MTH404: *Analysis II*
Homework: These will be periodically updated. Semester II, 2021/22

(1) Let (X, d) be a metric space and $D \subseteq X$. Show that the following are equivalent.
 (a) D is nowhere dense in X.
 (b) $cl(D)$ (closure of D) is nowhere dense in X.
 (c) $X \setminus D$ contains an open dense subset of X.

(2) Let A be the set of all $x \in [0, 1]$ whose decimal expansion contains only two digits: $0, 1$. Show that A is uncountable and nowhere dense in $\mathbb{R}$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$. Define

$$osc(f, x) = \lim_{h \to 0} \sup\{|f(a) - f(b)| : a, b \in (x - h, x + h)\}$$

 (a) Show that f is continuous at x iff $osc(f, x) = 0$.
 (b) Show that $\{x \in \mathbb{R} : osc(f, x) < a\}$ is open in $\mathbb{R}$ for each $a > 0$.

(4) Call $X \subseteq \mathbb{R}$ a G-delta subset of $\mathbb{R}$ iff X is the intersection of a countable family of open sets in $\mathbb{R}$. Show that for any $f : \mathbb{R} \to \mathbb{R}$, the set $\{x \in \mathbb{R} : f \text{ is continuous at } x\}$ is a G-delta subset of $\mathbb{R}$.

(5) Let $Y \subseteq \mathbb{R}$ be countable. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = \{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$.

(6) Show that there is no $f : \mathbb{R} \to \mathbb{R}$ such that $\{x \in \mathbb{R} : f \text{ is continuous at } x\} = \mathbb{Q}$.

(7) Show that the ternary Cantor set is Lebesgue null.

(8) Let $\mathcal{A}$ be a family of subsets of X such that $\emptyset, X \in \mathcal{A}$. Let $f : \mathcal{A} \to [0, \infty]$ be such that $f(\emptyset) = 0$. Define $m : \mathcal{P}(X) \to [0, \infty]$ by

$$m(X) = \inf\left\{\sum_{n \geq 1} f(A_n) : \langle A_n : n \geq 1\rangle \text{ is a sequence of members of } \mathcal{A} \text{ such that } X \subseteq \bigcup_{n \geq 1} A_n\right\}$$

Show that m is an outer measure on X.

(9) Let $\mathcal{F}$ be a σ-algebra on X. Show that $\mathcal{F}$ is either finite or uncountable.

(10) Suppose X is a nonempty set and $\mathcal{A}$ is a collection of subsets of X. Let $\mathcal{E}$ be the family of all σ-algebras $\mathcal{F}$ on X such that $\mathcal{A} \subseteq \mathcal{F}$. Show that $\bigcap \mathcal{E}$ is the smallest σ-algebra on X that contains every set in $\mathcal{A}$.

(11) Let $E \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable. Show the following.

(a) $\mu(E) = \inf\{\mu(U) : E \subseteq U$ and U is open$\}$.

(b) $\mu(E) = \sup\{\mu(K) : K \subseteq E$ and K is compact$\}$.

(c) There exists a G_δ-set $G \subseteq \mathbb{R}^n$ such that $E \subseteq G$ and $\mu(G \setminus E) = 0$.

(d) There exists an F_σ-set $F \subseteq \mathbb{R}^n$ such that $F \subseteq E$ and $\mu(E \setminus F) = 0$.

(12) Show that $\mathbf{Baire}(\mathbb{R}^n)$ is a σ-algebra on $\mathbb{R}^n$.

(13) Let $0 < a < 1$. Show that there does not exist a Lebesgue measurable $E \subseteq \mathbb{R}$ such that for every open interval J,

$$a\mu(J) \leq \mu(A \cap J) \leq (1-a)\mu(J)$$

(14) Suppose $P \subseteq \mathbb{R}^n$ is perfect. Show that $|P| = \mathfrak{c}$.

(15) Show that there exists $E \subseteq \mathbb{R}$ such that for every interval J, both $J \cap E$ and $J \cap (\mathbb{R} \setminus E)$ have positive measure.

(16) Suppose $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and $B \subseteq \mathbb{R}^n$ is a Bernstein set. Assume $\mu_n(E) > 0$. Show that $B \cap E$ is Lebesgue non-measurable. Conclude that for every $E \subseteq \mathbb{R}^n$, either $\mu_n(E) = 0$ or E has a Lebesgue non-measurable subset.

(17) Let $U \subseteq \mathbb{R}$ be open. Show that the characteristic function of U is a Baire class one function.

(18) Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be measurable spaces and $f : X \to Y$. Suppose $\mathcal{A} \subseteq \mathcal{F}$ and the σ-algebra generated by $\mathcal{A}$ is $\mathcal{F}$. Then f is $(\mathcal{E}, \mathcal{F})$-measurable iff for every $A \in \mathcal{A}$, $f^{-1}[A] \in \mathcal{E}$.

(19) Show that for every Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, there exists a Borel function $g : \mathbb{R}^n \to \mathbb{R}$ such that f and g are almost everywhere equal.

(20) Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are Lebesgue measurable functions, $h : \mathbb{R} \to \mathbb{R}$ is a Borel function and $a, b \in \mathbb{R}$. Then $|f|$, $af + bg$, fg and $h \circ f$ are also Lebesgue measurable. Furthermore, if $0 \notin \text{range}(f)$, then $1/f$ is also Lebesgue measurable.

(21) Suppose $E \subseteq \mathbb{R}^n$, and $f_k : E \to \mathbb{R}$ is Lebesgue measurable for every $k \geq 1$. Assume that for every $x \in E$, $g(x) = \limsup_k f_k(x)$ and $h(x) = \liminf_k f_k(x)$ are finite. Show that $g, h : E \to \mathbb{R}$ are Lebesgue measurable.

(22) Suppose $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable. Then for every $\varepsilon > 0$, there exists a continuous $g : [a, b] \to \mathbb{R}$ such that $\mu(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \varepsilon$.

(23) Suppose $(X, \mathcal{F}, m)$ is a measure space and $f : X \to [0, \infty)$ is an $\mathcal{F}$-measurable function.

(a) Show that for every $a > 0$, $m(\{x \in X : f(x) > a\}) \leq \frac{1}{a} \int f\, dm$.

(b) Show that $\int f\, dm = 0$ iff $m(\{x \in X : f(x) > 0\}) = 0$.

(24) Let $(X, \mathcal{F}, m)$ be a finite measure space. Assume $(X, \mathcal{F}, m)$ is **complete** which means the following: For every $E \in \mathcal{F}$, if $m(E) = 0$, then every subset of E is in $\mathcal{F}$. Suppose $f : X \to \mathbb{R}$ is a bounded function. Define

$$\bar{\int} f\, dm = \sup \left\{ \int h\, dm : h \leq f \text{ and } h : X \to \mathbb{R} \text{ is simple} \right\}$$

$$\underline{\int} f\, dm = \inf \left\{ \int h\, dm : f \leq h \text{ and } h : X \to \mathbb{R} \text{ is simple} \right\}$$

Show that $\bar{\int} f\, dm = \underline{\int} f\, dm$ iff f is $\mathcal{F}$-measurable.

(25) Suppose $(X, \mathcal{F}, m)$ is a measure space and $f_n : X \to \mathbb{R}$ is an $\mathcal{F}$-measurable function for every $n \geq 1$. Assume there exists an m-integrable function $g : X \to [0, \infty)$ such that $f_n \leq g$ for every $n \geq 1$. Define $f : X \to \mathbb{R}$ by $f(x) = \limsup_n f_n(x)$. Show that

$$\int f\, dm \geq \limsup_n \int f_n\, dm$$

Also give an example to show that the above inequality can be strict.

(27) Let $(X, \mathcal{F}, m)$ be a finite measure space and suppose $f : X \to \mathbb{R}$ is an m-integrable function. Show that $m(\{x \in X : f(x) \neq 0\}) = 0$ iff for every $E \in \mathcal{F}$, $\int_E f\, dm = 0$.

(28) Let Y be a nonempty set. Let $\mathcal{R}$ be a a family of subsets of Y such that

(a) $\emptyset \in \mathcal{R}$.

(b) If $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$.

(c) If $A \in \mathcal{R}$, then $Y \setminus A$ is a disjoint union of finitely many members of $\mathcal{R}$.

Let $\mathcal{A}$ be the family of all sets which are disjoint unions of finitely many members of $\mathcal{R}$. Show that $\mathcal{A}$ is an algebra on Y.

(29) Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces and let $\mathcal{R}$ be the family of all measurable rectangles in $Y = X_1 \times X_2$. Let $\mathcal{A}$ be the family of all sets which are disjoint unions of finitely many members of $\mathcal{R}$. Then $\mathcal{A}$ is an algebra on Y.

(30) Let $(X, \mathcal{F}, m)$ be any measure space. Put
$\mathcal{N} = \{Y \subseteq X : (\exists A \in \mathcal{F})(m(A) = 0 \text{ and } Y \subseteq A)\}$ and define
$\mathcal{E} = \{E \Delta Y : E \in \mathcal{F}, Y \in \mathcal{N}\}$ and $m' : \mathcal{E} \to [0, \infty]$ by $m'(E \Delta Y) = m(E)$ for every $E \in \mathcal{F}$ and $Y \in \mathcal{N}$. Then $(X, \mathcal{E}, m')$ is a complete measure space and $m' \restriction \mathcal{F} = m$.

(31) Show that $\mu_2 = \mu_1 \otimes \mu_1$. Here, μ_n denotes Lebesgue measure in $\mathbb{R}^n$.

(32) Let $E \subseteq \mathbb{R}^2$ be Lebesgue measurable. Show that the following are equivalent.

(a) $\mu_2(E) = 0$

(b) For almost every $x \in \mathbb{R}$, $\mu(E_x) = 0$.

(c) For almost every $y \in \mathbb{R}$, $\mu(E^y) = 0$.

(33) Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable. Show that for almost every $x \in E$,

$$\lim_{r \downarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} = 1$$

(34) Assume $E \subseteq \mathbb{R}^2$ is unbounded and $\mu(E) > 0$. Show that for every $a > 0$ there exist x, y, z in E such that the area of the triangle with vertices x, y, z is a.

(35) Let $(\mathbb{R}^n, \mathcal{M}, \mu)$ be the Lebesgue measure space. Let ν be another measure on $(\mathbb{R}^n, \mathcal{M})$ such that for every $x \in \mathbb{R}^n$ and $r > 0$, $\nu(B(x, r)) = \mu(B(x, r))$. Show that $\mu = \nu$.

(36) Let $(X, \mathcal{F})$ be a measurable space and suppose $m_1, m_2 : \mathcal{F} \to [0, 1]$ are two probability measures. Assume $m_1 \ll m_2$ and $m_2 \ll m_1$. Let $f_1 = \dfrac{dm_1}{dm_2}$ and $f_2 = \dfrac{dm_2}{dm_1}$. Show that for m_1-almost every $x \in X$, $f_1(x) f_2(x) = 1$.

(37) Let $(X, \mathcal{F}, m)$ be a **finite** measure space. Show that for every $1 \le p \le q < \infty$, $L^q(m) \subseteq L^p(m)$.

(38) Let μ be Lebesgue measure on $\mathbb{R}$ and $1 \le p < q < \infty$. Show that $L^p(\mu) \nsubseteq L^q(\mu)$ and $L^q(\mu) \nsubseteq L^p(\mu)$.

(39) Let $f : [a, b] \to \mathbb{R}$ be Lebesgue integrable. Define $h : [a, b] \to \mathbb{R}$ by $h(x) = \displaystyle\int_a^x f \, d\mu$. Show that for almost every $x \in [a, b]$, $h'(x) = f(x)$.