets of $\mathbb{R}^n$ is $2^{\mathfrak{c}}$. Since $|\mathbf{Borel}(\mathbb{R}^n)| = \mathfrak{c} < 2^{\mathfrak{c}}$, it follows that $\mathbf{Borel}(\mathbb{R})^n \subsetneq \text{dom}(\mu_n)$. $\square$

# Regularity of Lebesgue measure

$E \subseteq \mathbb{R}^n$ is **bounded** iff for some $0 < M < \infty$, $E \subseteq [-M, M]^n$. This implies that $\mu^\star(E) \leq (2M)^n$ is finite.

## Theorem

*Let $E \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable. Then for every $\varepsilon > 0$, there exists a compact set $K$ and an open set $U$ such that $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.*

**Proof**: Fix $0 < M < \infty$ such that $E \subseteq [-M, M]^n$. Let $\mu^\star(E) = a < \infty$. Choose a sequence $\langle B_k : k \geq 1 \rangle$ on open $n$-boxes such that $E \subseteq \bigcup_{k \geq 1} B_k$ and $\sum_{k \geq 1} \mu(B_k) < a + \varepsilon/2$. Let $U = \bigcup_{n \geq 1} B_k$. Then $U$ is open, $E \subseteq U$ and $\mu(U) < a + \varepsilon/2$. Hence $\mu(U \setminus E) = \mu(U) - \mu(E) < \varepsilon/2$. Repeating this argument with $F = [-M, M]^n \setminus E$, we can find an open set $V$ such that $F \subseteq V$ and $\mu(V \setminus F) < \varepsilon/2$. Put $K = [-M, M]^n \setminus V$ and note that $K$ is compact being both bounded and closed. Next as $F \subseteq V$, we get $K \subseteq E$. Also $E \setminus K = V \setminus F$. So $\mu(E \setminus K) = \mu(V \setminus F) < \varepsilon/2$. Finally, $\mu(U \setminus K) = \mu(U \setminus E) + \mu(E \setminus K) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

# Regularity of Lebesgue measure

The following is an immediate corollary. The proof is left to the reader.

## Corollary

*Let $E \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable.*

(1) $\mu(E) = \sup\{\mu(K) : K \subseteq E$ *and* $K$ *is compact*$\}$.

(2) $\mu(E) = \inf\{\mu(U) : E \subseteq U$ *and* $U$ *is open*$\}$.

(3) *There exists a* $G_\delta$*-set* $G \subseteq \mathbb{R}^n$ *such that* $E \subseteq G$ *and* $\mu(G \setminus E) = 0$.

(4) *There exists an* $F_\sigma$*-set* $F \subseteq \mathbb{R}^n$ *such that* $F \subseteq E$ *and* $\mu(E \setminus F) = 0$.

It also follows that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable iff there exist $B \subseteq \mathbb{R}^n$ Borel and $X \subseteq \mathbb{R}^n$ such that $\mu(X) = 0$ and $E = B\Delta X$. Here, $B\Delta X = (B \setminus X) \cup (X \setminus B)$ is the symmetric difference of $B$ and $X$.

# Baire property

Define

$$\textbf{Baire}(\mathbb{R}^n) = \{U \Delta X : X, U \subseteq \mathbb{R}^n \text{ where U is open and X is meager}\}$$

Members of $\textbf{Baire}(\mathbb{R}^n)$ are called **sets with property of Baire**.

**Exercise**: Show that $\textbf{Baire}(\mathbb{R}^n)$ is a $\sigma$-algebra on $\mathbb{R}^n$.

# Inner Lebesgue measure

## Definition (Inner measure)

*Let $X \subseteq \mathbb{R}^n$. The inner Lebesgue measure of $X$ is defined by*

$$\mu_\star(X) = \sup\{\mu(K) : K \subseteq X \text{ and } K \text{ is compact}\}$$

## Lemma

*Suppose $X \subseteq \mathbb{R}^n$ is bounded. Then $X$ is Lebesgue measurable iff $\mu_\star(X) = \mu^\star(X)$.*

**Proof**: The left to right implication follows from Clause (2) of the Corollary on the previous slide. For the other direction, suppose $X \subseteq \mathbb{R}^n$ is bounded and $\mu_\star(X) = \mu^\star(X) = a < \infty$. For each $m \geq 1$, choose $K_m, U_m$ such that $K_m$ is compact, $U_m$ is open, $K_m \subseteq X \subseteq U_m$ and $a - 1/m < \mu(K_m) \leq \mu(U_m) < a + 1/m$. Put $F = \bigcup_{m \geq 1} K_m$ and $G = \bigcap_{m \geq 1} U_m$. Then $F, G$ are Borel, $F \subseteq X \subseteq G$ and $\mu(X \setminus F) \leq \mu(G \setminus F) = 0$. So $X \setminus F$ is Lebesgue measurable. Hence $X = F \cup (X \setminus F)$ is also Lebesgue measurable. $\square$

# Lebesgue density theorem in $\mathbb{R}$

Suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable and $x \in \mathbb{R}$. The upper and lower Lebesgue densities of $E$ at $x$ are defined as follows

$$d_{up}(E, x) = \limsup_{\varepsilon \to 0} \frac{\mu(E \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon}$$

$$d_{low}(E, x) = \liminf_{\varepsilon \to 0} \frac{\mu(E \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon}$$

Note that $0 \leq d_{low}(E, x) \leq d_{up}(E, x) \leq 1$. If $d_{up}(E, x) = d_{low}(E, x) = d$, then we write $d(E, x) = d$ and say that the Lebesgue density of $E$ at $x$ is $d$.

## Theorem (Lebesgue density theorem)

*Suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable. Then $\{x \in E : d(E, x) \neq 1\}$ is Lebesgue null.*

# An application of Lebesgue density theorem

## Lemma

*There is no Lebesgue measurable $E \subseteq \mathbb{R}$ such that for every open interval $J$, we have $\mu(E \cap J) = \mu(J)/2$.*

**Proof**: Suppose not and fix an $E$ such that for every open interval $J$, we have $\mu(E \cap J) = \mu(J)/2$. Clearly, $\mu(E) > 0$. By Lebesgue density theorem, there exists $x \in E$ such that $d(E, x) = 1$. Choose $\varepsilon > 0$ such that

$$\frac{\mu(E \cap (x - \varepsilon, x + \varepsilon))}{2\varepsilon} > 0.9$$

Put $J = (x - \varepsilon, x + \varepsilon)$ and note that $\mu(J)/2 = \mu(E \cap J) > 0.9\mu(J)$. A contradiction. $\square$

# Proof of Lebesgue density theorem

Recall that every open $U \subseteq \mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals. These open intervals are called the **components of** $U$.

## Lemma

*Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Let* $U \subseteq (a, b)$ *be open. Let* $U_f = \{x \in U : (\exists y > x)(f(x) > f(y) \text{ and } (x, y) \subseteq U)\}$*. Then* $U_f$ *is open and for every component* $(c, d) \subseteq U_f$*, we have* $f(c) \geq f(d)$*.*

**Proof of Lemma**: See video/notes. $\square$

**Proof of Lebesgue density theorem**: See video/notes. $\square$

# Steinhaus theorem

## Theorem (Steinhaus)

*Let $E \subseteq \mathbb{R}$ be measurable and $\mu(E) > 0$. Then $E - E = \{x - y : x, y \in E\}$ contains an open interval around 0.*

**Proof**: By Lebesgue density theorem, there exists $x \in E$ such that $d(E, x) = 1$. Fix an interval $(a, b)$ centered at $x$ such that $\mu(E \cap (a, b)) > 0.9(b - a)$. Let $\delta = 0.1(b - a)$. We claim that $(-\delta, \delta) \subseteq E - E$. Suppose $0 \leq \varepsilon < \delta$. Note that

$$\mu((E + \varepsilon) \cap (a, b)) \geq \mu(E \cap (a, b)) - \varepsilon \geq 0.8(b - a)$$

It follows that $E \cap (E + \varepsilon) \neq \emptyset$. Choose $x \in E \cap (E + \varepsilon)$. Since $x \in E + \varepsilon$, we can choose $y \in E$ such that $x = y + \varepsilon$. So $x - y = \varepsilon$ and $x, y \in E$. Hence both $\varepsilon, -\varepsilon$ are in $E - E$. It follows that $(-\delta, \delta) \subseteq E - E$. $\square$

# Erdős similarity problem

Let $A \subseteq \mathbb{R}$. A **similar copy of** $A$ is a set of the form $sA + t = \{sa + t : a \in A\}$ where $s \neq 0$ (scaling factor) and $t \in \mathbb{R}$ (translation). The following fact can be proved using the Lebesgue density theorem.

## Fact

*Let $A \subseteq \mathbb{R}$ be finite. Then for every $E \subseteq \mathbb{R}$ with $\mu(E) > 0$, $E$ contains a similar copy of $A$.*

## Question (Erdős)

*Let $A$ be any infinite subset of $\mathbb{R}$. Must there exist $E \subseteq \mathbb{R}$ such that $\mu(E) > 0$ and $E$ does not contain any similar copy of $A$? What if $A = \{2^{-n} : n \geq 1\}$?*

# Uniqueness of Lebesgue measure

## Theorem

*Let $\mathcal{M}$ be the family of all Lebesgue measurable subsets of $\mathbb{R}$. Let $\nu : \mathcal{M} \to [0, \infty]$ be a measure such that for every open interval $J$, we have $\nu(J) = \mu(J)$. Then for every $E \in \mathcal{M}$, we have $\mu(E) = \nu(E)$.*

**Proof**: Since every open $U \subseteq \mathbb{R}$ is a a countable union of pairwise disjoint open intervals, it follows that $\nu(U) = \mu(U)$. Next suppose $K \subseteq \mathbb{R}$ is compact. Choose $N > 0$ such that $K \subseteq (-N, N)$ and let $V = (-N, N) \setminus K$. Since $V$ is open, $\mu(V) = \nu(V)$. Now observe that $\mu(K) = \mu((-N, N)) - \mu(V) = \nu((-N, N)) - \nu(V) = \nu(K)$. So $\mu$ and $\nu$ agree on every compact set. Let $E$ be a bounded measurable set. Then $\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ and U is open}\} = \inf\{\nu(U) : E \subseteq U \text{ and U is open}\} \geq \nu(E)$ where the last inequality follows from the monotonicity of $\nu$. Similarly, $\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and K is compact}\} \leq \nu(E)$. Hence $\mu(E) = \nu(E)$ for every bounded $E \in \mathcal{M}$. The general case follows by using countable additivity of $\mu, \nu$ and the fact that $E = \bigsqcup_{n \in \mathbb{Z}} [n, n+1) \cap E$. $\square$

# Perfect sets

We say that $P \subseteq \mathbb{R}^n$ is a **perfect set** iff $P$ is a nonempty closed set and $P$ has **no isolated points**.

## Lemma (Perfect kernel)

*Suppose $C$ is an uncountable closed subset of $\mathbb{R}^n$. Then there exists $P \subseteq C$ such that $P$ is perfect and $C \setminus P$ is countable.*

**Proof**: Let $U$ be the union of all open $n$-boxes of the form $B = J_1 \times \cdots \times J_n$ where $J_1, \ldots, J_n$ are open intervals with rational end-points such that $B \cap C$ is countable. The reader should check that $P = C \setminus U$ is as required. $\square$

The set $P$ in the above lemma is called the **perfect kernel of** $C$.

## Lemma

*Suppose $P \subseteq \mathbb{R}^n$ is perfect. Then $|P| = |\mathbb{R}| = \mathfrak{c}$. Hence every uncountable closed set in $\mathbb{R}^n$ has cardinality $\mathfrak{c}$.*

**Proof**: Homework. $\square$

# Fat Cantor sets in $\mathbb{R}$

## Definition

*$C \subseteq \mathbb{R}$ is a fat Cantor set iff $C$ is compact nowhere dense set and $\mu(C) > 0$.*

## Lemma

*Suppose $E \subseteq \mathbb{R}$ is Lebesgue measurable and $\mu(E) > 0$. Then $E$ contains a fat Cantor set.*

**Proof**: By thinning out $E$, we can clearly assume that $E$ is bounded and $\mathbb{Q} \cap E = \emptyset$. Since $\mu(E) = \sup\{\mu(K) : K \subseteq E$ and K is compact$\} > 0$, we can choose $K \subseteq E$ compact such that $\mu(K) > 0$. Since $K$ is closed and $K \cap \mathbb{Q} = \emptyset$, $K$ must be nowhere dense in $\mathbb{R}$. Hence $K$ is a fat Cantor set contained in $E$. $\square$

# Fat Cantor sets

**Theorem**

*There exists $E \subseteq \mathbb{R}$ such that for every interval $J$, both $J \cap E$ and $J \cap (\mathbb{R} \setminus E)$ have positive measure.*

**Proof**: Homework. $\square$

# Vitali sets revisited

Let $E$ be the equivalence relation on $\mathbb{R}$ defined by $xEy$ iff $x - y \in \mathbb{Q}$. We say that $V \subseteq \mathbb{R}$ is a **Vitali set** iff $V$ meets every $E$-equivalence class at exactly one point.

## Theorem

*Let $V$ be a Vitali set. Then $\mu_\star(V) = 0$ and $\mu^\star(V) > 0$. So every Vitali set is Lebesgue non-measurable.*

**Proof**: Let $K \subseteq V$ be compact. Observe that $K - K \subseteq V - V$ and $(V - V) \cap \mathbb{Q} = \{0\}$. So by Steinhaus theorem, $\mu(K) = 0$. Hence $\mu_\star(V) = 0$.
Next, towards a contradiction, assume $\mu^\star(V) = 0$. Observe that $\mathbb{R} = \bigcup\{V + r : r \in \mathbb{Q}\}$. Since $\mu^\star$ is translation invariant, $\mu^\star(V + r) = \mu^\star(V) = 0$. By countable subadditivity of $\mu^\star$, it follows that $\mu(\mathbb{R}) \leq \sum_{r \in \mathbb{Q}} \mu^\star(V + r) = 0$ which is a contradiction. So $\mu^\star(V) > 0$. $\square$

# Bernstein sets

$B \subseteq \mathbb{R}^n$ is called a Bernstein set iff for every perfect set $P \subseteq \mathbb{R}^n$, both $B \cap P$ and $(\mathbb{R}^n \setminus B) \cap P$ are nonempty.

## Theorem

*There exists a Bernstein set $B \subseteq \mathbb{R}^n$.*

**Proof**: See video/notes. $\square$

**Exercise**: Suppose $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and $B \subseteq \mathbb{R}^n$ is a Bernstein set. Assume $\mu_n(E) > 0$. Show that $B \cap E$ is Lebesgue non-measurable. Conclude that for every $E \subseteq \mathbb{R}^n$, either $\mu_n(E) = 0$ or $E$ has a Lebesgue non-measurable subset.

# Baire class one functions

$f : \mathbb{R}^n \to \mathbb{R}$ is a **Baire class one function** iff there exists a sequence $\langle f_k : k \geq 1 \rangle$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that for every $x \in \mathbb{R}^n$,

$$\lim_k f_k(x) = f(x)$$

**Example**: Let $g : \mathbb{R} \to \mathbb{R}$ be an everywhere differentiable function. Then for every $x \in \mathbb{R}$, $g'(x) = \lim_n n(g(x + 1/n) - g(x))$. So $g'$ is a Baire class one function.

**Exercise**: Let $U \subseteq \mathbb{R}$ be open. Show that the characteristic function of $U$, denoted $1_U$ is a Baire class one function.

# Points of continuity of Baire class one functions

## Theorem (Baire)

*Let $f : \mathbb{R} \to \mathbb{R}$ be a Baire class one function. Then the set of points of discontinuity of $f$ is meager.*

**Proof**: Fix a sequence $\langle f_n : n \geq 1 \rangle$ of continuous functions that pointwise converges to $f$. Let $\varepsilon > 0$ be arbitrary. It suffices to show that $W_\varepsilon = \{x \in \mathbb{R} : osc(f, x) < \varepsilon\}$ is a dense subset of $\mathbb{R}$. Let $I$ be a closed interval. We'll show that $I \cap W_\varepsilon \neq \emptyset$. For $i, j \geq 1$, define $C_{i,j} = \{x \in \mathbb{R} : |f_i(x) - f_j(x)| \leq \varepsilon/3\}$. Since $|f_i - f_j|$ is continuous, each $C_{i,j}$ is a closed set. Let $A_n = \bigcap\{C_{i,j} : i, j \geq n\}$. Then $A_n$'s form an increasing sequence of closed sets. Since for every $x \in \mathbb{R}$, $\lim_n f_n(x) = f(x)$, it follows that $\bigcup_{n\geq 1} A_n = \mathbb{R}$. By Baire category theorem, we can fix an $n \geq 1$ such that $A_n \cap I$ is not nowhere dense in $I$. Since $A_n \cap I$ is closed, there exists an open interval $J \subseteq A_n \cap I$. Let $x$ be the center of $J$. Choose $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq J$ and for every $y, z \in (x - \delta, x + \delta)$, we have $|f_n(y) - f_n(z)| < \varepsilon/4$. Now for any $k > n$, we have $|f_k(y) - f_k(z)| \leq |f_k(y) - f_n(y)| + |f_n(y) - f_n(z)| + |f_n(z) - f_k(z)| < \varepsilon/3 + \varepsilon/4 + \varepsilon/3$. Letting $k \to \infty$, we get $|f(y) - f(z)| < \varepsilon$. It follows that $x \in W_\varepsilon \cap I$. $\square$

# Pointwise limits

## Lemma

*The set of Baire class one functions is not closed under pointwise limits.*

**Proof**: Let $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the set of rationals $\mathbb{Q}$. Since $1_{\mathbb{Q}}$ is an everywhere discontinuous function, the previous theorem implies that $1_{\mathbb{Q}}$ is not a Baire class one function. Let $\{a_1, a_2, \ldots\}$ be an enumeration of $\mathbb{Q}$. Define $f_n$ to be the characteristic function of $\{a_1, \ldots, a_n\}$. Then $\langle f_n : n \geq 1 \rangle$ is a sequence of Baire class one functions that pointwise converges to $1_{\mathbb{Q}}$. $\square$

# The Baire hierarchy

Using transfinite recursion, for each $\alpha < \omega_1$, define the set $\mathbf{FBaire}_\alpha(\mathbb{R}^n)$ of **Baire class $\alpha$ functions** as follows.

(1) $\mathbf{FBaire}_0(\mathbb{R}^n)$ consists of all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

(2) $\mathbf{FBaire}_{\alpha+1}(\mathbb{R}^n)$ is the set of all functions that are pointwise limits of a sequence of functions in $\mathbf{FBaire}_\alpha(\mathbb{R}^n)$.

(3) If $\alpha$ is a limit ordinal, then $\mathbf{FBaire}_\alpha(\mathbb{R}^n) = \bigcup_{\beta<\alpha} \mathbf{FBaire}_\beta(\mathbb{R}^n)$.

Define $\mathbf{FBaire}(\mathbb{R}^n) = \bigcup_{\alpha<\omega_1} \mathbf{FBaire}_\alpha(\mathbb{R}^n)$. Lebesgue showed that this is a proper hierarchy in the sense that $\mathbf{FBaire}_\alpha(\mathbb{R}^n) \subsetneq \mathbf{FBaire}_{\alpha+1}(\mathbb{R}^n)$. The following should be clear.

## Fact

$\mathrm{FBaire}(\mathbb{R}^n)$ *is the smallest family of functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *that contains all continuous functions and is closed under pointwise limits.*

# Borel functions

If $f : X \to Y$ and $A \subseteq Y$, the **preimage of** $A$ **under** $f$ is defined by

$$f^{-1}[A] = \{x \in X : f(x) \in A\}$$

## Definition

*$f : \mathbb{R}^n \to \mathbb{R}$ is a Borel function iff for every Borel $B \subseteq \mathbb{R}$, $f^{-1}[B]$ is a Borel subset of $\mathbb{R}^n$.*

## Lemma

*$f : \mathbb{R}^n \to \mathbb{R}$ is a Borel function iff for every open interval $J \subseteq \mathbb{R}$, $f^{-1}[J]$ is Borel.*

**Proof**: Let $\mathcal{F} = \{A \subseteq \mathbb{R} : f^{-1}[A] \text{ is Borel}\}$. Check that $\mathcal{F}$ is a $\sigma$-algebra on $\mathbb{R}$ that contains every open interval. $\square$

# Borel equals Baire

## Lemma

FBorel$(\mathbb{R}^n)$ *is closed under pointwise limits. Hence,* FBaire$(\mathbb{R}^n) \subseteq$ FBorel$(\mathbb{R}^n)$.

**Proof**: Suppose $f_k : \mathbb{R}^n \to \mathbb{R}$ is Borel for every $k \geq 1$, and $\lim_k f_k(x) = f(x)$. To see that $f$ is Borel, it is enough to show that $f^{-1}[(a,b)]$ is Borel for every $a < b$ in $\mathbb{R}$. Now $a < f(x) < b$ iff $a < \lim_k f_k(x) < b$ iff there exists $M \geq 1$ such that for all sufficiently large $k$, $a + 1/M < f_k(x) < b - 1/M$. It follows that

$$f^{-1}[(a,b)] = \bigcup_{M \geq 1} \bigcup_{N \geq 1} \bigcap_{k \geq N} f_k^{-1}[(a + 1/M, b - 1/M)]$$

is Borel. Since every continuous function is Borel, it follows that **FBaire**$(\mathbb{R}^n) \subseteq$ **FBorel**$(\mathbb{R}^n)$. □

## Fact (Lebesgue, Hausdorff)

FBorel$(\mathbb{R}^n) =$ FBaire$(\mathbb{R}^n)$. *Hence* FBorel$(\mathbb{R}^n)$ *is the smallest class of real valued functions on* $\mathbb{R}^n$ *that contains all continuous functions and is closed under pointwise limits.*

# Lebesgue measurable functions

Suppose $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are measurable spaces and $f : X \to Y$. We say that $f$ is $(\mathcal{E}, \mathcal{F})$-**measurable** iff for every $B \in \mathcal{F}$, $f^{-1}[B] \in \mathcal{E}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. We say that $f$ is **Lebesgue measurable** iff it is $(\mathcal{M}, \mathbf{Borel}(\mathbb{R}^m))$-measurable where $\mathcal{M}$ is the $\sigma$-algebra of all Lebesgue measurable subsets of $\mathbb{R}^n$.

## Lemma

*Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $f$ is Lebesgue measurable iff for every open interval $J$, $f^{-1}[J]$ is Lebesgue measurable in $\mathbb{R}^n$.*

**Proof**: Let $\mathcal{F} = \{A \subseteq \mathbb{R} : f^{-1}[A] \text{ is Leb. measurable}\}$. Check that $\mathcal{F}$ is a $\sigma$-algebra on $\mathbb{R}$ that contains every open interval. $\square$

# Lebesgue vs Borel

## Lemma

*Let $f : \mathbb{R}^n \to \mathbb{R}$. If $f$ is Borel, then it is Lebesgue measurable. The converse is false.*

**Proof**: The first part easily follows from the fact that every Borel set is Lebesgue measurable. Next, let $X \subseteq \mathbb{R}^n$ be a non-Borel set such that $\mu(X) = 0$. Then the characteristic function of $X$ is Lebesgue measurable but not Borel. □

## Definition

*Let $f, g : \mathbb{R}^n \to \mathbb{R}$. We say that $f$ and $g$ are almost everywhere equal iff $\mu(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) = 0$.*

**Exercise**: For every Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, there exists a Borel function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and $g$ are almost everywhere equal.

# Measurable functions

Let $\mathcal{F}$ be the smallest family of functions from $\mathbb{R}^n$ to $\mathbb{R}$ satisfying the following.

(a) $\mathcal{F}$ contains all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

(b) $\mathcal{F}$ is closed under pointwise limits.

(c) If $f \in \mathcal{F}$, $g : \mathbb{R}^n \to \mathbb{R}$ and $f$ and $g$ are almost everywhere equal, then $g \in \mathcal{F}$.

Then $\mathcal{F}$ is the family of all Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

# Closure properties

**Lemma**

*Suppose* $f, g : \mathbb{R}^n \to \mathbb{R}$ *are Lebesgue measurable functions,* $h : \mathbb{R} \to \mathbb{R}$ *is a Borel function and* $a, b \in \mathbb{R}$*. Then* $|f|$*,* $af + bg$*,* $fg$ *and* $h \circ f$ *are also Lebesgue measurable. Furthermore, if* $0 \notin range(f)$*, then* $1/f$ *is also Lebesgue measurable.*

**Proof**: Homework. $\square$

# Restrictions

Suppose $E \subseteq \mathbb{R}^n$ and $f : E \to \mathbb{R}$. We say that $f$ is Lebesgue measurable iff $E$ is Lebesgue measurable and there exists a Lebesgue measurable $g : \mathbb{R}^n \to \mathbb{R}$ such that $f = g \restriction E$.

Let $(X, d)$ be a metric space, $f : X \to \mathbb{R}$ and $K \subseteq X$. Recall that $f \restriction K$ is continuous iff either one of the following holds.

(i) For every sequence $\langle x_n : n \geq 1 \rangle$ of points in $K$ if $\lim_n x_n = x$ and $x \in K$, then $\lim_n f(x_n) = f(x)$.

(ii) For every open interval $J$ with rational end-points, there exists an open $U \subseteq X$ such that $f^{-1}[J] = U \cap K$ (so $f^{-1}[J]$ is relatively open in $K$).

## Theorem (Lusin)

*Suppose $E \subseteq \mathbb{R}^n$ is bounded and $f : E \to \mathbb{R}$ is Lebesgue measurable. Then for every $\varepsilon > 0$, there exists a compact $K \subseteq E$ such that $\mu(E \setminus K) < \varepsilon$ and $f \restriction K$ is continuous.*

**Proof**: Let $\langle J_n : n \geq 1 \rangle$ list all open intervals with rational end-points. Then $E_n = f^{-1}[J_n]$ is a bounded Lebesgue measurable. Choose $K_n, U_n$ such that $K_n \subseteq E_n \subseteq U_n$, $K_n$ is compact, $U_n$ is open and $\mu(U_n \setminus K_n) < \varepsilon/2^{n+1}$. Put $A = \bigcup_{n \geq 1}(U_n \setminus K_n)$. Then $\mu(A) \leq \varepsilon/2$. Let $g = f \restriction (E \setminus A)$. Then $g^{-1}[J_n] = U_n \cap (E \setminus A)$ is relatively open in $E \setminus A$ for every $n \geq 1$. So $g$ is continuous. Choose a compact $K \subseteq (E \setminus A)$ such that $\mu((E \setminus A) \setminus K) < \varepsilon/2$. Then $\mu(E \setminus K) < \varepsilon$ and $f \restriction K$ is continuous. □

**Exercise**: Suppose $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable. Then for every $\varepsilon > 0$, there exists a continuous $g : [a, b] \to \mathbb{R}$ such that $\mu(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \varepsilon$.

# Almost uniform convergence

## Theorem (Egoroff)

*Suppose $E \subseteq \mathbb{R}^n$ is bounded and for every $n \geq 1$, $f_n : E \to \mathbb{R}$ is Lebesgue measurable. Assume that $f_n$'s pointwise converge to $f : E \to \mathbb{R}$. Then for each $\varepsilon > 0$, there exists a compact $K \subseteq E$ such that $\mu(E \setminus K) < \varepsilon$ and $f_n \restriction K$ uniformly converges to $f \restriction K$.*

**Proof**: For each $k, m \geq 1$, define

$$E_{k,m} = \{x \in E : (\forall j \geq k)(|f_j(x) - f(x)| < 1/m)\}$$

Since $|f_j - f|$ is Lebesgue measurable, it follows that each $E_{k,m}$ is Lebesgue measurable. Note that $E_{k,m}$'s are increasing with $k$ and since $f_k$'s pointwise converge to $f$, we have $\bigcup_{k \geq 1} E_{k,m} = E$. Fix $k(m) \geq 1$ such that $\mu(E \setminus E_{k(m),m}) < \varepsilon/2^{m+1}$. Put $F = \bigcap_{m \geq 1} E_{k(m),m}$. Then $\mu(E \setminus F) < \varepsilon/2$ and it is easily checked that $f_n \restriction F$ uniformly converges to $f \restriction F$. Choose a compact $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon/2$. Then $\mu(E \setminus K) < \varepsilon$ and $f_n \restriction K$ uniformly converges to $f \restriction K$. $\square$

# Measurable functions on $(X, \mathcal{F})$

Let $(X, \mathcal{F})$ be a measurable space and $f : X \to \mathbb{R}$. We say that $f$ is $\mathcal{F}$-**measurable** iff for every Borel $B \subseteq \mathbb{R}$, $f^{-1}[B] \in \mathcal{F}$.
The following can be proved exactly like Homework problems 20-21.

(1) Suppose $f, g : X \to \mathbb{R}$ are $\mathcal{F}$-measurable functions, $h : \mathbb{R} \to \mathbb{R}$ is a Borel function and $a, b \in \mathbb{R}$. Then $|f|$, $af + bg$, $fg$ and $h \circ f$ are also $\mathcal{F}$-measurable. Furthermore, if $0 \notin \text{range}(f)$, then $1/f$ is also $\mathcal{F}$- measurable.

(2) Suppose $f_k : X \to \mathbb{R}$ are $\mathcal{F}$-measurable for every $k \geq 1$. Assume that for every $x \in X$, $g(x) = \limsup_k f_k(x)$ and $h(x) = \liminf_k f_k(x)$ are finite. Show that $g, h : X \to \mathbb{R}$ are $\mathcal{F}$-measurable.

# Simple functions

## Definition (Simple functions)

*Suppose $(X, \mathcal{F})$ is a measurable space and $h : X \to \mathbb{R}$ is $\mathcal{F}$-measurable. We say that $h$ is simple iff range$(h)$ is finite.*

Suppose $h : X \to \mathbb{R}$ is a simple function and range$(h) = \{a_1, a_2, \ldots, a_n\}$. Define $X_k = h^{-1}[\{a_k\}]$. Then $\{X_k : k \leq n\}$ is a partition of $X$ into sets in $\mathcal{F}$ and

$$h = \sum_{k \leq n} a_k 1_{X_k}$$

where $1_{X_k} : X \to \mathbb{R}$ is the characteristic function of $X_k$. It is easy to see that the family of simple functions is closed under linear combinations and products.

# Approximations via simple functions

## Theorem

*Suppose $(X, \mathcal{F})$ is a measurable space and $f : X \to [0, \infty)$ is $\mathcal{F}$-measurable. Then there exists a sequence $\langle h_n : n \geq 1\rangle$ of simple functions such that $h_n \leq h_{n+1}$ and for every $x \in X$, $\lim_n h_n(x) = f(x)$. Furthermore, if $f$ is bounded, then $h_n$'s uniformly converge to $f$.*

**Proof**: For each $n \geq 1$ and $0 \leq k < 4^n$, define $B_n = f^{-1}[[2^n, \infty)]$,

$$A(k, n) = f^{-1}\left[\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right]$$

and

$$h_n = 2^n 1_{B_n} + \sum_{0 \leq k < 4^n} \left(\frac{k}{2^n}\right) 1_{A(k,n)}$$

It is easy to check that $h_n \leq h_{n+1}$ and $0 \leq f(x) - h_n(x) \leq 2^{-n}$ for every $x \in f^{-1}[[0, 2^n)]$. It follows that $h_n$'s pointwise converge to $f$. Furthermore, if range$(h) \subseteq [0, N)$, then $f^{-1}[[0, 2^n)] = X$ for all $n \geq N$ and therefore the convergence is uniform. $\square$

# Integrating non-negative simple functions

Let $(X, \mathcal{F}, m)$ be a measure space. Let $h = \sum_{k \le n} a_k 1_{X_k}$ be a non-negative simple function on $X$ (So each $a_k \ge 0$). We define the **Lebesgue integral of** $h$ as follows

$$\int h \, dm = \sum_{k \le n} a_k m(X_k)$$

where, by definition, $0 \cdot \infty = 0$. For $A \in \mathcal{F}$, define the **Lebesgue integral of** $h$ **on** $A$ by

$$\int_A h \, dm = \int 1_A h \, dm$$

# Integrating non-negative simple functions

**Lemma**

*Let $(X, \mathcal{F}, m)$ be a measure space. Suppose $h_1, h_2$ are non-negative simple functions on $X$.*

(a) *For every $a \geq 0$,* $\int (ah_1 + h_2)\, dm = a \int h_1\, dm + \int h_2\, dm$

(b) *If $h_1 \leq h_2$, then* $\int h_1\, dm \leq \int h_2\, dm$

**Proof**: Exercise. $\square$

# Integrating non-negative simple functions

## Lemma

*Let $(X, \mathcal{F}, m)$ be a measure space and let $h$ be a non-negative simple function on $X$. Define $\nu : \mathcal{F} \to \mathbb{R}$ by $\nu(E) = \int_E h\, dm$. Then $\nu$ is a measure.*

**Proof**: Let $h = \sum_{k \le n} a_k 1_{X_k}$ where $\{X_k : k \le n\}$ is a partition of $X$ into sets in $\mathcal{F}$ and each $a_k \ge 0$. It is clear that $\nu(E) \ge 0$ for every $E \in \mathcal{F}$ and $\nu(\emptyset) = 0$. So it suffices to show that $\nu$ is countably additive. Fix a countable family $\{E_j : j \ge 1\}$ of pairwise disjoint sets in $\mathcal{F}$ and let $E = \bigcup_{j \ge 1} E_j$. For each $k \le n$, consider $\int_E a_k 1_{X_k}\, dm = \int a_k 1_E 1_{X_k}\, dm = \int a_k 1_{E \cap X_k}\, dm = a_k \mu(E \cap X_k)$. As $\mu$ is countably additive, $a_k \mu(E \cap X_k) = a_k \sum_{j \ge 1} \mu(E_j \cap X_k) = \sum_{j \ge 1} \int_{E_j} a_k 1_{X_k}\, dm$. Hence for every $k \le n$,

$$\int_E a_k 1_{X_k}\, dm = \sum_{j \ge 1} \int_{E_j} a_k 1_{X_k}\, dm$$

Summing over $k \le n$ and using part (a) of the previous lemma, we get

$$\nu(E) = \int_E h\, dm = \sum_{j \ge 1} \int_{E_j} h\, dm = \sum_{j \ge 1} \nu(E_j)$$

$\square$

# Lebesgue integral of non-negative functions

Let $(X, \mathcal{F}, m)$ be a measure space and suppose $f : X \to [0, \infty)$ is $\mathcal{F}$-measurable.

(1) The **Lebesgue integral of** $f$ is defined by

$$\int f\, dm = \sup \left\{ \int h\, dm : h \text{ is simple and } 0 \leq h \leq f \right\}$$

(2) For $A \in \mathcal{F}$, the **Lebesgue integral of** $f$ **on** $A$ is defined by

$$\int_A f\, dm = \int 1_A f\, dm$$

If $f : X \to [0, \infty)$ is simple, then part (b) of the previous lemma implies that this definition agrees with the old definition.

**Exercise**: Suppose $f, g : X \to [0, \infty)$ are $\mathcal{F}$-measurable. Show that $f \leq g$ implies $\displaystyle\int f\, dm \leq \int g\, dm$ and for every constant $c \geq 0$, $\displaystyle\int cf\, dm = c \int f\, dm$.

# Monotone convergence theorem

## Theorem

*Let* $(X, \mathcal{F}, m)$ *be a measure space and suppose for each* $n \geq 1$, $f_n : X \to [0, \infty)$ *is* $\mathcal{F}$*-measurable. Assume* $f_n \leq f_{n+1}$ *for every* $n \geq 1$ *and for every* $x \in X$, $\lim_n f_n(x) = \sup_n f_n(x) < \infty$. *Define* $f : X \to [0, \infty)$ *by* $f(x) = \lim_n f_n(x)$. *Then* $f$ *is* $\mathcal{F}$*-measurable and*

$$\int f\, dm = \lim_n \int f_n\, dm$$

**Proof**: As $f \geq f_n$, we get $\int f\, dm \geq \int f_n\, dm$. Taking limits as $n \to \infty$, we get $\int f\, dm \geq \lim_n \int f_n\, dm$. For the other inequality, it suffices to show that for every $0 < \varepsilon < 1$ and a simple function $h : X \to [0, \infty)$ such that $0 \leq h \leq f$, we have $\lim_n \int f_n\, dm \geq (1 - \varepsilon) \int h\, dm$. Put $E_n = \{x \in X : f_n(x) \geq (1 - \varepsilon) h(x)\}$. Then $E_n$'s are increasing with $n$ and $\bigcup_n E_n = X$. Since the map $E \mapsto \int_E h\, dm$ is a measure on $(X, \mathcal{F})$ (by Slide 63), it follows that $\lim_n \int_{E_n} h\, dm = \int h\, dm$. Furthermore, $\int f_n\, dm \geq \int_{E_n} f_n\, dm \geq (1 - \varepsilon) \int_{E_n} h\, dm$. It follows that

$$\lim_n \int f_n\, dm \geq \lim_n \int_{E_n} f_n\, dm \geq (1 - \varepsilon) \lim_n \int_{E_n} h\, dm = (1 - \varepsilon) \int h\, dm$$

□

# Linearity for non-negative functions

## Theorem

*Let $(X, \mathcal{F}, m)$ be a measure space and suppose for each $n \geq 1$, $f_n : X \to [0, \infty)$ is $\mathcal{F}$-measurable. Let $a \geq 0$.*

(1) $$\int af_1 + f_2 \, dm = a \int f_1 \, dm + \int f_2 \, dm$$

(2) *Assume* $\sum_{n \geq 1} f_n(x) < \infty$ *for every* $x \in X$. *Then* $\int \sum_{n \geq 1} f_n \, dm = \sum_{n \geq 1} \int f_n \, dm$

**Proof**: (1) Using the theorem on Slide 60, we can fix simple functions $h_k, g_k$ for $k \geq 1$ such that $0 \leq h_k \leq h_{k+1}$, $0 \leq g_k \leq g_{k+1}$, $h_k$'s pointwise converge to $f_1$ and $g_k$'s pointwise converge to $f_2$. It follows that $ah_k + g_k$ pointwise converges to $af_1 + f_2$. By the monotone convergence theorem, $\int (af_1 + f_2) \, dm = \lim_k \int (ah_k + g_k) \, dm = \lim_k (a \int h_k \, dm + \int g_k \, dm) = a \int f_1 \, dm + \int f_2 \, dm$.

(2) Put $g_n = \sum_{k \leq n} f_k$ and $f = \sum_{k \geq 1} f_k$. Then $g_n$'s are monotonically increasing and they pointwise converge to $f$. So by the monotone convergence theorem, $\lim_n \int g_n \, dm = \int f \, dm$. By part (1), $\int g_n \, dm = \sum_{k \leq n} \int f_k \, dm$. Hence $\sum_{k \geq 1} \int f_k \, dm = \int f \, dm$. $\square$

# Integrable functions

Let $(X, \mathcal{F})$ be a measurable space and suppose $f : X \to \mathbb{R}$ is $\mathcal{F}$-measurable. Define $f^+ : X \to [0, \infty)$ and $f^- : X \to [0, \infty)$ as follows: $f^+(x) = \max(0, f(x))$ and $f^-(x) = \max(0, -f(x))$. Note that $f^+$ and $f^-$ are both $\mathcal{F}$-measurable and $f = f^+ - f^-$.

Suppose $(X, \mathcal{F}, m)$ is a measure space and $f : X \to \mathbb{R}$ is $\mathcal{F}$-measurable. If at least one of $\int f^+ \, dm$, $\int f^- \, dm$ is finite, then we define

$$\int f \, dm = \int f^+ \, dm - \int f^- \, dm$$

Suppose $(X, \mathcal{F}, m)$ is a measure space and $f : X \to \mathbb{R}$ is $\mathcal{F}$-measurable. We say that $f$ **is integrable** iff $\displaystyle\int |f| \, dm < \infty$ iff both $\displaystyle\int f^+ \, dm$, $\displaystyle\int f^- \, dm$ are finite.

The set of all integrable functions $f : X \to \mathbb{R}$ is denoted by $L^1(m)$. The following should be clear.

**Exercise**: Let $(X, \mathcal{F}, m)$ be a measure space and suppose $f, g \in L^1(m)$. Then $af + g \in L^1(m)$ and $\displaystyle\int (af + g) \, dm = a \int f \, dm + \int g \, dm$.

# Fatou's lemma

## Theorem

*Let $(X, \mathcal{F}, m)$ be a measure space. Suppose $f_n : X \to [0, \infty)$ is $\mathcal{F}$-measurable for every $n \geq 1$. Define $f : X \to [0, \infty)$ by $f(x) = \liminf_n f_n(x)$. Then $f$ is $\mathcal{F}$-measurable and*

$$\int f\, dm \leq \liminf_n \int f_n\, dm$$

**Proof**: That $f$ is $\mathcal{F}$-measurable is clear (see Slide 58). Put $g_n = \inf_{k \geq n} f_k$. Then $g_n \leq g_{n+1}$ and $g_n$'s pointwise converge to $f$. By the monotone convergence theorem,

$$\int f\, dm = \lim_n \int g_n\, dm$$

Also for every $n$, $g_n \leq f_n$. Therefore, $\int g_n\, dm \leq \int f_n\, dm$. Taking $\liminf_n$ on both sides, we get

$$\int f\, dm = \lim_n \int g_n\, dm = \liminf_n \int g_n\, dm \leq \liminf_n \int f_n\, dm$$

$\square$

# Interchanging limits and Lebesgue integral

Suppose $f, f_n : \mathbb{R} \to [0, \infty)$ are Lebesgue integrable functions for every $n \geq 1$ and $f_n$'s pointwise converge to $f$. Must the following hold?

$$\lim_n \int f_n \, d\mu = \int \lim_n f_n \, d\mu$$

(1) Define $f_n : \mathbb{R} \to [0, \infty)$ as follows: $f_n(x) = n$ if $x \in (0, 1/n)$ and 0 otherwise. Then $f_n$'s pointwise converge to 0 everywhere on $\mathbb{R}$ but $\int f_n \, d\mu = 1$ does not converge to $\int 0 \, d\mu = 0$.

(2) Define $f_n : \mathbb{R} \to [0, \infty)$ as follows: $f_n(x) = 1$ if $x \in [n, n+1]$ and 0 otherwise. Then $f_n$'s pointwise converge to 0 everywhere on $\mathbb{R}$ but $\int f_n \, d\mu = 1$ does not converge to $\int 0 \, d\mu = 0$.

Note that in both examples, there is no function $g \in L^1(\mu)$ such that $f_n \leq g$ for every $n \geq 1$.

# Dominated convergence theorem

## Theorem

*Let $(X, \mathcal{F}, m)$ be a measure space. Let $f_n : X \to \mathbb{R}$ be $\mathcal{F}$-measurable for every $n \geq 1$. Assume $f_n$'s pointwise converge to $f : X \to \mathbb{R}$. Suppose there exists $g \in L^1(m)$ such that $|f_n| \leq g$ for every $n \geq 1$. Then $f \in L^1(m)$ and $\int f \, dm = \lim_n \int f_n \, dm$.*

**Proof**: $f$ is clearly $\mathcal{F}$-measurable. As $|f_n| \leq g$ for every $n \geq 1$, taking limits as $n \to \infty$, we get $|f| \leq g$. So $\int |f| \, dm \leq \int g \, dm < \infty$ and hence $f \in L^1(m)$.

Next observe that $g - f_n \geq 0$ and $g + f_n \geq 0$ for every $n \geq 1$ and $g + f_n$ and $g - f_n$ pointwise converge to $g + f$ and $g - f$ respectively. So we can apply Fatou's lemma to the sequences $g + f_n$ and $g - f_n$ to get the following.

# Dominated convergence theorem

$$\int g\,dm + \int f\,dm \le \liminf_n \int (g+f_n)\,dm = \int g\,dm + \liminf_n \int f_n\,dm$$

$$\int g\,dm - \int f\,dm \le \liminf_n \int (g-f_n)\,dm = \int g\,dm - \limsup_n \int f_n\,dm$$

where we used $\liminf_n -a_n = -\limsup_n a_n$. Since $\int g\,dm < \infty$, we can cancel it to get

$$\limsup_n \int f_n\,dm \ge \liminf_n \int f_n\,dm \ge \int f\,dm \ge \limsup_n \int f_n\,dm$$

But this means that all inequalities are equalities here and the result follows. $\square$

# Dominated convergence theorem

## Corollary

*Let $(X, \mathcal{F}, m)$ be a finite measure space. Let $0 < M < \infty$ and suppose $f_n : X \to [-M, M]$ is $\mathcal{F}$-measurable for every $n \geq 1$. Assume $f_n$'s pointwise converge to $f : X \to [-M, M]$. Then $f \in L^1(m)$ and $\int f\, dm = \lim_n \int f_n\, dm$.*

**Proof** Apply the dominated convergence theorem with $g = M$. $\square$

# Riemann Integral

An **interval partition of** $[a, b]$ is a finite $P \subseteq [a, b]$ with $a, b \in P$. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and

$$P = \{a = a_0 < a_1 < \cdots < a_n = b\}$$

is an interval partition of $[a, b]$. For each $0 \leq k < n$, let $m_k = \inf\{f(x) : x \in [a_k, a_{k+1}]\}$ and $M_k = \sup\{f(x) : x \in [a_k, a_{k+1}]\}$. Define $L(P, f) = \sum_{0 \leq k < n} m_k(a_{k+1} - a_k)$ and $U(P, f) = \sum_{0 \leq k < n} M_k(a_{k+1} - a_k)$.

The **lower Riemann integral of** $f$ is defined by

$$\underline{\int_a^b} f(x)\, dx = \sup\{L(P, f) : P \text{ is an interval partition of } [a, b]\}$$

The **upper Riemann integral of** $f$ is defined by

$$\overline{\int_a^b} f(x)\, dx = \inf\{U(P, f) : P \text{ is an interval partition of } [a, b]\}$$

# Riemann Integrability

We say that $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** iff $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ **is finite** and this common value is denoted by $\int_a^b f(x)\,dx$.

## Definition

*Let* $f : [a, b] \to \mathbb{R}$*. For* $A \subseteq [a, b]$*, define the* oscillation of $f$ on $A$ *by*

$$osc(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$$

**Exercise**: Show that $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for every $\varepsilon > 0$, there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that

$$\sum_{k=0}^{n-1} \mathrm{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) < \varepsilon$$

# Oscillations

## Lemma

*Let $f : [c, d] \to [-M, M]$ where $0 < M < \infty$. Let $\varepsilon > 0$. Assume that for every $x \in (c, d)$, $osc(f, x) < \varepsilon$. Then there exist $c = c_0 < c_1 < \cdots < c_n = d$ such that*

$$\sum_{k=0}^{n-1} osc(f, [c_k, c_{k+1}])(c_{k+1} - c_k) < 2\varepsilon(d - c)$$

**Proof**: For each $x \in (c, d)$ choose an open interval $J_x \subseteq (c, d)$ centered at $x$ such that $\text{osc}(f, cl(J_x)) < \varepsilon$. Let $I_1 = (c - \frac{\varepsilon(d-c)}{4M}, c + \frac{\varepsilon(d-c)}{4M})$ and $I_2 = (d - \frac{\varepsilon(d-c)}{4M}, d + \frac{\varepsilon(d-c)}{4M})$. Then $\mathcal{U} = \{J_x : x \in (c, d)\} \cup \{I_1, I_2\}$ is an open cover of $[c, d]$. As $[c, d]$ is compact, $\mathcal{U}$ has a finite subcover $\mathcal{F}$. Let $c_1 < c_2 < \cdots < c_{n-1}$ list the set of end points of the intervals in $\mathcal{F} \setminus \{I_1, I_2\}$. As $I_1, I_2 \in \mathcal{F}$, we must have $c_1 \leq c + \frac{\varepsilon(d-c)}{4M}$ and $c_{n-1} \geq d - \frac{\varepsilon(d-c)}{4m}$. Furthermore, for every $1 \leq k < n_1$, $[c_k, c_{k+1}] \subseteq J_x$ for some $J_x \in \mathcal{F}$. Therefore, $\text{osc}(f, [c_k, c_{k+1}]) < \varepsilon$. It is now easy to check that

$$\sum_{k=0}^{n-1} \text{osc}(f, [c_k, c_{k+1}])(c_{k+1} - c_k) < 2\varepsilon(d - c) \quad \square$$

# Riemann Integrability

## Theorem

*Let $f : [a, b] \to [-M, M]$ where $0 < M < \infty$. Then $f$ is Riemann integrable iff the set of points of discontinuity of $f$ has measure zero.*

**Proof**: Let $D = \{x \in [a, b] : f \text{ is discontinuous at } x\}$. First assume that $D$ has measure zero. Fix $\varepsilon > 0$. Let $D_\varepsilon = \{x \in (a, b) : \operatorname{osc}(f, x) \geq \varepsilon\} \cup \{a, b\}$. Then $D_\varepsilon$ is a closed set of measure zero. Let $\langle I_k : k \geq 1 \rangle$ be a sequence of open intervals such that $D_\varepsilon \subseteq \bigcup_{k \geq 1} I_k$ and $\sum_{k \geq 1} |I_k| < \varepsilon$. Since $D_\varepsilon$ is compact, we can fix $N \geq 1$ such that $\{I_k : k \leq N\}$ already covers $D_\varepsilon$. Let $U_\varepsilon = [a, b] \setminus \bigcup_{k \leq N} cl(I_k)$. Then $U_\varepsilon$ is a finite union of open intervals. For each such open interval $(c, d)$, by the previous lemma, we can fix a finite $W(c, d) = \{c = c_0 < c_1 < \cdots < c_m = d\}$ such that

$$\sum_{k=0}^{m-1} \operatorname{osc}(f, [c_k, c_{k+1}])(c_{k+1} - c_k) < 2\varepsilon(d - c)$$

Let $a = a_0 < a_1 < \cdots < a_n = b$ list all members of these $W(c, d)$'s together with the end points of $\{I_k : k \leq N\}$.

# Riemann Integrability

Note that for each $0 \le k < n$,

(i) Either $\operatorname{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) < 2\varepsilon(a_{k+1} - a_k)$, or

(ii) $[a_k, a_{k+1}] \subseteq I_j$ for some $1 \le j \le N$. And therefore, $\operatorname{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) < 2M|I_j|$.

It follows that

$$\sum_{k=0}^{n-1} \operatorname{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) \le 2\varepsilon\mu(U_\varepsilon) + \sum_{j \le N} 2M(|I_j|) < 2\varepsilon(M + b - a)$$

which goes to zero as $\varepsilon \to 0$. So $f$ is Riemann integrable.

Next suppose $D$ is not Lebesgue null. We will show that $f$ is not Riemann integrable. For each $\delta > 0$, let $D_\delta = \{x \in (a, b) : \operatorname{osc}(f, x) \ge \delta\}$. Since $D \subseteq \bigcup_{\delta \in \mathbb{Q}^+} D_\delta$, we can fix $\delta > 0$ such that $\mu(D_\delta) > 0$. Let $\varepsilon = \delta\mu(D_\delta)/2$. It suffices to show that for every $a = a_0 < a_1 < \cdots < a_n = b$,

$$\sum_{k=0}^{n-1} \operatorname{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) \ge \varepsilon$$

# Riemann Integrability

Note that if $(a_k, a_{k+1}) \cap D_\delta \neq \emptyset$, then $\operatorname{osc}(f, [a_k, a_{k+1}]) \geq \delta/2$. Let $T = \{k < n : (a_k, a_{k+1}) \cap D_\delta \neq \emptyset\}$. Then $D_\delta \subseteq \bigcup_{k \in T} [a_k, a_{k+1}]$. So $\sum_{k \in T} (a_{k+1} - a_k) \geq \mu(D_\delta)$. It follows that

$$\sum_{k=0}^{n-1} \operatorname{osc}(f, [a_k, a_{k+1}])(a_{k+1} - a_k) \geq \sum_{k \in T} \frac{\delta}{2}(a_{k+1} - a_k) \geq \frac{\delta \mu(D_\delta)}{2} = \varepsilon \quad \square$$

# Riemann vs Lebesgue Integral

## Corollary

*Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. Then $f$ is Lebesgue integrable and*

$$\int_{[a,b]} f \, d\mu = \int_a^b f(x) \, dx$$

**Proof**: Let $C$ be the set of points of continuity of $f$. Then $C$ is Borel and $f \restriction C$ is continuous. So $f \restriction C$ is Lebesgue measurable. As $[a, b] \setminus C$ has measure zero, $f$ is also Lebesgue measurable. Fix $M > 0$ such that $|f| \leq M$. As $\int_{[a,b]} |f| \, d\mu \leq M(b - a) < \infty$, it follows that $f$ is also Lebesgue integrable. If $P$ is an interval partition of $[a, b]$, then then $L(P, f) \leq \int_{[a,b]} f \, d\mu$. Taking supremum over all $P$'s we get $\underline{\int}_a^b f(x) \, dx \leq \int_{[a,b]} f \, d\mu$. A similar argument shows that $\int_{[a,b]} f \, d\mu \leq \overline{\int}_a^b f(x) \, dx$. Since $\underline{\int}_a^b f(x) \, dx = \overline{\int}_a^b f(x) \, dx$, it follows that $\int_{[a,b]} f \, d\mu = \int_a^b f(x) \, dx$. $\square$

# Algebras

## Definition (Algebra)

*Let $X$ be a nonempty set. An algebra on $X$ is a family $\mathcal{F}$ of subsets of $X$ that satisfies the following.*

(1) $\emptyset \in \mathcal{F}$ *and* $X \in \mathcal{F}$.

(2) *(Closed under complements) If* $E \in \mathcal{F}$, *then* $X \setminus E \in \mathcal{F}$.

(3) *(Closed under finite unions) If* $E_1, E_2 \in \mathcal{F}$, *then* $E_1 \cup E_2 \in \mathcal{F}$.

## Lemma

*Let $Y$ be a nonempty set. Let $\mathcal{R}$ be a a family of subsets of $Y$ such that*

(a) $\emptyset \in \mathcal{R}$.

(b) *If* $A, B \in \mathcal{R}$, *then* $A \cap B \in \mathcal{R}$.

(c) *If* $A \in \mathcal{R}$, *then* $Y \setminus A$ *is a disjoint union of finitely many members of* $\mathcal{R}$.

*Let $\mathcal{A}$ be the family of all sets which are disjoint unions of finitely many members of $\mathcal{R}$. Then $\mathcal{A}$ is an algebra on $Y$.*

**Proof**: Homework $\square$

# Premeasures on algebras

Let $\mathcal{A}$ be an algebra on a nonempty set $X$. We say that $m : \mathcal{A} \to [0, \infty]$ is a **premeasure on** $\mathcal{A}$ iff $m$ satisfies the following.

(i) $m(\emptyset) = 0$.

(ii) If $\{E_n : n \geq 1\}$ is a countable family of pairwise disjoint sets in $\mathcal{A}$ and $E = \bigcup_{n\geq 1} E_n \in \mathcal{A}$, then $m(E) = \sum_{n\geq 1} m(E_n)$.

Note that if $A_1, A_2 \in \mathcal{A}$ are disjoint, then $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ and if $A \subseteq B$ are in $\mathcal{A}$, then $m(A) \leq m(B)$.

## Theorem (Extending premeasures)

*Let $\mathcal{A}$ be an algebra on a nonempty set $X$ and $m : \mathcal{A} \to [0, \infty]$ be a premeasure on $\mathcal{A}$. Let $\mathcal{F}$ be the $\sigma$-algebra generated by $\mathcal{A}$. Then there exists a measure $\nu$ on $\mathcal{F}$ such that $\nu \restriction \mathcal{A} = m$. Also, if $m(X) < \infty$, then $\nu$ is unique.*

**Proof**: Define $\nu^\star : \mathcal{P}(X) \to [0, \infty]$ by

$$\nu^\star(E) = \inf \left\{ \sum_{n\geq 1} m(A_n) : A_n \in \mathcal{A} \text{ for every } n \geq 1 \text{ and } E \subseteq \bigcup_{n\geq 1} A_n \right\}$$

# Extending premeasures to measures

By HW problem 8, $\nu^\star$ is an outer measure on $X$. First, we claim that $\nu^\star \restriction \mathcal{A} = m$. Suppose $E, A_n \in \mathcal{A}$ for every $n \geq 1$ and $E \subseteq \bigcup_{n\geq 1} A_n$. Define $B_n = E \cap (A_n \setminus \bigcup_{k\leq n-1} A_k)$. Then each $B_n \in \mathcal{A}$ and $E$ is a disjoint union of $B_n$'s. So $\sum_{n\geq 1} m(A_n) \geq \sum_{n\geq 1} m(B_n) = m(E)$. Hence $\nu^\star(E) \geq m(E)$. The other inequality is trivial because $E \in \mathcal{A}$. So $\nu^\star(E) = m(E)$ for every $E \in \mathcal{A}$.

Next, we claim that every set in $\mathcal{A}$ is $\nu^\star$-measurable. Suppose $E \in \mathcal{A}$ and $A \subseteq X$. We need to check $\nu^\star(E) \geq \nu^\star(A \cap E) + \nu^\star(A \cap E^c)$ where $E^c = X \setminus E$ is the complement of $E$ in $X$. Let $\varepsilon > 0$ be arbitrary. Choose $F_n \in \mathcal{A}$ such that $A \subseteq \bigcup_{n\geq 1} F_n$ and $\nu^\star(A) + \varepsilon \geq \sum_{n\geq 1} m(F_n)$. Now

$$\sum_{n\geq 1} m(F_n) = \sum_{n\geq 1} m(F_n \cap E) + \sum_{n\geq 1} m(F_n \cap E^c) \geq \nu^\star(A \cap E) + \nu^\star(A \cap E^c)$$

Hence $\nu^\star(A) + \varepsilon \geq \nu^\star(A \cap E) + \nu^\star(A \cap E^c)$. Letting $\varepsilon \to 0$, we get $\nu^\star(E) \geq \nu^\star(A \cap E) + \nu^\star(A \cap E^c)$. So every set in $\mathcal{A}$ is $\nu^\star$-measurable.

# Extending premeasures to measures

Let $\nu_1$ be the restriction of $\nu^\star$ to $\nu^\star$-measurable sets. By Caratheodory's theorem, $\nu_1$ is a measure. As each set in $\mathcal{A}$ is $\nu^\star$-measurable, $\mathcal{A} \subseteq \mathrm{dom}(\nu_1)$. Finally, since $\nu^\star \restriction \mathcal{A} = m$, we also have $\nu_1 \restriction \mathcal{A} = m$. Put $\nu = \nu_1 \restriction \mathcal{F}$. This completes the proof of existence of $\nu$.

For uniqueness, assume $m(X) < \infty$ and let $\nu' : \mathcal{F} \to [0, \infty]$ be another measure on $\mathcal{F}$ such that $\nu' \restriction \mathcal{A} = m$. Let $E \in \mathcal{F}$ and $\varepsilon > 0$. Choose $\langle A_n : n \geq 1 \rangle$ such that each $A_n \in \mathcal{A}$, $E \subseteq \bigcup_{n \geq 1} A_n$ and $\nu(E) + \varepsilon \geq \sum_{n \geq 1} m(A_n) = \sum_{n \geq 1} \nu'(A_n) \geq \nu'(E)$. Letting $\varepsilon \to 0$, we get $\nu(E) \geq \nu'(E)$. So $\nu \geq \nu'$. For the reverse inequality, put $A = \bigcup_{n \geq 1} A_n$ and observe that

$$\nu(A) = \lim_{n \to \infty} \nu \left( \bigcup_{k \leq n} A_k \right) = \lim_{n \to \infty} \nu' \left( \bigcup_{k \leq n} A_k \right) = \nu'(A)$$

Since $\nu$ is finite, $\nu(A \setminus E) = \nu(A) - \nu(E) \leq \sum_{n \geq 1} \nu(A_n) - \nu(E) \leq \varepsilon$. As $E \subseteq A$, $\nu(E) \leq \nu(A) = \nu'(A) = \nu'(E) + \nu'(A \setminus E) \leq \nu'(E) + \nu(A \setminus E) \leq \nu'(E) + \varepsilon$. Letting $\varepsilon \to 0$, we get $\nu(E) \leq \nu'(E)$. So $\nu = \nu'$. $\square$

# $\sigma$-finite measures

(1) Suppose $\mathcal{A}$ is an algebra on $X$ and $m : \mathcal{A} \to [0, \infty]$ is a premeasure. We say that $m$ is $\sigma$-**finite** iff there exists a countable family $\{E_n : n \geq 1\} \subseteq \mathcal{A}$ such that $\bigcup_{n \geq 1} E_n = X$ and $m(E_n) < \infty$ for every $n \geq 1$.

(2) A measure space $(X, \mathcal{F}, m)$ is $\sigma$-**finite** iff there exists a countable family $\{E_n : n \geq 1\} \subseteq \mathcal{F}$ such that $\bigcup_{n \geq 1} E_n = X$ and $m(E_n) < \infty$ for every $n \geq 1$.

**Exercise**: Let $\mathcal{A}$ be an algebra on a nonempty set $X$ and $m : \mathcal{A} \to [0, \infty]$ be a $\sigma$-**finite** premeasure on $\mathcal{A}$. Let $\mathcal{F}$ be the $\sigma$-algebra generated by $\mathcal{A}$. Show that there exists a **unique** measure $\nu$ on $\mathcal{F}$ such that $\nu \restriction \mathcal{A} = m$.

# Product of $\sigma$-algebras

Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces. Let $Y = X_1 \times X_2$. We say that $S \subseteq Y$ is an $(\mathcal{A}_1, \mathcal{A}_2)$-**measurable rectangle** iff $S = E_1 \times E_2$ for some $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. The **product algebra** $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the $\sigma$-algebra on $Y$ generated by the family of all measurable rectangles. Define the **product measurable space** by

$$(X_1, \mathcal{A}_1) \otimes (X_2, \mathcal{A}_2) = (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$$

## Lemma

*Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces and let $\mathcal{R}$ be the family of all measurable rectangles in $Y = X_1 \times X_2$. Let $\mathcal{A}$ be the family of all sets which are disjoint unions of finitely many members of $\mathcal{R}$. Then $\mathcal{A}$ is an algebra on $Y$.*

**Proof**: Just check that conditions (a)-(c) in the the lemma on Slide 80 hold for the family $\mathcal{R}$. $\square$

# Product of power-set algebra

**Question**

*Is* $(\mathbb{R}, \mathcal{P}(\mathbb{R})) \otimes (\mathbb{R}, \mathcal{P}(\mathbb{R})) = (\mathbb{R}^2, \mathcal{P}(\mathbb{R}^2))$*? In other words, does every subset of plane belong to the* $\sigma$*-algebra generated by all rectangles?*

B. V. Rao (On discrete Borel spaces and projective sets, Bull Amer. Math. Soc. 75 (1969), 614–617)) showed that the answer is "Yes" under the continuum hypothesis and K. Kunen (PhD Thesis, Stanford, 1968) showed that the answer is consistently "No". So this question is undecidable in ZFC.

# Product measures: Existence and Uniqueness

## Theorem

*Let $(X_1, \mathcal{A}_1, m_1)$ and $(X_2, \mathcal{A}_2, m_2)$ be $\sigma$-finite measure spaces. Let $Y = X_1 \times X_2$ and $\mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Then there is a unique measure $m : \mathcal{F} \to [0, \infty]$ such that for every $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$, $m(E_1 \times E_2) = m_1(E_1)m_2(E_2)$.*

**Proof**: Let $\mathcal{R}$ be the family of all measurable rectangles in $Y = X_1 \times X_2$. Start by defining $m(E_1 \times E_2) = m_1(E_1)m_2(E_2)$ for every $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. Let $\mathcal{A}$ be the family of all sets which are disjoint unions of finitely many members of $\mathcal{R}$. Then $\mathcal{A}$ is an algebra on $Y$. Suppose $S \in \mathcal{A}$, and
$S = \bigsqcup\limits_{1 \le i \le n} (A_i \times B_i) = \bigsqcup\limits_{1 \le j \le p} (C_j \times D_j)$ where $A_i, C_j \in \mathcal{A}_1$ and $B_i, D_j \in \mathcal{A}_2$. Then it is easy to check that

$$\sum_{1 \le i \le n} m(A_i \times B_i) = \sum_{1 \le j \le p} m(C_j \times D_j)$$

So we can extend $m$ to $\mathcal{A}$ by defining $m(S) = \sum_{1 \le i \le n} m(A_i \times B_i)$ for every $S = \bigsqcup_{1 \le i \le n}(A_i \times B_i) \in \mathcal{A}$. By the theorem on Slide 81 (Extending premeasures to measures), it suffices to show that $m$ is a premeasure on $\mathcal{A}$.

# Product measure: Existence and Uniqueness

Suppose $S = A \times B \in \mathcal{R}$, $S_n = A_n \times B_n \in \mathcal{R}$ for $n \geq 1$ and $S$ is a disjoint union of $S_n$'s. We will show that $m(S) = \sum_{n\geq 1} m(S_n)$. Define $f_n : X_1 \to [0, \infty]$ by $f_n(x) = m_2(B_n)$ if $x \in A_n$ and $f_n(x) = 0$ if $x \notin A_n$. Then $f_n$'s are $\mathcal{A}_1$-measurable and it is easy to see that $\sum_{n\geq 1} f_n(x) = m_2(B)$ if $x \in A$ and 0 if $x \notin A$. So by the monotone convergence theorem,

$$\sum_{n\geq 1} \int_A f_n \, dm_1 = \int_A \sum_{n\geq 1} f_n \, dm_1 = \int_A m_2(B) \, dm_1 = m_1(A)m_2(B) = m(S)$$

Since $\int_A f_n \, dm_1 = \int_A 1_{A_n} m_2(B_n) \, dm_1 = m_1(A_n)m_2(B_n) = m(S_n)$, it follows that $m(S) = \sum_{n\geq 1} m(S_n)$.

For the general case, let $S = \bigsqcup_{1\leq j\leq N} T_j$ where each $T_j \in \mathcal{R}$. Suppose $S = \bigsqcup_{n\geq 1} E_n$ where each $E_n = \bigsqcup_{1\leq k\leq K_n} E_{n,k}$ and $E_{n,k} \in \mathcal{R}$. Note that each $T_j$ is a disjoint union of $\{E_{n,k} \cap T_j : n < \omega, k \leq K_n\}$. Now use the previous case. $\square$

# Definition of product measure

A measure space $(X, \mathcal{F}, m)$ is **complete** iff for every $E \in \mathcal{F}$ and $A \subseteq E$, if $m(E) = 0$, then $A \in \mathcal{F}$.

**Exercise**: (**Completion of a measure**) Let $(X, \mathcal{F}, m)$ be any measure space. Put $\mathcal{N} = \{Y \subseteq X : (\exists A \in \mathcal{F})(m(A) = 0 \text{ and } Y \subseteq A)\}$ and define $\mathcal{E} = \{E \Delta Y : E \in \mathcal{F}, Y \in \mathcal{N}\}$ and $m' : \mathcal{E} \to [0, \infty]$ by $m'(E \Delta Y) = m(E)$ for every $E \in \mathcal{F}$ and $Y \in \mathcal{N}$. Then $(X, \mathcal{E}, m')$ is a complete measure space and $m' \restriction \mathcal{F} = m$. We say that $(X, \mathcal{E}, m')$ is a **completion** of $(X, \mathcal{F}, m)$.

Let $(X_1, \mathcal{A}_1, m_1)$ and $(X_2, \mathcal{A}_2, m_2)$ be $\sigma$-**finite measure spaces**. By the previous theorem, there exists a unique measure $m : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, \infty]$ satisfying $m(A \times B) = m_1(A)m_2(B)$ for every $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$. We define the **product measure** $m_1 \otimes m_2$ to be the **completion of** $m$.

# Product of several measures

Let $(X_i, \mathcal{A}_i)$ be measurable spaces for $1 \le i \le n$. Put

$$Y = \prod_{1 \le i \le n} X_i = \{(x_1, x_2, \cdots, x_n) : (\forall i \le n)(x_i \in X_i)\}$$

We say that $S \subseteq Y$ is an $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$-**measurable box** iff $S = \prod_{1 \le i \le n} E_i$ for some $E_i \in \mathcal{A}_i$. The **product algebra** $\bigotimes_{1 \le i \le n} \mathcal{A}_i$ is the $\sigma$-algebra on $Y$ generated by the family of all $(\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n)$-measurable boxes.

## Theorem

*Let $(X_i, \mathcal{A}_i, m_i)$ be $\sigma$-finite measure spaces for $1 \le i \le n$. Let $Y = \prod_{1 \le i \le n} X_i$ and $\mathcal{F} = \bigotimes_{1 \le i \le n} \mathcal{A}_i$. Then there is a unique measure $m : \mathcal{F} \to [0, \infty]$ such that whenever $E_i \in \mathcal{A}_i$ for $1 \le i \le n$,*

$$m\left(\prod_{1 \le i \le n} E_i\right) = \prod_{1 \le i \le n} m_i(E_i)$$

# Measurability in product spaces

Let $E \subseteq X_1 \times X_2$, $F : X_1 \times X_2 \to \mathbb{R}$, $x \in X_1$ and $y \in X_2$. The **vertical section of** $E$ **at** $x$ is $E_x = \{y \in X_2 : (x, y) \in E\}$ and the **horizontal section of** $E$ **at** $y$ is $E^y = \{x \in X_1 : (x, y) \in E\}$. Define $F_x : X_2 \to \mathbb{R}$ and $F^y : X_1 \to \mathbb{R}$ by $F_x(y) = F^y(x) = F(x, y)$.

## Lemma

*Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces and let $(Y, \mathcal{F})$ be their product. So $Y = X_1 \times X_2$ and $\mathcal{F} = \mathcal{A}_1 \otimes \mathcal{A}_2$.*

(1) *Let $E \in \mathcal{F}$. Then for every $x \in X_1$, $E_x \in \mathcal{A}_2$ and for every $y \in X_2$, $E^y \in \mathcal{A}_1$.*

(2) *Let $F : X_1 \times X_2 \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then for every $x \in X_1$, $F_x$ is $\mathcal{A}_2$-measurable and for every $y \in X_2$, $F^y$ is $\mathcal{A}_1$-measurable.*

**Proof**: (1) Let $\mathcal{E}$ be the family of all $E \subseteq X \times Y$ such that for every $x \in X_1$, $E_x \in \mathcal{A}_2$ and for every $y \in X_2$, $E^y \in \mathcal{A}_1$. Check that $\mathcal{E}$ is a $\sigma$-algebra that contains all $(\mathcal{A}_1, \mathcal{A}_2)$-measurable rectangles. For (2), use (1) and $f_x^{-1}[B] = (f^{-1}[B])_x$, $(f^y)^{-1}[B] = (f^{-1}[B])^y$. $\square$

# Monotone class lemma

Suppose $X$ is a non-empty set. $\mathcal{M}$ is a **monotone class on** $X$ iff $\mathcal{M}$ is a family of subsets of $X$ satisfying the following.

(a) If $A_n \in \mathcal{M}$ and $A_n \subseteq A_{n+1}$ for every $n \geq 1$, then $\bigcup_{n\geq 1} A_n \in \mathcal{M}$.

(a) If $A_n \in \mathcal{M}$ and $A_{n+1} \subseteq A_n$ for every $n \geq 1$, then $\bigcap_{n\geq 1} A_n \in \mathcal{M}$.

Note that the intersection of any family of monotone classes on $X$ is also a monotone class on $X$. It follows that for every family $\mathcal{A}$ of subsets of $X$, there is a smallest monotone class $\mathcal{M}$ on $X$ such that $\mathcal{A} \subseteq \mathcal{M}$. We say that $\mathcal{M}$ is the **monotone class generated by** $\mathcal{A}$.

## Lemma (Monotone class lemma)

*Let $\mathcal{A}$ be an algebra on $X$. Then the monotone class generated by $\mathcal{A}$ coincides with the $\sigma$-algebra generated by $\mathcal{A}$.*

**Proof**: See notes. $\square$

# Fubini's theorem for sets

## Lemma

*Let $(X_1, \mathcal{A}_1, m_1)$ and $(X_2, \mathcal{A}_2, m_2)$ be $\sigma$-finite measure spaces and let $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then*

(a) *the map $x \mapsto m_2(E_x)$ is $\mathcal{A}_1$-measurable,*

(b) *the map $y \mapsto m_1(E^y)$ is $\mathcal{A}_2$-measurable and*

(c) $(m_1 \otimes m_2)(E) = \displaystyle\int m_2(E_x)\, dm_1(x) = \int m_1(E^y)\, dm_2(y).$

**Proof**: See notes. □

# Tonelli's theorem: Non-negative functions

## Theorem

*Let $(X_1, \mathcal{A}_1, m_1)$ and $(X_2, \mathcal{A}_2, m_2)$ be $\sigma$-finite measure spaces and $m = m_1 \otimes m_2$. Put $dom(m) = \mathcal{A}$ and suppose $F : X_1 \times X_2 \to [0, \infty)$ is an $\mathcal{A}$-measurable function. Then the following hold.*

(a) $F_x : X_2 \to [0, \infty)$ *is* $\mathcal{A}_2$*-measurable for* $m_1$*-almost every* $x \in X_1$ *and* $g(x) = \int F_x \, dm_2$ *is* $\mathcal{A}_1$*-measurable.*

(b) $F^y : X_1 \to [0, \infty)$ *is* $\mathcal{A}_1$*-measurable for* $m_2$*-almost every* $y \in X_2$ *and* $h(y) = \int F^y \, dm_1$ *is* $\mathcal{A}_2$*-measurable.*

(c) $\displaystyle\int F \, dm = \int g(x) \, dm_1(x) = \int h(y) \, dm_2(y).$

**Proof**: See notes. $\square$

# Fubini's theorem: Integrable functions

## Theorem

*Let $(X_1, \mathcal{A}_1, m_1)$ and $(X_2, \mathcal{A}_2, m_2)$ be $\sigma$-finite measure spaces and $m = m_1 \otimes m_2$. Suppose $F : X_1 \times X_2 \to \mathbb{R}$ is an $m$-integrable function. Then the following hold.*

(a) $F_x : X_2 \to \mathbb{R}$ *is* $m_2$*-integrable for* $m_1$*-almost every* $x \in X_1$ *and* $g(x) = \int F_x \, dm_2$ *is* $m_1$*-integrable.*

(b) $F^y : X_1 \to \mathbb{R}$ *is* $m_1$*-integrable for* $m_2$*-almost every* $y \in X_2$ *and* $h(y) = \int F^y \, dm_1$ *is* $m_2$*-integrable.*

(c) $\int F \, dm = \int g(x) \, dm_1(x) = \int h(y) \, dm_2(y).$

**Proof**: See notes. $\square$

# 1 Radon-Nikodym Theorem

**Lemma 1.1.** *Let $(X, \mathcal{F})$ be a measurable space. Suppose $m, \nu : \mathcal{F} \to [0, \infty)$ are finite measures and $\nu \ll m$. Assume $\nu(X) > 0$ and put $K = m(X)/\nu(X)$. Then there exists $A \in \mathcal{F}$ such that $\nu(A) > 0$ and $K(\nu \restriction A) \geq m \restriction A$. Recall that $(\nu \restriction A)(E) = \nu(A \cap E)$.*

*Proof.* Suppose not. Then for every $A \in \mathcal{F}$ with $\nu(A) > 0$, there exists $B \subseteq A$ such that $\nu(B) > 0$ and $K\nu(B) < m(B)$. Let $\mathcal{A}$ be a maximal disjoint family of those $B \in \mathcal{F}$ for which $\nu(B) > 0$ and $K\nu(B) < m(B)$. Then $\mathcal{A}$ is countable (as $\nu$ is finite) and $\nu(X \setminus \bigcup \mathcal{A}) = 0$ (as $\mathcal{A}$ is maximal). It follows that

$$\nu(X) = \sum_{B \in \mathcal{A}} \nu(B) < \sum_{B \in \mathcal{A}} \frac{m(B)}{K} \leq \frac{m(X)}{K}$$

which contradicts the fact that $K = m(X)/\nu(X)$.

□

**Theorem 1.2.** *Let $(X, \mathcal{F}, m)$ be a $\sigma$-finite measure space. Let $\nu : \mathcal{F} \to [0, \infty]$ be a $\sigma$-finite measure such that $\nu \ll m$. Then there exists an $\mathcal{F}$-measurable $h : X \to [0, \infty)$ such that for every $E \in \mathcal{F}$,*

$$\nu(E) = \int_E h \, dm$$

*Furthermore, if $g : X \to [0, \infty)$ is another such function then $h$ and $g$ agree $m$-almost everywhere.*

*Proof.* We will first prove the theorem assuming that $m, \nu$ are both finite. We can also assume that $\nu(X) > 0$ otherwise $h = 0$ is as required. Define

$$\mathcal{E} = \left\{ h : X \to [0, \infty) : h \text{ is } \mathcal{F} \text{ measurable and for every } E \in \mathcal{F}, \int_E h \, dm \leq \nu(E) \right\}$$

1. If $h_1, h_2 \in \mathcal{E}$, then $\max(f_1, f_2) \in \mathcal{E}$.
2. Let $s = \sup \left\{ \int h \, dm : h \in \mathcal{E} \right\}$. Then there exists a sequence $\langle h_n : n \geq 1 \rangle$ of functions in $\mathcal{E}$ such that $h_n \leq h_{n+1}$, and $\int h_n \, dm$ converges to $s$. Let $h = \lim h_n$. Then it is easy to check that $h \in \mathcal{E}$ and by monotone convergence theorem, $\int h \, dm = s$.
3. We claim that $s = \nu(X)$. Suppose not and we'll get a contradiction. Define $\nu' : \mathcal{F} \to [0, \infty)$ by $\nu'(E) = \nu(E) - \int_E h \, dm$ and observe that $\nu'$ is a measure, $\nu' \ll m$ and $\nu'(X) > 0$.

   Put $K = m(X)/\nu'(X)$. By Lemma 1.1, we can find $A \in \mathcal{F}$ such that $\nu'(A) > 0$ and $K(\nu' \restriction A) \geq m \restriction A$. Let $g = h + (1/K)1_A$. Then $g$ is $\mathcal{F}$-measurable and for every $E \in \mathcal{F}$, $\int_E g\,dm = \int_{E\cap A} h\,dm + (1/K)m(E\cap A) + \int_{E\cap A^c} h\,dm \leq \int_{E\cap A} h\,dm + \nu'(E\cap A) + \int_{E\cap A^c} h\,dm = \int_{E\cap A} h\,dm + \left(\nu(E\cap A) - \int_{E\cap A} h\,dm\right) + \int_{E\cap A^c} h\,dm = \nu(E\cap A) + \int_{E\cap A^c} h\,dm \leq \nu(E\cap A) + \nu(E\cap A^c) = \nu(E)$.

Hence $\int_E g\,dm \leq \nu(E)$. So $g \in \mathcal{E}$. Note that $m(A) > 0$ because $\nu'(A) > 0$ and $\nu' \ll m$. But now $\int g\,dm = \int h\,dm + m(A)/K = s + m(A)/K > s$ which is impossible.

4. We claim that for every $E \in \mathcal{F}$, $\int_E h\,dm = \nu(E)$. Suppose not and fix $E \in \mathcal{F}$ such that $\int_E h\,dm < \nu(E)$. Then $s = \int h\,dm = \int_E h\,dm + \int_{X\setminus E} h\,dm < \nu(E) + \nu(X \setminus E) = \nu(X) = s$: A contradiction.

5. To see that $h$ is unique, let $g : X \to [0,\infty)$ be an $\mathcal{F}$-measurable function such that for every $E \in \mathcal{F}$, $\int_E g\,dm = \int_E h\,dm = \nu(E)$. Let $W_n = \{y \in X : h(y) \geq g(y) + 2^{-n}\}$. Then $\int_{W_n} h\,dm - \int_{W_n} g\,dm \geq 2^{-n} m(W_n)$. So $m(W_n) = 0$ for every $n \geq 1$. Hence $m(\{y \in X : h(y) > g(y)\}) = 0$. Similarly, $m(\{y \in X : h(y) < g(y)\}) = 0$ and therefore $h$ and $g$ agree $m$-almost everywhere.

This completes the proof of the theorem when $m, \nu$ are both finite. To deal with the $\sigma$-finite case, first decompose $X = \bigsqcup_{n\geq 1} E_n$ such that both $m \upharpoonright E_n$ and $\nu \upharpoonright E_n$ are finite measures. Apply the finite case of the theorem to the measures $m \upharpoonright E_n$ and $\nu \upharpoonright E_n$ to get $h_n : X \to [0,\infty)$ as in the conclusion of the theorem. Finally, put $h = \sum_{n\geq 1} h_n$ and check that it works for $\nu, m$. $\square$

The function $h$ in Theorem 1.2 called the **Radon-Nikodym derivative of $\nu$ w.r.t.** $m$ and is denoted as follow

$$\frac{d\nu}{dm} = h$$

# AC, BV etc.

April 20, 2022

## 1 Fundamental theorem of calculus for absolutely continuous functions

**Definition 1.1.** *Let $J \subseteq \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$. We say that $f$ is absolutely continuous on $J$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite sequence $\langle (a_i, b_i) : 1 \le i \le n \rangle$ of pairwise disjoint subintervals of $J$,*

$$\sum_{i \le n} (b_i - a_i) < \delta \implies \sum_{i \le n} |f(b_i) - f(a_i)| < \varepsilon$$

We'll denote the set of all absolutely continuous functions $f : J \to \mathbb{R}$ by $AC(J)$. The following lemma says that $AC(J)$ is a vector space over $\mathbb{R}$. The easy proof is left to the reader.

**Lemma 1.2.** *For every $f, g \in AC(J)$ and $a \in \mathbb{R}$, $f + g \in AC(J)$ and $af \in AC(J)$.*

Denote the set of all Lebesgue integrable functions $f : [a, b] \to \mathbb{R}$ by $L^1([a, b])$. For $f \in L^1([a, b])$, define $h_f : [a, b] \to \mathbb{R}$ by

$$h_f(x) = \int_a^x f \, d\mu$$

**Lemma 1.3.** *$h_f : [a, b] \to \mathbb{R}$ is absolutely continuous.*

*Proof.* Note that $h_f = h_{f^+} - h_{f^-}$. Since the difference of two absolutely continuous functions is absolutely continuous, it suffices to show that $h_f$ is absolutely continuous for every $f \in L^1([a, b])$ with $f : [a, b] \to [0, \infty)$.

Let $\mathcal{M}$ be the set of all Lebesgue measurable subsets of $[a, b]$. Define $m : \mathcal{M} \to [0, \infty)$ by $m(E) = \int_E f \, d\mu$. Then $m$ is a measure and $m \ll \mu$. Since $\mu, m$ are finite measures, by a previous lemma, it follows that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $E \in \mathcal{M}$,

$$\mu(E) < \delta \implies m(E) < \varepsilon$$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that for every $E \in \mathcal{M}$, $\mu(E) < \delta \implies m(E) < \varepsilon$. Suppose $\langle (a_i, b_i) : 1 \le i \le n \rangle$ is a finite sequence of pairwise disjoint subintervals of $[a, b]$

such that $\sum_{i\le n}(b_i - a_i) < \delta$. Put $E = \bigcup_{i\le n}(a_i, b_i)$. Then $\mu(E) < \delta$. Hence $m(E) < \delta$. But

$$m(E) = \sum_{i\le n} m((a_i, b_i)) = \sum_{i\le n} \int_{(a_i,b_i)} f\, d\mu = \sum_{i\le n} |h_f(b_i) - h_f(a_i)|$$

It follows that $h_f$ is absolutely continuous on $[a, b]$. □

**Theorem 1.4.** *Let $h : [a, b] \to \mathbb{R}$ be absolutely continuous and montonically increasing. Then the following hold.*

*(1) For every $E \subseteq \mathbb{R}$, if $\mu(E) = 0$, then $\mu(h[E]) = 0$.*

*(2) $h$ is almost everywhere differentiable on $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = h'(x)$ if $h'(x)$ exists and $f(x) = 0$ otherwise. Then $f \in L^1([a, b])$ and for every $x \in [a, b]$,*

$$h(x) = h(a) + \int_a^x f\, d\mu$$

*Proof.* (1) Suppose $E \subseteq [a, b]$ is null. We can assume $E \subseteq (a, b)$ since this removes at most two points from $f[E]$. Let $\varepsilon > 0$ be arbitrary. We will show that $\mu(f[E]) \le \varepsilon$. Since $h$ is absolutely continuous, there exists $\delta > 0$ witnessing the absolute continuity of $f$ for this $\varepsilon$. Choose an open $U \subseteq (a, b)$ such that $E \subseteq U$ and $\mu(U) < \delta$. Let $\{(a_i, b_i) : i \ge 1\}$ list all components of $U$. Then $\mu(U) = \sum_{i\ge1}(b_i - a_i) < \delta$. Now for every $n \ge 1$, $\langle(a_i, b_i) : i \le n\rangle$ is a finite sequence of pairwise disjoint subintervals of $[a, b]$ with $\sum_{i\le n}(b_i - a_i) < \delta$. Hence $\sum_{i\le n} |h(b_i) - h(a_i)| < \varepsilon$. Taking supremum over all $n$, we get $\sum_{i\ge1} |h(b_i) - h(a_i)| \le \varepsilon$. Since $h$ is monotonically increasing, $h[U] = \bigcup_{i\ge1} h[(a_i, b_i)] = \bigcup_{i\ge1}(h(a_i), h(b_i))$. Hence $\mu(h[E]) \le \mu(h[U]) = \sum_{i\ge1} \mu((h(a_i), h(b_i))) = \sum_{i\ge1} |h(b_i) - h(a_i)| \le \varepsilon$.

(2) Define $g : [a, b] \to \mathbb{R}$ by $g(x) = x + h(x)$. Then $g$ is absolutely continuous on $[a, b]$ and **strictly** increasing.

Let $\mathcal{M}$ be the set of all Lebesgue measurable subsets of $[a, b]$. We claim that for every $E \in \mathcal{M}$, $g[E] \in \mathcal{M}$. Let $E \in \mathcal{M}$. Choose a sequence of compact subsets $K_n \subseteq E$ such that $\mu(E \setminus K_n) < 1/n$. Put $K = \bigcup_{n\ge1} K_n$. Then $\mu(E \setminus K) = 0$. Note that $g[K] = \bigcup_{n\ge1} g[K_n]$ is $F_\sigma$ as each $g[K_n]$ is compact. Also by (1), $g[E \setminus K]$ is null. So $g[E] = g[K] \cup g[E \setminus K]$ is the union of an $F_\sigma$-set and a null set. Hence $g[E] \in \mathcal{M}$.

Define $m : \mathcal{M} \to [0, \infty)$ by $m(E) = \mu(g[E])$. Since $g$ is one-one, $m$ is a measure on $\mathcal{M}$. Also $m \ll \mu$ since $g[E]$ is null for every null $E \in \mathcal{M}$. Let $f_1 = \frac{dm}{d\mu}$ be the Radon-Nikodym derivative of $m$ w.r.t. $\mu$. Then for every $x \in [a, b]$, $m([a, x]) = \int_a^x f_1\, d\mu$. Also, $m([a, x]) = \mu(g[[a, x]]) = \mu([g(a), g(x)]) = g(x) - g(a) = h(x) - h(a) + (x - a)$. It follows that for every $x \in [a, x]$,

$$h(x) = h(a) - \int_a^x (f_1 - 1)\, d\mu$$

Put $f = f_1 - 1$. Then $f \in L^1([a, b])$ as $f_1 \in L^1([a, b])$. Finally, by Homework problem 39, it follows that $h(x) = h(a) + \int_a^x f\, d\mu$ is differentiable for almost every $x \in [a, b]$ and its derivative is equal to $f(x)$. □

We would next like to extend Theorem 1.4 to all AC functions. This will be done by showing that every AC function is the difference of two increasing AC functions.

**Definition 1.5.** *Let $f : [a, b] \to \mathbb{R}$ and $[c, d] \subseteq [a, b]$. The total variation of $f$ on $[c, d]$ is defined by*

$$Var_f(c, d) = \sup\{\sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)| : c = x_0 < x_1 < \cdots < x_n = d\}$$

*We say that $f$ is of bounded variation on $[a, b]$ iff $Var_f(a, b) < \infty$. Let $BV([a, b])$ denote the set of all functions $f : [a, b] \to \mathbb{R}$ of bounded variation on $[a, b]$.*

Note that every monotonically increasing/decreasing function $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ and $BV([a, b])$ is a vector space over $\mathbb{R}$.

**Lemma 1.6.** *Let $f : [a, b] \to \mathbb{R}$. Define $V_f : [a, b] \to [0, \infty]$ by $V_f(x) = Var_f(a, x)$. The following hold.*

*(a) If $f \in BV([a, b])$, then $V_f : [a, b] \to [0, \infty)$ and both $V_f$ and $V_f - f$ are monotonically increasing functions. So every function in $BV([a, b])$ is the difference of two monotonically increasing functions.*

*(b) If $f \in AC([0, 1])$, then $f \in BV([a, b])$ and both $V_f$ and $V_f - f$ are absolutely continuous monotonically increasing functions. So every function in $AC([a, b])$ is the difference of two monotonically increasing absolutely continuous functions.*

*Proof.* Will be covered in lecture. □

It is clear that $BV([a, b])$ is not a subset of $AC([a, b])$ since every increasing function $f : [a, b] \to \mathbb{R}$ is in $BV([a, b])$ and an increasing function can have jump discontinuities. Let $C([a, b])$ denote the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Is $C([a, b]) \cap BV([a, b]) \subseteq AC([a, b])$? The answer is no: See `https://en.wikipedia.org/wiki/Cantor_function`.

**Theorem 1.7** (Fundamental theorem of calculus for Lebesgue integrals)**.** *Let $h : [a, b] \to \mathbb{R}$. Then the following are equivalent.*

*(1) $h \in AC([a, b])$.*

*(2) $h$ is almost everywhere differentiable on $[a, b]$, $h' \in L^1([a, b])$ and for every $x \in [a, b]$,*

$$h(x) = h(a) + \int_a^x h' \, d\mu$$

*Proof.* (2) $\implies$ (1): Follows from Lemma 1.3.

(1) $\implies$ (2): By Lemma 1.6(b), there are monotonically increasing AC functions $h_1, h_2 : [a, b] \to \mathbb{R}$ such that $h = h_1 - h_2$. Applying Theorem 1.4 to $h_1$ and $h_2$ gives (2). □