MTH727 Notes*

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1 Introduction to ZFC

ZFC (shorthand for Zermelo-Fraenkel set theory with the axiom of Choice) is a first order theory in the language with one binary relation symbol: $\in$. The theory attempts to capture the true facts in the structure $(V, \in)$ where V is the "universe of all sets". The axioms of ZFC are: Extensionality, Comprehension/Separation, Pairing, Union, Replacement, Infinity, Power Set, Foundation/Regularity and Choice. Most of these axioms say that the universe of sets is closed under several operations (For example, the axiom of pairing says that for any two sets x, y, there is a set with x, y as members).

Axiom 1.1 (Existence).

$$\exists x(x = x)$$

Axiom 1.2 (Extensionality).

$$\forall x \forall y[\forall z(z \in x \iff z \in y)) \implies x = y]$$

Extensionality says that if two sets have the same members, then they are equal. Note that the converse is automatically true by the substitution rule in first order logic.

Definition 1.3. *$(\forall x \in y)(\phi)$ abbreviates $(\forall x)(x \in y \implies \phi)$ and $(\exists x \in y)(\phi)$ abbreviates $(\exists x)(x \in y \wedge \phi)$. Define $x \subseteq y$ iff $\forall v(v \in x \implies v \in y)$.*

Axiom 1.4 (Comprehension Scheme). *For every formula ϕ in which y is not free,*

$$\forall x \exists y \forall v(v \in y \iff (v \in x \wedge \phi))$$

Axiom of comprehension is an axiom scheme, one axiom for each ϕ. Intuitively, it says that for every set x and a property $\phi(v)$, x has a subset y whose members are precisely those $v \in x$ for which $\phi(v)$ holds. We write

$$\{v \in x : \phi\}$$

to denote this subset. The empty set is defined by $0 = \{x : x \neq x\}$. Define the difference of x, y by $x \setminus y = \{v \in x : v \notin y\}$.

Definition 1.5 (Intersection). *For $X \neq 0$, define the intersection of sets in X, $\bigcap X = \{v : (\forall y \in X)(v \in y)\}$.*

Note that $\bigcap X$ exists: Let $w \in X$ and use comprehension to get $\{v \in w : (\forall y \in X)(v \in y)\} = \bigcap X$.

Axiom 1.6 (Pairing).

$$\forall x \forall y \exists z(x \in z \wedge y \in z)$$

It follows that $\{x, y\}$ exists: Let z be any set that has x, y as members and apply comprehension to get $\{x, y\} = \{v \in z : v = x \vee v = y\}$.

Definition 1.7 (Ordered pair). $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$

*These notes will be periodically revised. If you find an error, please let me know at krashu@iitk.ac.in

It is easy to check that $\langle x, y\rangle = \langle u, v\rangle$ iff $x = u$ and $y = v$. We'll use sometimes write (x, y) instead of $\langle x, y\rangle$ for ease of typing.

Axiom 1.8 (Union)**.**

$$\forall X \exists y \forall v (v \in X \implies v \subseteq y)$$

The axiom of union says that for every set X, there is a set y which contains the union of those sets which are members of X. Together with axiom of comprehension, it implies that every set x has a union

$$\bigcup x = \{v : \exists w (w \in x \wedge v \in w)\}$$

Axiom 1.9 (Replacement Scheme)**.** *For each formula ϕ in which B is not free,*

$$\forall A \left[(\forall x \in A)(\exists! y)\phi(x, y) \implies \exists B (\forall x \in A)(\exists y \in B)(\phi(x, y))\right]$$

Here, $(\exists! x)(\phi(x))$ ("there exists a unique x such that $\phi(x)$") abbreviates $\exists x \forall y (\phi(y) \iff x = y)$. Note that, like comprehension, this is also an axiom scheme. Replacement says that, if a formula $\phi(x, y)$ defines a "function" $x \mapsto y$ on the set A (in the sense that for each $x \in A$, there is a unique y such that $\phi(x, y)$ holds), then there exists a set B which contains the range of this "function" on A.

Before we introduce the next axiom, we need to define some notions.

2 Relations, functions and well orderings

Definition 2.1 (Cartesian Product)**.** *The Cartesian product of A and B is*

$$A \times B = \{\langle x, y\rangle : x \in A \wedge y \in B\}$$

The existence of $A \times B$ is justified by the following.

Theorem 2.2.

$$\forall A \forall B \exists C (C = A \times B)$$

Proof. If either one of A, B is empty, then $A \times B = 0$ and therefore it exists. So assume A, B are both non empty. First note that, by replacement, for each $x \in A$, we can form the set $p(x, B) = \{(x, y) : y \in B\}$. Applying replacement once more, the set $\{p(x, B) : x \in A\}$ also exists. Now apply the axiom of union to get $A \times B = \bigcup\{p(x, B) : x \in A\}$. ☐

Definition 2.3 (Relation)**.** *R is a relation iff every member of R is an ordered pair:*

$$(\forall z \in R)(\exists x)(\exists y)(z = \langle x, y\rangle)$$

We sometimes write xRy in place of $\langle x, y\rangle \in R$. Define $\mathrm{dom}(R) = \{x : (\exists y)(\langle x, y\rangle \in R)\}$ and $\mathrm{range}(R) = \{y : (\exists x)(\langle x, y\rangle \in R)\}$.

Definition 2.4 (Function)**.** *f is a function iff f is a relation and*

$$\forall x \forall y \forall z \left[(\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f) \implies y = z\right]$$

If f is a function and $\langle x, y\rangle \in f$, we write $f(x) = y$ (y is the (unique) value of f at x). We write $f : A \to B$ iff f is a function, $\mathrm{dom}(f) = A$ and $\mathrm{range}(f) \subseteq B$. A function f is injective or one-to-one iff for every $x_1, x_2 \in \mathrm{dom}(f)$, $f(x_1) \neq f(x_2)$. A function $f : A \to B$ is surjective or onto iff for every $y \in B$, there exists $x \in A$ such that $f(x) = y$. If f is a function and w is a set, we define the restriction of f to w, $f \restriction w$, by $\{(x, y) \in f : x \in w \cap \mathrm{dom}(f)\}$.

Definition 2.5 (Isomorphism)**.** *Suppose R, S are relations and A, B are sets. We write $(A, R) \cong (B, S)$ iff there is a bijection $f : A \to B$ such that for every $x, y \in A$, xRy iff $f(x)Sf(y)$.*

Definition 2.6 (Linear ordering)**.** *We say that $(X, \prec)$ is a (strict) linear/total ordering if $\prec$ is a relation that satisfies the following.*

- *(Irreflexive) For all $x \in X$, $x \not\prec x$.*
- *(Transitive) For all $x, y, z \in X$, if $x \prec y$ and $y \prec z$, then $x \prec z$.*
- *(Total) For every $x, y \in X$ if $x \neq y$, then either $x \prec y$ or $y \prec x$.*

If $(X, \prec)$ is a linear order, and $x, y \in X$, we sometimes write $x \preceq y$ to denote $x = y$ or $x \prec y$. Define the set of $\prec$-predecessors of x in X by $\mathsf{pred}(X, \prec, x) = \{y \in X : y \prec x\}$.

Definition 2.7 (Well ordering)**.** *$(X, \prec)$ is a well ordering iff $(X, \prec)$ is a linear ordering and every non empty subset of X has a $\prec$-least member.*

Note that if $(X, \prec)$ is a well ordering then for every $x \in X$, either x is $\prec$-largest member of X or x has a $\prec$-successor y which means that $x \prec y$ and for every $z \prec y$, $z \preceq x$.

Lemma 2.8. *Suppose $(X, \prec)$ is a well ordering. Then $(X, \prec)$ is not isomorphic to $(\mathsf{pred}(X, \prec, x), \prec)$ for any $x \in X$.*

Proof. Suppose not and let $f : X \to \mathsf{pred}(X, \prec, x)$ be an isomorphism. Note that $f(x) \prec x$ so the set $W = \{y \in X : f(y) \prec y\}$ is non empty. Let z be $\prec$-least member of W. So $f(z) \prec z$. Since f preserves $\prec$, we also get $f(f(z)) \prec f(z)$. So $w = f(z) \in W$ and $f(w) \prec w$ which is a contradiction. □ □

Lemma 2.9. *Suppose $(X, \prec)$ is a well ordering and $f : X \to X$ is an isomorphism. Then f is the identity function on X.*

Proof. Let $f : X \to X$ be an isomorphism and, towards a contradiction, suppose for some $x \in X$, $f(x) \neq x$. Since $\prec$ is a well order, there exists $x \in X$ such that that $f(x) \neq x$ and x is $\prec$-least such member of X. Put $y = f(x)$. Then either $y \prec x$ or $x \prec y$.

If $y \prec x$, then $f(y) = y$ as x was $\prec$-least non fixed point of f. But since f preserves $\prec$ and $y \prec x$, $y = f(y) \prec f(x) = y$ which is impossible.

Next suppose $x \prec y$. Since f is surjective, there is some $w \in X$ such that $f(w) = x$. Clearly, $w \not\preceq x$ so $x \prec w$. But then $y = f(x) \prec f(w) = x$ which contradicts $x \prec y$. □ □

Corollary 2.10. *Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well orderings and $f : X \to Y$ and $g : X \to Y$ are isomorphisms. Then $f = g$.*

Theorem 2.11. *Suppose $(X, \prec_1)$ and $(Y, \prec_2)$ are well orderings. Then exactly one of the following holds.*

(1) $(X, \prec_1) \cong (Y, \prec_2)$.

(2) For some $x \in X$, $(\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (Y, \prec_2)$.

(3) For some $y \in Y$, $(\mathsf{pred}(Y, \prec_2, y), \prec_2) \cong (X, \prec_1)$.

Furthermore, in each of the three cases, the isomorphism is unique.

Proof. Note that Lemma 2.8 implies that at most one of the three cases can occur. Put

$$f = \{\langle x, y\rangle : (x \in X) \wedge (y \in Y) \wedge (\mathsf{pred}(X, \prec_1, x), \prec_1) \cong (\mathsf{pred}(Y, \prec_2, y), \prec_2)\}$$

Use Lemma 2.8 to show that f is a function, $\mathrm{dom}(f)$ is a $\prec_1$-initial segment of X, $\mathrm{range}(f)$ is a $\prec_2$-initial segment of Y and f is an isomorphism from $(\mathrm{dom}(f), \prec_1)$ to $(\mathrm{range}(f), \prec_2)$. Towards a contradiction, suppose that all of (1), (2) and (3) fail. Then $\mathrm{dom}(f) \neq X$ and $\mathrm{range}(f) \neq Y$. Let u be the $\prec_1$-least member of $X \setminus \mathrm{dom}(f)$ and let v be the $\prec_2$-least member of $Y \setminus \mathrm{range}(f)$. But then $\langle v, w\rangle \in f$ which is impossible. The uniqueness of the isomorphism in each of the cases follows from Corollary 2.10.

□

We can now state the axiom of choice.

Axiom 2.12 (Choice).

$$\forall X \exists \prec ((X, \prec) \textit{ is a well ordering})$$

Lemma 2.13. *Suppose X, Y are non empty sets, $0 \notin Y$ and $F : X \to Y$. Then there exists $h : X \to \bigcup Y$ such that for every $x \in X$, $h(x) \in F(x)$.*

Proof. Let $\prec$ be a well ordering on $Z = \bigcup Y$. Define $h(x)$ to be the $\prec$-least member of $F(x)$. More formally, $h = \{(x, y) \in X \times Z : y \in F(x) \wedge (\forall z \in F(x))(y \preceq z)\}$. □

3 Ordinals

Definition 3.1 (Transitive sets). *A set x is transitive if every member of x is a subset of x.*

Definition 3.2 (Ordinals). *x is an ordinal iff x is transitive and $(x, \in)$ is a well ordering.*

We are slightly abusing the notation here since $\in$ is not a set. Nevertheless, for any set x, the relation $\varepsilon_x = \{\langle y, z\rangle : y \in x \wedge z \in x \wedge y \in z\}$ is the restriction of the membership relation on x. So $\in$ stands for ε_x in the pair $(x, \in)$.

The proof of the following facts are left to the reader.

Fact 3.3. *(a) If x is an ordinal and $y \in x$, then y is an ordinal and $y = \mathsf{pred}(x, \in, y)$.*

(b) If x, y are ordinals and $(x, \in) \cong (y, \in)$, then $x = y$.

(c) If x is an ordinal, then $x \notin x$.

(d) If x, y are ordinals, then exactly one of the following holds: $x = y$, $x \in y$, $y \in x$.

(e) If C is a non empty set of ordinals, then there exists $x \in C$ such that $\forall y \in C(y = x \vee x \in y)$.

Theorem 3.4. *There is no set that contains every ordinal.*

Proof. Suppose not and (using comprehension) let X be the set of all ordinals. Then by Fact 3.3(a) X is transitive and by Fact 3.3(c, d, e), $(X, \in)$ is a linear ordering. Finally, by Fact 3.3(e), $(X, \in)$ is a well ordering. So X is an ordinal. Hence $X \in X$. But this contradicts Fact 3.3(c). □

Lemma 3.5. *If A is a set of ordinals, then $(A, \in)$ is a well ordering. Hence if A is a transitive set of ordinals, then A is an ordinal.*

Proof. Use Fact 3.3. □

Theorem 3.6. *For every well ordering $(X, \prec)$, there is a unique ordinal A such that $(X, \prec) \cong (A, \in)$.*

Proof. Let Y be the set of all $x \in X$ such that $(\mathsf{pred}(X, \prec, x), \prec)$ is isomorphic to an ordinal. Using the axiom of replacement, define a function f on Y by letting $f(x)$ to be the unique ordinal which is isomorphic to $(\mathsf{pred}(X, \prec, x), \prec)$. Let $A = \text{range}(f)$. Note that A is a transitive set of ordinals. Hence, by Lemma 3.5, A is an ordinal. It is also easy to check that $f : Y \to A$ is an isomorphism from $(Y, \prec)$ to $(A, \in)$.

So we would be done if $Y = X$. Suppose $Y \neq X$. Note that Y is a $\prec$-initial segment of X. Let b be the $\prec$-least member of $X \setminus Y$. Then $Y = \mathsf{pred}(X, \prec, b)$. But $(\mathsf{pred}(X, \prec, b), \prec)$ is isomorphic to the ordinal A. So $b \in Y$ which is a contradiction.

The uniqueness of A follows from Fact 3.3(b).

□

Definition 3.7 (Order type). *If $(X, \prec)$ is a well ordering, let $\mathsf{type}(X, \prec)$ be the unique ordinal A such that $(X, \prec) \cong (A, \in)$.*

We denote ordinals by Greek letters: α, β, γ, etc. and from now on we'll write $\alpha < \beta$ instead of $\alpha \in \beta$. In what follows, we write $(\forall\alpha)(\phi)$ instead of $(\forall\alpha)(\alpha \text{ is an ordinal } \implies \phi)$ and $(\exists\alpha)(\phi)$ instead of $(\exists\alpha)(\alpha \text{ is an ordinal } \wedge \phi)$.

Definition 3.8 (sup, min). *For a set of ordinals A, define* $\sup(A) = \bigcup A$ *and, if* $A \neq 0$, $\min(A) = \bigcap A$.

Check that $\sup(A)$ is the least ordinal $\geq$ every ordinal in A and $\min(A)$ is the least ordinal in A.

Definition 3.9 (Successor and limit). *The successor of α is defined by*

$$S(\alpha) = \alpha \cup \{\alpha\}$$

An ordinal α is called a successor ordinal if for some ordinal β, $\alpha = S(\beta)$. Otherwise α is a limit ordinal.

Note that $S(\alpha)$ is the least ordinal bigger than α.

Definition 3.10 (Natural numbers). *α is a natural number iff for every $\beta \leq \alpha$, β is a successor ordinal.*

We define $1 = S(0)$, $2 = S(1)$, $3 = S(2)$ etc. To ensure the existence of the set of all natural numbers, we need the following.

Axiom 3.11 (Infinity).

$$\exists x(0 \in x \wedge (\forall y \in x)(S(y) \in x))$$

If $0 \in x \wedge (\forall y \in x)(S(y) \in x)$, then (using comprehension) we define $\omega = \{y \in x : y \text{ is a natural number}\}$.

Given two linear orderings $(L_1, \prec_1)$ and $(L_2, \prec_2)$, one can define another linear ordering by putting a copy of $(L_2, \prec_2)$ after a copy of $(L_1, \prec_1)$. The following definition makes this precise.

Definition 3.12. *Suppose $(L_1, \prec_1)$ and $(L_2, \prec_2)$ are linear orderings. We define the sum*

$$(L, \prec) = (L_1, \prec_1) \oplus (Ł_2, \prec_2)$$

as follows.

(1) $L = (L_1 \times \{0\}) \bigcup (L_2 \times \{1\})$.

(2) For every $x, y \in L$, $x \prec y$ iff one of the following holds

- *(a)* $x = (a, 0)$, $y = (b, 0)$ *and* $a \prec_1 b$.
- *(b)* $x = (a, 1)$, $y = (b, 1)$ *and* $a \prec_2 b$.
- *(c)* $x = (a, 0)$ *and* $y = (b, 1)$.

It is easy to check that the $(L, \prec) = (L_1, \prec_1) \oplus (Ł_2, \prec_2)$ is also a linear ordering. Note that we defined $L = (L_1 \times \{0\}) \bigcup (L_2 \times \{1\})$ (and not $L = L_1 \bigcup L_2$) because L_1, L_2 may not be disjoint.

Definition 3.13 (Ordinal addition).

$$\alpha + \beta = \textit{type}((\alpha, <) \oplus (\beta, <))$$

It is easy to check that $\alpha + \beta$ is an ordinal. Note that $S(\alpha) = \alpha + 1$ and if $m, n < \omega$, then $m + n$ is the usual sum. Ordinal addition is not commutative in general: For example $\omega = 1 + \omega \neq \omega + 1$. The first few ordinals are: $0, 1, 2, \ldots, \omega, S(\omega) = \omega + 1, \omega + 2, \ldots, \omega + \omega, \omega + \omega + 1, \ldots, \omega + \omega, \ldots$.

Definition 3.14 (Lexicographic order). *Suppose $(L_1, \prec_1)$ and $(L_2, \prec_2)$ are linear orderings. We define the product*

$$(L, \prec) = (L_1, \prec_1) \otimes (Ł_2, \prec_2)$$

as follows.

(1) $L = L_1 \times L_2$.

(2) For every (x_1, y_1) and (x_2, y_2) in L, $(x_1, y_1) \prec (x_2, y_2)$ iff

- *(a) Either $x_1 \prec_1 x_2$ or*
- *(b) $x_1 = x_2$ and $y_1 \prec_2 y_2$.*

Definition 3.15 (Ordinal multiplication)**.**

$$\alpha \cdot \beta = \mathit{type}((\beta, <) \otimes (\alpha, <))$$

It is again easy to check that $\alpha \cdot \beta$ is an ordinal. If $m, n < \omega$, then $m \cdot n$ is the usual product. Ordinal multiplication is not commutative in general: $\omega \cdot 2 = \omega + \omega \neq 2 \cdot \omega = \omega$. We leave the following as an exercise for the reader.

Lemma 3.16. *For any α, β, γ*

(i) (Associativity) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ and $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

(ii) $\alpha + 0 = \alpha$, $\alpha \cdot 0 = 0$ and $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

(iii) (Continuity at limits) If β is a limit ordinal, $\alpha + \beta = \sup\{\alpha + \eta : \eta < \beta\}$ and $\alpha \cdot \beta = \sup\{\alpha \cdot \eta : \eta < \beta\}$

(iv) (Left distributivity) $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$

Definition 3.17 (Finitary functions)**.** *${}^{B}A$ is the set of all functions from A to B. If $n < \omega$, we write A^n instead of ${}^{n}A$. $A^{<\omega} = \bigcup\{A^n : n < \omega\}$. A function f is a finitary function on A, if for some $n < \omega$, $f : A^n \to A$.*

Definition 3.18 (Sequences, enumerations)**.** *We say that $\langle x_\xi : \xi < \alpha\rangle$ is a sequence in X iff there is a function $f : \alpha \to X$ such that for every $\xi < \alpha$, $f(\xi) = x_\xi$. We say that $\langle x_\xi : \xi < \alpha\rangle$ an enumeration of X iff $\langle x_\xi : \xi < \alpha\rangle$ is a sequence and $(\forall x \in X)(\exists \xi < \alpha)(x = x_\xi)$.*

4 Classes and Transfinite Recursion

A class is an expression of the form $\mathbf{A} = \{x : \phi\}$ where ϕ is a formula. If $\mathbf{A} = \{x : \phi\}$, we write $v \in \mathbf{A}$ to denote $\phi(v/x)$ where $\phi(v/x)$ is obtained by replacing every free occurrence of x in ϕ, by v. We'll denote classes by bold font upper case Roman letters. Every set is a class: $x = \{v : v \in x\}$. We say that a class $\mathbf{A}$ is a set iff $\exists y(y = \mathbf{A})$. For example, $\bigcup X$ and $A \times B$ are sets. A class is a proper class if it is not a set.

Definition 4.1. $\mathbf{V} = \{x : x = x\}$ *and* $\mathbf{ORD} = \{x : x \text{ is an ordinal}\}$.

Note that $\mathbf{V}$ and $\mathbf{ORD}$ are proper classes. The following is an example of a theorem scheme: For each class $\mathbf{C} = \{x : \phi\}$ we get one theorem.

Theorem 4.2 (Transfinite induction)**.** *If $\mathbf{C} \subseteq \mathbf{ORD}$ and $\mathbf{C} \neq 0$, then $\mathbf{C}$ has a least member.*

Proof. Since $\mathbf{C} \neq 0$, fix $\alpha \in \mathbf{C}$. By comprehension, $X = \{\beta \in \alpha : \beta \in \mathbf{C}\}$ is a set of ordinals. If $X = \emptyset$, then it is clear that α is the least ordinal in $\mathbf{C}$. So assume $X \neq \emptyset$. Now by Fact 3.3(e), X has a least ordinal β. It is easy to see that β is also the least ordinal in $\mathbf{C}$. ☐

Definition 4.3. *Let $\mathbf{A} = \{x : \phi(x)\}$ and $\mathbf{B} = \{x : \psi(x)\}$. We define*

- $\mathbf{A} \bigcup \mathbf{B} = \{x : \phi(x) \vee \psi(x)\}$
- $\mathbf{A} \bigcap \mathbf{B} = \{x : \phi(x) \wedge \psi(x)\}$
- $\mathbf{A} \setminus \mathbf{B} = \{x : \phi(x) \wedge \neg\psi(x)\}$
- $\bigcup \mathbf{A} = \{v : \exists x(\phi(x) \wedge v \in x)\}$
- $\mathbf{A} \subseteq \mathbf{C}$ *iff* $(\forall x)(\phi(x) \implies \psi(x))$

Using classes, we can restate the axiom of comprehension as follows: For each class $\mathbf{A} = \{v : \phi(v)\}$, $\mathbf{A} \cap x = \mathbf{A} \cap \{v : v \in x\} = \{v : v \in x \wedge \phi(v)\}$ is a set.

Definition 4.4. *Suppose* $\mathbf{F} = \{z : \phi(z)\}$, $\mathbf{A} = \{x : \psi(x)\}$ *and* $\mathbf{B} = \{y : \chi(y)\}$ *are classes. We say that* $\mathbf{F}$ *is a function from* $\mathbf{A}$ *to* $\mathbf{B}$ *and write* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *iff*

$$(\forall x \in \mathbf{A})(\exists! y \in \mathbf{B})(\exists z \in \mathbf{F})[z = (x, y)]$$

We write $\mathbf{F}(x) = y$ *to denote* $(x, y) \in \mathbf{F}$.

For example, we can think of the successor operation $\alpha \mapsto S(\alpha)$ as the class function $\mathbf{S} = \{z : \exists\alpha(z = (\alpha, S(\alpha))\}$ from $\mathbf{ORD}$ to $\mathbf{ORD}$. If $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{A}$, then we define $\mathbf{F} \restriction \mathbf{C} = \{(x, y) \in \mathbf{F} : x \in \mathbf{C}\}$.

Theorem 4.5 (Transfinite recursion)**.** *Suppose* $\mathbf{F} : \mathbf{V} \to \mathbf{V}$. *Then there exists a unique* $\mathbf{G} : \mathbf{ORD} \to \mathbf{V}$ *such that* $(\forall\alpha)(\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha))$.

Proof. Put

$$\mathbf{G} = \bigcup\{g : g \text{ is a function } \wedge \exists\alpha[\mathrm{dom}(g) = \alpha \wedge \forall\beta < \alpha(g(\beta) = \mathbf{F}(g \restriction \beta))]\}$$

Use transfinite induction (Theorem 4.2) to check that $\mathbf{G} : \mathbf{ORD} \to \mathbf{V}$ and $(\forall\alpha)(\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha))$. Uniqueness follows from Theorem 4.2 as well. □

As an application of transfinite recursion (Theorem 4.5), let us define ordinal exponentiation α^β. By transfinite recursion on β, define α^β as follows.

(i) $\alpha^0 = 1$.

(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.

(iii) If β is a limit ordinal, then $\alpha^\beta = \sup(\{\alpha^\gamma : \gamma < \beta\})$.

As an application of transfinite induction (Theorem 4.2), by induction on γ, show that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

5 Cardinals

Theorem 5.1 (Schröder-Bernstein)**.** *Suppose there are injective functions* $f : A \to B$ *and* $g : B \to A$. *Then there exists a bijection* $f : A \to B$.

The following easily follows from Theorem 2.11.

Lemma 5.2. *Assume AC (axiom of choice). For any pair of sets* X *and* Y, *either there is an injective function from* X *to* Y *or there is an injective function from* Y *to* X.

Definition 5.3 (Cardinality)**.** *If* X *can we well ordered, we define the cardinality of* X, *denoted* $|X|$, *as the least ordinal* α *such that there is a bijection from* α *to* X.

Under the axiom of choice, $|X|$ is defined for every set X. If X, Y can be well ordered, then $|X| = |Y|$ iff there is a bijection from X to Y.

Definition 5.4 (Cardinals)**.** *A cardinal is an ordinal* κ *such that* $|\kappa| = \kappa$. *The class of all cardinals is denoted by* $\mathbf{CARD} = \{x : x \textit{ is a cardinal}\}$.

$0, 1, 2, \ldots,$ are the finite cardinals. ω is the first infinite cardinal. $\omega + 1$ is not a cardinal since $|\omega + 1| = \omega$.

Definition 5.5 (Countable)**.** *A set* x *is countable iff there exists a* $f : \omega \to x$ *such that range*$(f) = x$. *Otherwise,* x *is uncountable*

As an exercise, show that $\omega \times \omega$ is countable. The power set axiom is needed to guarantee the existence of uncountable sets.

Axiom 5.6 (Power set)**.**

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

Define the power set of x, $\mathcal{P}(x) = \{z : z \subseteq x\}$.

Theorem 5.7 (Cantor). *For every set x, there is no surjective fnction $f : x \to \mathcal{P}(x)$.*

Proof. Let $f : x \to \mathcal{P}(x)$. Define $y = \{v \in x : v \notin f(v)\}$. We claim that $y \notin \text{range}(f)$. Suppose not and let $a \in x$ be such that $f(a) = y$. Then $a \in y$ iff $a \notin f(a)$ iff $a \notin y$ which is impossible. $\square$

It follows from the power set axiom and the axiom of choice that $|\mathcal{P}(\kappa)| > \kappa$. The next theorem says that the axiom of choice is not needed for this.

Theorem 5.8 (ZF). *For every α, there exists κ such that $\kappa > |\alpha|$.*

Proof. If α is finite, this is clear since we can take $\kappa = \alpha + 1 > \alpha$. So assume α is infinite. Let $W = \{R \in \mathcal{P}(\alpha \times \alpha) : ((\alpha, R) \text{ is a well ordering}\}$. By replacement, $X = \{\mathsf{type}((\alpha, R)) : R \in W\}$ is a set of ordinals. Let $\kappa = \sup(X)$. We claim that $\kappa > |\alpha|$. Suppose not. Then $\kappa \leq |\alpha|$. Note that $\kappa \geq \alpha$ since $\alpha \in X$ (being the type of the usual well order on α). By Schröder-Bernstein theorem, it follows that $|\kappa| = |\alpha|$. Since α is infinite and $\kappa \geq \alpha$, $|\kappa+1| = |\kappa| = |\alpha|$. Let $f : \alpha \to \kappa+1$ be a bijection and define $R = \{(\xi, \eta) \in \alpha \times \alpha : f(\xi) < f(\eta)\}$. Then (α, R) is a well ordering and $\mathsf{type}((\alpha, R)) = \kappa + 1$. But this means $\kappa + 1 \in X$ while $\sup(X) = \kappa$ which is impossible. Note that the proof doesn't use AC. $\square$

Definition 5.9 (Successor/Limit cardinals). *Suppose α is an ordinal and κ is a cardinal.*

(a) α^+ *is the least cardinal* $> \alpha$.

(b) κ *is a successor cardinal iff* $\kappa = \alpha^+$ *for some* α.

(c) κ *is a limit cardinal iff* κ *is not a successor cardinal.*

Definition 5.10 (Aleph/Omega hierarchy). *Using transfinite recursion, $\omega_\alpha = \aleph_\alpha$ is defined as follows.*

(i) $\omega_0 = \omega$.

(ii) $\omega_{\alpha+1} = (\omega_\alpha)^+$.

(iii) If α is a limit ordinal, then $\omega_\alpha = \sup(\{\omega_\beta : \beta < \alpha\})$.

The reader should check that $\{(\alpha, \omega_\alpha) : \alpha \in \mathbf{ORD}\}$ is a definable class using Theorem 4.5. The following is easy to check using transfinite induction on α.

Lemma 5.11. *The following hold for every ordinal α.*

(a) $\omega_\alpha \geq \alpha$.

(b) $\mathbf{CARD} = \{\omega_\alpha : \alpha \in \mathbf{ORD}\}$.

(c) For every $\beta < \alpha$, $\omega_\beta < \omega_\alpha$.

(d) ω_α *is a successor cardinal iff* α *is a successor ordinal.*

(e) ω_α *is a limit cardinal iff* α *is a limit ordinal.*

Lemma 5.12 (ZF). *Suppose κ is an infinite cardinal. Then $|\kappa \times \kappa| = \kappa$.*

Proof. By transfinite induction on κ. If $\kappa = \omega$, then this holds. So assume $\kappa > \omega$ and for every cardinal $\theta < \kappa$, $|\theta \times \theta| = \theta$. Define an ordering $\prec$ (called the max-lexicographic order) on $\kappa \times \kappa$ as follows: $(\alpha_1, \beta_1) \prec (\alpha_2, \beta_2)$ iff

- either $\mathsf{max}(\{\alpha_1, \beta_1\}) < \mathsf{max}(\{\alpha_2, \beta_2\})$ or
- $\mathsf{max}(\{\alpha_1, \beta_1\}) = \mathsf{max}(\{\alpha_2, \beta_2\})$ and $\alpha_1 < \alpha_2$ or
- $\mathsf{max}(\{\alpha_1, \beta_1\}) = \mathsf{max}(\{\alpha_2, \beta_2\})$ and $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$.

It is easy to check that $\prec$ is a well ordering on $\kappa \times \kappa$. If $\alpha < \kappa$ is infinite, then the set $\mathsf{pred}(\kappa \times \kappa, \prec, (\alpha, \alpha))$ of $\prec$-predecessors of (α, α) is contained in $(\alpha+1) \times (\alpha+1)$ and hence, by inductive hypothesis, has cardinality $\leq |(\alpha+1) \times (\alpha+1)| = ||\alpha| \times |\alpha|| = |\alpha| \leq \alpha < \kappa$. It follows that every $\prec$-initial segment of $(\kappa \times \kappa, \prec)$ has type $< \kappa$. So $\mathsf{type}(\kappa \times \kappa, \prec) = \kappa$. Hence $|\kappa \times \kappa| = \kappa$. □

Definition 5.13 (Cardinal addition and multiplication)**.** *For cardinals κ and λ define the cardinal sum and product as follows.*

- $\kappa \oplus \lambda = |(\kappa \times \{0\}) \bigcup (\lambda \times \{1\})|$
- $\kappa \otimes \lambda = |\kappa \times \lambda|$

For finite cardinals m, n, $m \oplus n = m + n$ and $m \otimes n = m \cdot n$ are the usual sums and products. For infinite cardinals, the sums and products are just the maximum.

Lemma 5.14. *Suppose κ and λ are cardinals and one of them is infinite. Then, $\kappa \oplus \lambda = \kappa \otimes \lambda = \mathsf{max}\{\kappa, \lambda\}$.*

Proof. We can assume that $\lambda \leq \kappa$ and κ is infinite. Note that it is enough to show that $\kappa \otimes \kappa = \kappa$. But this follows from Lemma 5.12. □

Lemma 5.15 (AC)**.** *Suppose κ is an infinite cardinal and $|X_\alpha| \leq \kappa$ for every $\alpha < \kappa$. Then $|\bigcup\{X_\alpha : \alpha < \kappa\}| \leq \kappa$.*

Proof. Put $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. Fix a well ordering $\prec$ of $\mathcal{P}(X \times \kappa)$. Let h be a function with domain κ such that for every $\alpha < \kappa$, $h(\alpha)$ is the $\prec$-least injective function from X_α to κ. It follows that there is a injective function from $g : X \to \kappa \times \kappa$ – Given $x \in X$, pick the least α such that $x \in X_\alpha$ and define $g(x) = (\alpha, h(\alpha)(x))$. It follows that $|X| \leq |\kappa \times \kappa| = \kappa$. □

Definition 5.16 (Closure)**.** *Suppose $f : A^n \to A$ is a finitary function on A and $B \subseteq A$.*

(a) We say that B is closed under f iff range$(f \restriction B^n) \subseteq B$.

(b) We define the closure of B under A to be the set $\bigcap\{C \subseteq A : B \subseteq C \wedge C$ is closed under $f\}$

Theorem 5.17 (AC)**.** *Let κ be an infinite cardinal. Suppose $B \subseteq A$, $|B| \leq \kappa$ and $\mathcal{F}$ is a set of $\leq \kappa$ finitary functions on A. Then there exists $C \subseteq A$ such that*

- *$B \subseteq C \subseteq A$,*
- *$|C| \leq \kappa$ and*
- *for every $f \in \mathcal{F}$, C is closed under f.*

Proof. For $f \in \mathcal{F}$ and $D \subseteq A$, define $f \star D = \mathrm{range}(f \restriction D^n)$ where $f : A^n \to A$. Inductively, define $C_0 = B$ and $C_{n+1} = C_n \cup \bigcup\{f \star C_n : f \in \mathcal{F}\}$. By Lemma 5.15, for every $n < \omega$, $|C_n| \leq \kappa$. Put $C = \bigcup\{C_n : n < \omega\}$. By Lemma 5.15 again, $|C| \leq \kappa$. It is easy to see that $B \subseteq C \subseteq A$ and C is closed under every function in $\mathcal{F}$. □

6 Cardinal exponentiation and cofinality

Definition 6.1 (AC)**.** *For cardinals κ and λ, define*

$$\kappa^\lambda = |{}^\lambda\kappa|$$

Recall that ${}^\lambda\kappa$ is the set of all functions from λ to κ. Note that there is a natural bijection from $\mathcal{P}(\lambda)$ to ${}^\lambda 2$ that maps $A \subseteq \lambda$ to its characteristic function $1_A : \lambda \to 2$. So $|\mathcal{P}(\lambda)| = 2^\lambda$

Lemma 6.2. *For every $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$. Then there is a bijection $f : {}^\lambda\kappa \to {}^\lambda 2$.*

Proof. Let us write $A \preceq B$ iff there is an injective function from A to B. Then

$$^{\lambda}2 \preceq {}^{\lambda}\kappa \preceq {}^{\lambda}\lambda \preceq \mathcal{P}(\lambda \times \lambda) \preceq \mathcal{P}(\lambda) \preceq {}^{\lambda}2$$

Now apply Schröder-Bernstein theorem (Theorem 5.1). $\square$

The following is left as an exercise for the reader.

Lemma 6.3 (AC)**.** *Suppose κ, λ, μ are cardinals. Then $\kappa^{\lambda \oplus \mu} = \kappa^{\lambda} \otimes \kappa^{\mu}$ and $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \otimes \mu}$.*

CH (Continuum hypothesis) is the statement $2^{\omega} = \omega_1$ and GCH (Generalized continuum hypothesis) is the statement: For every cardinal κ, $2^{\kappa} = \kappa^{+}$. Note that, assuming AC, Theorem 5.7 implies that $2^{\kappa} \geq \kappa^{+}$.

Definition 6.4 (Cofinality)**.** *$X \subseteq \alpha$ is cofinal in α iff $\sup(X) = \alpha$. A function $f : \alpha \to \beta$ is cofinal iff range(f) is cofinal in β. The cofinality of β, denoted cf(β), is the least ordinal α such that there is a cofinal $f : \alpha \to \beta$.*

Note that $\mathrm{cf}(\beta) \leq \beta$ and if $\beta = \alpha + 1$ is a successor ordinal, then $\mathrm{cf}(\beta) = 1$.

Lemma 6.5. *Suppose cf$(\beta) = \alpha$. Then there is a strictly increasing cofinal function $f : \alpha \to \beta$.*

Proof. Let $g : \alpha \to \beta$ be a cofinal function. Define $f : \alpha \to \beta$ inductively by

$$f(\xi) = \max(g(\xi), \sup(\{f(\eta) + 1 : \eta < \xi\}))$$

It is easy to check that $\xi < \eta < \alpha \implies f(\xi) < f(\eta)$ and $\mathrm{range}(f) \subseteq \beta$ is cofinal. $\square$

Lemma 6.6. *Suppose α is a limit ordinal and $f : \alpha \to \beta$ is strictly increasing cofinal function. Then cf$(\alpha) =$ cf(β).*

Proof. Let $f_1 : \mathrm{cf}(\alpha) \to \alpha$ be a cofinal function. Then $f_2 : \mathrm{cf}(\alpha) \to \beta$ defined by $f_2(\xi) = f(f_1(\xi))$ is cofinal. So $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$.

Next, let $g : \mathrm{cf}(\beta) \to \beta$ be cofinal. Define $h : \mathrm{cf}(\beta) \to \alpha$ by $h(\xi) = \min(\{\eta < \alpha : f(\eta) > g(\xi)\})$. Since f is strictly increasing, $h : \mathrm{cf}(\beta) \to \alpha$ is a cofinal function. So $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$. $\square$

It follows that if α is a limit ordinal, then $\mathrm{cf}(\omega_{\alpha}) = \mathrm{cf}(\alpha)$.

Corollary 6.7. *cf(cf$(\beta)) =$ cf(β).*

Proof. By Lemma 6.5, there is a strictly increasing cofinal function $f : \mathrm{cf}(\beta) \to \beta$. Apply Lemma 6.6. $\square$

Definition 6.8 (Regular/Singular)**.** *β is regular iff cf$(\beta) = \beta$. Otherwise it is singular.*

It is easy to check that ω is regular and if β is regular, then β is a cardinal.

Lemma 6.9 (AC)**.** *Every successor cardinal is regular.*

Proof. Let κ^{+} be a successor cardinal and let $\alpha < \kappa^{+}$. Then $|\alpha| \leq \kappa$. Towards a contradiction, suppose $f : \alpha \to \kappa^{+}$ is cofinal. Then $\kappa = \bigcup(\{f(\xi) : \xi < \alpha\})$. But this contradicts Lemma 5.15 which says that a union of fewer than κ sets, each of size $\leq \kappa$ has size $\leq \kappa$. $\square$

It follows that for every $n < \omega$, ω_n is regular. The first infinite singular cardinal is ω_{ω} whose cofinality is ω.

Definition 6.10 (Weakly/Strongly inaccessible)**.** *κ is weakly inaccessible if κ is a regular limit cardinal. κ is strongly inaccessible if for every $\lambda < \kappa$, $2^{\lambda} < \kappa$.*

We cannot prove the existence of weakly/strongly inaccessible cardinals in ZFC. This will be discussed later in the semester.

Lemma 6.11 (König)**.** *Assuming AC, $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.*

Proof. Let $f : \mathrm{cf}(\kappa) \to \kappa$ be cofinal. Suppose $G : \kappa \to {}^{\mathrm{cf}(\kappa)}\kappa$. It suffice to show that G is not onto. Define $h : \mathrm{cf}(\kappa) \to \kappa$ by

$$h(\alpha) = \min(\kappa \setminus \{G(\mu)(\alpha) : \mu < f(\alpha)\})$$

It is easy to check that $h \notin \mathrm{range}(G)$. □

Corollary 6.12 (AC). *$cf(2^\lambda) > \lambda$.*

Proof. Suppose $\mathrm{cf}(2^\lambda) \leq \lambda$. Then $(2^\lambda)^{\mathrm{cf}(2^\lambda)} \leq (2^\lambda)^\lambda = 2^{\lambda \otimes \lambda} = 2^\lambda$ which contradicts Lemma 6.11. □

Definition 6.13 (Beth hierarchy). *Using transfinite induction, $\beth_\alpha$ is defined as follows.*

(i) $\beth_0 = \omega$

(ii) $\beth_{\alpha+1} = 2^{\beth_\alpha}$

(iii) *If γ is limit ordinal, then* $\beth_\gamma = \sup(\{\beth_\alpha : \alpha < \gamma\})$

One can restate GCH as follows: $\forall\alpha(\beth_\alpha = \omega_\alpha)$.

7 The non-stationary ideal

Definition 7.1 (Ideals). *For a non empty set X, an ideal on X is a subset $\mathcal{I} \subseteq \mathcal{P}(X)$ satisfying the following conditions.*

(i) $0 \in \mathcal{I}$ *and* $X \notin \mathcal{I}$.

(ii) *For every* $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$.

(iii) *For every* $A \subseteq B \subseteq X$, *if* $B \in \mathcal{I}$, *then* $A \in \mathcal{I}$.

Two examples of ideals follow.

Example 7.2. *(1) Let κ be an infinite cardinal and $\mathcal{I} = \{A \subseteq \kappa : |A| < \kappa\}$. Then $\mathcal{I}$ is an ideal on κ.*

(2) A subset $X \subseteq \mathbb{R}$ is called nowhere dense if for every open interval $J \subseteq \mathbb{R}$, there is an open subinterval K such that $K \cap X = 0$. Let $\mathcal{I}$ be the family of those subsets of $\mathbb{R}$ that can be covered by the union of a countable family of nowhere dense sets. Then $\mathcal{I}$ is an ideal on $\mathbb{R}$. Members of $\mathcal{I}$ are called meager *sets.*

Definition 7.3 (Filters). *For a non empty set X, a filter on X is a subset $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following conditions.*

(i) $0 \notin \mathcal{F}$ *and* $X \in \mathcal{F}$.

(ii) *For every* $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

(iii) *For every* $A \subseteq B \subseteq X$, *if* $A \in \mathcal{F}$, *then* $B \in \mathcal{F}$.

Like the notions of closed and open sets in a topological space, ideals and filters are dual notions.

Definition 7.4 (Duals). *For an ideal $\mathcal{I}$ on X, the* dual filter *$\mathcal{I}^\star$ on X is defined by $\mathcal{I}^\star = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$. For a filter $\mathcal{F}$ on X, the* dual ideal *$\mathcal{F}^\star$ is defined by $\mathcal{F}^\star = \{A \subseteq X : X \setminus A \in \mathcal{F}\}$.*

The following is easy to check.

Lemma 7.5. *Suppose $\mathcal{I}$ is an ideal on X and $\mathcal{F}$ is a filter on X. Then $\mathcal{I}^\star$ is a filter on X and $\mathcal{F}^\star$ is an ideal on X. Furthermore, $(\mathcal{I}^\star)^\star = \mathcal{I}$ and $(\mathcal{F}^\star)^\star = \mathcal{F}$.*

Definition 7.6 (Completeness)**.** *An ideal $\mathcal{I}$ is κ-complete iff*

$$(\forall \mathcal{A} \subseteq \mathcal{I})(|\mathcal{A}| < \kappa \implies \bigcup \mathcal{A} \in \mathcal{I})$$

A filter $\mathcal{F}$ is κ-complete iff

$$(\forall \mathcal{A} \subseteq \mathcal{F})(|\mathcal{A}| < \kappa \implies \bigcap \mathcal{A} \in \mathcal{F})$$

Every ideal/filter is ω-complete. An ω_1-complete ideal/filter is sometimes also called a σ-complete ideal/filter. The meager ideal on $\mathbb{R}$ (Example 7.2(2)) is a σ-ideal. The ideal in Example 7.2(1) is κ-complete iff κ is a regular cardinal.

Definition 7.7 (Closed, Unbounded)**.** *Suppose α is an ordinal and $X \subseteq \alpha$.*

(1) X is unbounded in α iff $\sup(X \cap \alpha) = \alpha$.

(2) X is closed in α iff for every limit ordinal $\beta < \alpha$, if $X \cap \beta$ is unbounded in β, then $\beta \in X$.

(3) X is a club in α iff it is closed and unbounded in α.

Definition 7.8 (Club filter)**.** *If $cf(\mu) > \omega$, we define the club filter on μ by*

$$\mathsf{Club}(\mu) = \{X \subseteq \mu : (\exists C \subseteq X)(C \text{ is closed and unbounded in } \mu)\}$$

It is easy to check that $\{\alpha < \omega_1 : \alpha \text{ is a limit ordinal}\} \in \mathsf{Club}(\omega_1)$.

Lemma 7.9. *Suppose $cf(\mu) > \omega$ and $\mathcal{A}$ is a family of closed unbounded subsets of μ. Assume $|\mathcal{A}| < cf(\mu)$. Then $\bigcap \mathcal{A}$ is closed unbounded in μ.*

Proof. Let $|\mathcal{A}| = \theta < \mathrm{cf}(\mu)$. Fix an enumeration $\langle C_\alpha : \alpha < \theta\rangle$ of $\mathcal{A}$. Put $D = \bigcap\{C_\alpha : \alpha < \theta\}$. It is easy to check that D is closed. So it suffices to show that D is unbounded in μ.

For each $\alpha < \theta$, define $g_\alpha : \mu \to \mu$ by setting $g_\alpha(\xi)$ to be the least member of C_α strictly bigger than ξ. Since C_α is unbounded in μ, g_α is well defined. Note that since $\theta < \mathrm{cf}(\mu)$, for every $\xi < \mu$, $\sup(\{g_\alpha(\xi) : \alpha < \theta\}) < \mu$. Define $h : \mu \to \mu$ by $h(\xi) = \sup(\{g_\alpha(\xi) : \alpha < \theta\})$. Note that $h(\xi) > \xi$ for every $\xi < \mu$.

Let $\xi < \mu$ be arbitrary. For $1 \le n < \omega$, define $h^n(\xi)$ by $h^1(\xi) = h(\xi)$ and $h^{n+1}(\xi) = h(h^n(\xi))$. Observe that $h^n(\xi)$'s are strictly increasing with n. Define $\gamma = \sup(\{h^n(\xi) : 1 \le n < \omega\})$. We claim that $\xi < \gamma < \mu$ and $\gamma \in D$. As $\gamma > h(\xi)$ and $h(\xi) > \xi$, it follows that $\gamma > \xi$. Since $\mathrm{cf}(\mu) > \omega$, the set $\{h^n(\xi) : 1 \le n < \omega\}$ is bounded in μ. Hence its supremum $\gamma < \mu$. Finally note that, for each $\alpha < \theta$ and $1 \le n < \omega$, C_α contains an ordinal between $h^n(\xi)$ and $h^{n+1}(\xi)$. Therefore $C_\alpha \cap \gamma$ is unbounded in γ. Since C_α is closed in μ, $\gamma \in C_\alpha$. So $\gamma \in D = \bigcap\{C_\alpha : \alpha < \theta\}$. Hence D is unbounded in μ. $\square$

Lemma 7.10. *Suppose $cf(\mu) > \omega$. Then $\mathsf{Club}(\mu)$ is a $cf(\mu)$-complete filter on μ.*

Proof. It is clear that $0 \notin \mathsf{Club}(\mu)$ and $\mu \in \mathsf{Club}(\mu)$. It is also clear that if $X \in \mathsf{Club}(\mu)$ and $X \subseteq Y \subseteq \mu$, then $Y \in \mathsf{Club}(\mu)$. Finally, by Lemma 7.9, $\mathsf{Club}(\mu)$ is $\mathrm{cf}(\mu)$-complete. $\square$

Lemma 7.11. *Suppose κ is a regular uncountable cardinal and $\mathcal{F}$ is a family of $< \kappa$ finitary functions on κ. Then*

$$C = \{\alpha < \kappa : \alpha \text{ is closed under every function in } \mathcal{F}\}$$

is closed unbounded in κ.

Proof. It is easy to check that C is a closed subset of κ. To see that it is unbounded in κ, let $|\mathcal{F}| < \xi < \kappa$ be arbitrary. Inductively define $\langle \xi_n : n < \omega\rangle$ as follows. Put $\xi_0 = \xi$. Suppose ξ_n has been defined. Using Theorem 5.17, choose $B \subseteq \kappa$ such that $\xi_n \subseteq B$, $|B| \le |\xi_n| < \kappa$ and B is closed under every function in $\mathcal{F}$ and define $\xi_{n+1} = \sup(B)$. Observe that $\alpha = \sup(\{\xi_n : n < \omega\}) < \kappa$ (as κ is regular and uncountable) and α is closed under every function in $\mathcal{F}$. So $\xi < \alpha \in C$. Hence C is unbounded in μ. $\square$

It follows that if $(\kappa, \circ)$ is a group, then the set of $\alpha < \kappa$ such that $(\alpha, \circ)$ is a subgroup of $(\kappa, \circ)$ is closed unbounded in κ.

Definition 7.12 (Stationary). *Suppose $cf(\mu) > \omega$ and $S \subseteq \mu$. We say that S is stationary in μ iff for every $C \in \mathsf{Club}(\mu)$, $C \cap S \neq 0$. The non stationary ideal on μ is defined by*

$$\mathsf{NS}(\mu) = \{X \subseteq \mu : X \text{ is not stationary in } \mu\}$$

Note that $\mathsf{NS}(\mu)$ is the dual of $\mathsf{Club}(\mu)$.

Lemma 7.13. *Suppose $cf(\mu) > \lambda$ where λ is a regular infinite cardinal. Then $S = \{\alpha < \mu : cf(\alpha) = \lambda\}$ is stationary in μ.*

Proof. Let C be an arbitrary club in μ. Inductively construct $\langle \gamma_\alpha : \alpha < \lambda \rangle$ such that each $\gamma_\alpha \in C$ and γ_α's are strictly increasing with α. Put $\gamma = \sup(\{\gamma_\alpha : \alpha < \lambda\})$. Since $\mathrm{cf}(\mu) > \lambda$, $\gamma < \mu$. Since C is closed in μ, $\gamma \in C$. As $\mathrm{cf}(\gamma) = \lambda$, $\gamma \in S \cap C$. Therefore, S meets every closed unbounded subset of μ. Hence S is stationary in μ. $\square$

Suppose f is a function such that $\mathrm{dom}(f)$ and $\mathrm{range}(f)$ are sets of ordinals. We say that f is regressive if $(\forall \alpha \in \mathrm{dom}(f))(f(\alpha) < \alpha)$. The name "stationary" originates from the following fact about regressive function defined on stationary sets.

Lemma 7.14 (Pressing-down/Fodor's Lemma). *Suppose κ is a regular uncountable cardinal $S \subseteq \kappa$ is stationary in κ. Let $f : S \to \kappa$ be regressive. Then there exists $T \subseteq S$ such that T is stationary in κ and $f \restriction T$ is constant.*

Proof. We need a lemma about the diagonal intersection of clubs.

Lemma 7.15. *Suppose κ is regular uncountable and $\langle C_\alpha : \alpha < \kappa \rangle$ is a sequence of clubs in κ. Let $D = \{\gamma < \kappa : (\forall \alpha < \gamma)(\gamma \in C_\alpha)\}$. Then D is a club in κ.*

Proof. It is easy to check that D is closed in κ. To see that it is unbounded in κ, fix a function $g : \kappa \to \kappa$ such that for every $\xi < \kappa$, $g(\xi) \in \bigcap\{C_\alpha : \alpha < \xi\}$ and $g(\xi) > \xi$. Since $\mathsf{Club}(\kappa)$ is a κ-complete filter, g is well defined. Let $\gamma = \sup(\{g^n(\xi) : 1 \leq n < \omega\})$. Note that $\gamma < \kappa$ (as κ is regular uncountable) and for every $\alpha < \gamma$, $C_\alpha \cap \gamma$ is unbounded in γ. As each C_α is closed, it follows that for every $\alpha < \gamma$, $\gamma \in C_\alpha$. So $\xi < \gamma \in D$. Therefore D is unbounded in κ. $\square$

The set D in Lemma 7.15 is called the diagonal intersection of $\langle C_\alpha : \alpha < \kappa \rangle$.

We now prove Lemma 7.14. Let $f : S \to \kappa$ be regressive. Towards a contradiction, suppose there is no stationary $T \subseteq S$ such that $f \restriction T$ is constant. Then for each $\alpha < \kappa$, we can find a club C_α in κ such that for every $\xi \in C_\alpha$, $f(\xi) \neq \alpha$. Let $D = \{\gamma < \kappa : (\forall \alpha < \gamma)(\gamma \in C_\alpha)\}$. Then by Lemma 7.15, D is a club in κ. Since S is stationary in κ, we can choose $\gamma \in D \cap S$. Since $\gamma \in D$ and $\xi < \gamma$, $\gamma \in C_{f(\gamma)}$. But this contradicts the choice of $C_{f(\gamma)}$. $\square$

8 Diamond and Suslin line

Definition 8.1 (Diamond). *The diamond principle, denoted $\diamondsuit$, says the following: There is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ satisfying the following.*

(a) For every $\alpha < \omega_1$, $A_\alpha \subseteq \alpha$.

(b) For every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A_\alpha = A \cap \alpha\}$ is stationary in ω_1.

The diamond principle implies CH.

Lemma 8.2. $\diamondsuit \implies 2^\omega = \omega_1$.

Proof. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-witnessing sequence. Define $f : \mathcal{P}(\omega) \to \omega_1$ by letting $f(X)$ to be the least $\alpha \geq \omega$ such that $X \cap \alpha = A_\alpha$. Note that f is injective. Hence $\omega_1 \leq |\mathcal{P}(\omega)| = 2^\omega \leq \omega_1$. So $2^\omega = \omega_1$. □

We shall later see that $\diamondsuit$ is independent of ZFC. $\diamondsuit$ can be used for various combinatorial constructions. One of the early examples is that of a Suslin line.

Definition 8.3 (Suslin line)**.** *A Suslin line is a linear ordering $(L, \prec)$ satisfying the following.*

(a) (Dense) For every $a \prec b$, there exists $c \in L$ such that $a \prec c \prec b$.

(b) (No end points) L has no $\prec$-least or $\prec$-largest member.

(c) (Countable chain condition) Every family of pairwise disjoint open intervals in L is countable.

(d) (Non separable) For every $E \subseteq L$, if E is countable, then there exist a, b in L such that $a \prec b$ and $E \cap (a, b) = 0$ where $(a, b) = \{x \in L : a \prec x \prec b\}$ is the open interval from a to b.

Suslin hypothesis (abbreviated SH) is the statement that there is no Suslin line. Another way to state SH is the following: If $(L, \prec)$ is a dense linear ordering without end-points that satisfies the countable chain condition, then the Dedekind completion of $(L, \prec)$ is isomorphic to the real line $(\mathbb{R}, <)$. SH turned out to be independent of ZFC. We'll show below (Theorem 8.8) that $\diamondsuit$ implies the negation of SH. Before starting its proof, it will be useful to introduce the following notions.

Definition 8.4 (Dedekind cuts)**.** *Suppose $(L, \prec)$ is a linear ordering and $D \subseteq L$. We say that $D \subseteq L$ is a proper Dedekind cut in $(L, \prec)$ iff the following hold.*

(i) (Initial segment) For every $d \in D$ and $a \in L$, if $a \prec d$, then $a \in D$.

(ii) (Proper) Both $D, L \setminus D$ are nonempty, D does not have a $\prec$-largest member and $L \setminus D$ does not have a $\prec$-least member.

For example, $D = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ is a proper Dedekind cut in the rationals $(\mathbb{Q}, <)$ while $E = \{x \in \mathbb{Q} : x < 1\}$ isn't. Note that $(\mathbb{Q}, <)$ has 2^ω proper Dedekind cuts (for each irrational a, consider $D_a = \{x \in \mathbb{Q} : x < a\}$). Recall that if $(A, \prec)$ is a countable dense linear order without end points, then $(A, \prec)$ is order isomorphic to the rationals. Hence $(A, \prec)$ also has 2^ω proper Dedeking cuts.

Definition 8.5 (Open dense, Closed nowhere dense)**.** *Suppose $(L, \prec)$ is a dense linear ordering without end points.*

(i) We say that $U \subseteq L$ is open dense in L iff

(a) for every $x \in U$, there are $y, z \in L$ such that $y \prec x \prec z$ and $(y, z) \subseteq U$ and

(b) for every $a \prec b$ in L, there exists $c \in U$ such that $a \prec c \prec b$.

(ii) We say that $X \subseteq L$ is closed nowhere dense in L iff $L \setminus X$ is open dense in L.

We leave the following two lemmas as exercises for the reader.

Lemma 8.6. *Suppose $(L, \prec)$ is a dense linear ordering without end points in which every closed nowhere dense set is countable. Then $(L, \prec)$ satisfies the countable chain condition; i.e., every family of pairwise disjoint open intervals in L is countable.*

Lemma 8.7 (Baire Category theorem)**.** *Suppose $(L, \prec)$ is a countable dense linear order without end points. Suppose $\mathcal{F}$ is a countable family of closed nowhere dense subsets of $(L, \prec)$. Then, there exists a proper Dedekind cut $D \subseteq L$ such that D **avoids** every $X \in \mathcal{F}$ which means the following: For every $X \in \mathcal{F}$, there exist $a, b \in L$ such that $a \in D$, $b \in L \setminus D$ and $[a, b] \cap X = 0$.*

Theorem 8.8. *$\diamondsuit$ implies that there is a Suslin line.*

Proof. Using $\diamondsuit$, we'll construct a linear order $\prec$ on ω_1 such that $(\omega_1, \prec)$ is a Suslin line. Using transfinite recursion, we'll first construct $\langle \prec_\alpha : \alpha < \omega_1, \alpha \text{ limit} \rangle$ such that

(a) for every limit $\alpha < \omega_1$, $(\alpha, \prec_\alpha)$ is a dense linear order without end points,

(b) for every $\beta < \alpha < \omega_1$, $\prec_\beta = \prec_\alpha \bigcap (\beta \times \beta)$

Start by choosing $\prec_\omega$ such that $(\omega, \prec_\omega) \cong (\mathbb{Q}, <)$ where $(\mathbb{Q}, <)$ is the usual ordering of rationals. Note that (b) implies that if α is a limit of limit ordinals, then we must define $\prec_\alpha = \bigcup\{\prec_\beta : \beta < \alpha\}$. So the only stages where we have freedom are of the form $\beta + \omega$.

Once $\prec_\alpha$'s have been constructed, we'll define $\prec = \bigcup\{\prec_\alpha : \alpha < \omega_1, \alpha \text{ limit}\}$. Note that $\prec$ will automatically be a dense linear order on ω_1 without end points. To ensure that $(\omega_1, \prec)$ is not separable, it'll be sufficient to guarantee that no β is dense in $(\beta + \omega, \prec_{\beta+\omega})$. Given β, let D_β be a proper Dedekind cut in $(\beta, \prec_\beta)$. Define $\prec_{\beta+\omega}$ by **inserting** a copy of $\mathbb{Q}$ (rationals) into D_β; i.e., the ordinals in the set $\{\beta + n : n < \omega\}$ are first ordered isomorphically to the rationals and then inserted after the elements of D_β and before the elements of $\beta \setminus D_\beta$. It is clear that the resulting $(\beta + \omega, \prec_{\beta+\omega})$ will also be a dense linear order without endpoints and β will not be dense in $(\beta + \omega, \prec_{\beta+\omega})$.

So the "only" thing to ensure is that $(\omega_1, \prec)$ satisfy ccc (countable chain condition). By Lemma 8.6, it will be enough to guarantee that every closed nowhere dense subset of $(\omega_1, \prec)$ is countable. This will be done by using $\diamondsuit$ to judiciously choose the Dedekind cuts D_β's during the stages $\beta + \omega$ of the construction. Fix a $\diamondsuit$-witnessing sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ and define $\langle \prec_\alpha : \alpha < \omega_1, \alpha \text{ limit} \rangle$ as follows.

(1) $(\omega, \prec_\omega)$ is a dense linear ordering without end points.

(2) If $\alpha < \omega_1$ is a limit of limits $\prec_\alpha = \bigcup\{\prec_\beta : \beta < \alpha\}$.

(3) If $\beta < \omega_1$ is limit and $\alpha = \beta + \omega$, then (using Lemma 8.7) choose a proper Dedekind cut D_β in $(\beta, \prec_\beta)$ such that for every limit $\gamma \leq \beta$, if A_γ is closed nowhere dense in $(\beta, \prec_\beta)$, then D_β **avoids** A_γ (such D_β exists by Lemma 8.7 above) and define $\prec_{\beta+\omega}$ by inserting a copy of $\mathbb{Q}$ into D_β as described above.

Let us check that $\prec = \bigcup\{\prec_\alpha : \alpha < \omega_1, \alpha \text{ limit}\}$ is as required. As noted above, it suffices to check that every closed nowhere dense set in $(\omega_1, \prec)$ is countable. Fix $A \subseteq \omega_1$ such that A is a closed nowhere dense subset of $(\omega_1, \prec)$. Let $f_1, f_2 : \omega_1 \to \omega_1$ be such that for every $\xi \in \omega_1 \setminus A$, we have $f_1(\xi) \prec \xi \prec f_2(\xi)$ and $(f_1(\xi), f_2(\xi)) \cap A = 0$. Let $g, h : \omega_1^2 \to \omega_1$ be such that for every $\xi \prec \eta$, we have $\xi \prec g(\xi, \eta) \prec \eta$, $\xi \prec h(\xi, \eta) \prec \eta$, $g(\xi, \eta) \notin A$ and if $(\xi, \eta) \cap A \neq 0$, then $h(\xi, \eta) \in A$. Note that such f_1, f_2, g, h exist because A is closed nowhere dense in $(\omega_1, \prec)$.

Let C be the set of limit ordinals $\delta < \omega_1$ such that δ is closed under f_1, f_2, g, h. Then C is a club in ω_1 by Lemma 7.11. Note that for every $\delta \in C$, using the fact that δ is closed under f_1, f_2 and g, $A \cap \delta$ is closed nowhere dense in $(\delta, \prec_\delta)$.

Using $\diamondsuit$, choose $\delta \in C$ such that $A_\delta = A \cap \delta$. We'll show that $A = A \cap \delta$ and hence A is countable. To show this, by induction on β, we'll show that for every limit $\beta \geq \delta$, (a) and (b) below hold.

(a) $A \cap \beta = A \cap \delta$.

(b) For every $\gamma \in \beta \setminus A$, there are $\xi, \eta \in \delta$ such that $\xi \prec \gamma \prec \eta$ and $(\xi, \eta) \cap A = 0$.

If $\beta = \delta$, (a) clearly holds. For (b) using the fact that δ is closed under f_1, f_2, take $\xi = f_1(\gamma)$ and $\eta = f_2(\gamma)$. If β' is a limit of limits, and (a) and (b) hold for every $\beta < \beta'$, then they also hold at $\beta = \beta'$.

So assume that (a) and (b) hold for $\beta \geq \delta$ and we'll show that they also hold at $\beta + \omega$. Since $A \cap \beta = A \cap \delta = A_\delta$, (b) implies that A_δ is closed nowhere dense in $(\beta, \prec_\beta)$. Hence D_β avoids A_δ. Choose $\gamma \prec \gamma'$ in β such that $[\gamma, \gamma'] \cap (A \cap \beta) = 0$ and every ordinal in $\{\beta + n : n < \omega\}$ lies in (γ, γ'). Applying (b), get $\xi, \eta, \xi', \eta' \in \delta$ for γ, γ' there. Then $(\xi, \eta') \cap A \cap \delta = 0$. Since δ is closed under h, it follows that $(\xi, \eta') \cap A = 0$. Since every member of $\{\beta + n : n < \omega\}$ lies in (γ, γ') hence also in (ξ, η'), both (a) and (b) follow at $\beta + \omega$. $\square$

9 Saturated ideals and Ulam's dichotomy

Definition 9.1. *Let $\mathcal{I}$ be an ideal on X. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. We say that $\mathcal{I}$ is κ-saturated iff for every $\mathcal{F} \subseteq \mathcal{I}^+$,* ***if*** *for every distinct $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{I}$,* ***then*** *$|\mathcal{F}| < \kappa$.*

Theorem 9.2. *Suppose λ is a successor cardinal and $\mathcal{I}$ is a λ-complete ideal on λ that contains every finite subset of λ. Then $\mathcal{I}$ is not λ-saturated.*

The proof of Theorem 9.2 uses a combinatorial device called Ulam matrix.

Lemma 9.3 (Ulam matrix)**.** *Let κ be an infinite cardinal. For each $\alpha < \kappa^+$, let $f_\alpha : \alpha \to \kappa$ be an injective function. For each $\xi < \kappa$ and $\alpha < \kappa^+$, define $A^\xi_\alpha = \{\beta < \kappa^+ : \alpha < \beta \text{ and } f_\beta(\alpha) = \xi\}$. Then the following hold.*

(1) For every $\xi < \kappa$, $\langle A^\xi_\alpha : \alpha < \kappa^+\rangle$ is a sequence of disjoint subsets of κ^+.

(2) For every $\alpha < \kappa^+$, $\bigcup\{A^\xi_\alpha : \xi < \kappa\} = \kappa \setminus (\alpha + 1)$.

Proof of Theorem 9.2: Let $\lambda = \kappa^+$. Fix $\langle A^\xi_\alpha : \xi < \kappa \text{ and } \alpha < \kappa^+\rangle$ satisfying Clauses (1)+(2) in Lemma 9.3. Since $\bigcup\{A^\xi_\alpha : \xi < \kappa\} = \kappa \setminus (\alpha+1)$, for each $\alpha < \kappa^+$, we can fix $\xi(\alpha) < \kappa$ such that $A^{\xi(\alpha)}_\alpha \in \mathcal{I}^+$. Choose $X \subseteq \kappa^+$ and $\xi_\star < \kappa$ such that $|X| = \kappa^+$ and for every $\alpha \in X$, $\xi(\alpha) = \xi_\star$. Now $\{A^{\xi_\star}_\alpha : \alpha \in X\}$ is a disjoint family of sets in $\mathcal{I}^+$ of size $\kappa^+ = \lambda$. Therefore $\mathcal{I}$ is not λ-saturated. □

Corollary 9.4. *Let κ be an infinite cardinal and $S \subseteq \kappa^+$ be stationary. Then S can be partitioned into κ^+ stationary subsets of κ^+.*

Proof. Apply Theorem 9.2 to $\mathcal{I} = \{A \subseteq \kappa^+ : A \cap S \text{ is non-stationary}\}$. □

Definition 9.5. *A total diffused probability measure on a set X is a function $m : \mathcal{P}(X) \to [0,1]$ satisfying the following.*

(i) (Diffused) For every $a \in X$, $m(\{a\}) = 0$.

(ii) (Probability) $m(X) = 1$.

(iii) (Countably additive) For every sequence $\langle A_n : n < \omega\rangle$ of pairwise disjoint subsets of X, letting $A = \bigcup\{A_n : n < \omega\}$, we have

$$m(A) = \sum_{n<\omega} m(A_n)$$

Furthermore, m is called atomless if it also satisfies: For every $A \subseteq X$ with $m(A) > 0$, there exists $B \subseteq A$ such that $0 < m(B) < m(A)$. The null ideal of m is defined by $\mathsf{Null}(m) = \{A \subseteq X : m(A) = 0\}$. We say that m is κ-complete iff $\mathsf{Null}(m)$ is κ-complete.

Fact 9.6. *Suppose m is a total atomless probability measure on some set. Then range$(m) = [0,1]$.*

Definition 9.7. *An uncountable cardinal κ is weakly inaccessible iff it is a regular limit cardinal. κ is strongly inaccessible iff it is regular and satisfies $(\forall\theta < \kappa)(2^\theta < \kappa)$.*

Theorem 9.8 (Ulam's dichotomy)**.** *Suppose κ is the least cardinal such that there is a total diffused probability measure on κ. Then κ is a weakly inaccessible cardinal and there is a total diffused κ-complete probability measure on κ. Furthermore, exactly one of the following holds.*

(1) $\kappa \le \mathfrak{c}$ and every total diffused probability measure $m : \mathcal{P}(\kappa) \to [0,1]$ is atomless and therefore satisfies range$(m) = [0,1]$.

(2) $\kappa > \mathfrak{c}$ is strongly inaccessible and there is a 2-valued diffused κ-complete probability measure $m : \mathcal{P}(\kappa) \to \{0,1\}$. Therefore, the null ideal of m is a κ-complete ultrafilter on κ.

Proof. Let $m : \mathcal{P}(\kappa) \to [0,1]$ be any diffused probability measure on κ.

Claim 9.9. *m is a κ-complete measure.*

Proof. Suppose not. Fix $\theta < \kappa$ and $\{A_i : i < \theta\}$ such that for every $i < \theta$, $m(A_i) = 0$ and letting $A = \bigcup\{A_i : i < \theta\}$, we have $m(A) > 0$. Define $B_i = A_i \setminus \bigcup\{A_j : j < i\}$ and observe that B_i's are pairwise disjoint and $\bigcup\{B_i : i < \theta\} = A$. Define $m' : \mathcal{P}(\theta) \to [0, 1]$ by

$$m'(E) = \frac{m\left(\bigcup_{i \in E} B_i\right)}{m(A)}$$

Then m' is a total diffused probability measure on $\theta < \kappa$ which is impossible. $\square$

Claim 9.10. *κ is weakly inaccessible.*

Proof. κ is regular: Suppose not. Let $\mu = \mathrm{cf}(\kappa) < \kappa$. Fix an increasing cofinal sequence $\langle \kappa_i : i < \mu \rangle$ in κ. Since m is a diffused κ-complete measure, we must have $m(\kappa_i) = 0$ for every $i < \mu$. Moreover since $\mu < \kappa$, we also get $m(\bigcup\{\kappa_i : i < \kappa\}) = 0$ which is impossible. So κ is regular.

κ is a limit cardinal: Note that $\mathsf{Null}(m)$ is an ω_1-saturated (and hence also κ-saturated) κ-complete ideal on κ that contains every finite subset of κ. By Theorem 9.2, κ cannot be a successor cardinal. $\square$

It follows that κ is weakly inaccessible and every diffused total probability measure on κ is κ-complete.

Claim 9.11. *Let $m : \mathcal{P}(\kappa) \to [0, 1]$ be any κ-complete diffused probability measure on κ. Then m is atomless iff $\kappa \leq \mathfrak{c}$. Furthermore, if m is not atomless, then there is a 2-valued total diffused κ-complete probability measure on κ.*

Proof. First suppose m is atomless. By Fact 9.6, range$(m) = [0, 1]$. Recursively construct $\langle A_\sigma : \sigma \in 2^{<\omega} \rangle$ as follows. $A_{\langle\rangle} = \kappa$ and for every $\sigma \in 2^{<\omega}$, $A_\sigma = A_{\sigma 0} \sqcup A_{\sigma 1}$ and $m(A_{\sigma 0}) = m(A_{\sigma 1}) = 0.5 m(A_\sigma)$. For each $x \in 2^\omega$, define $A_x = \bigcap\{A_{x \restriction n} : n < \omega\}$. Then $\{A_x : x \in 2^\omega\}$ is a partition of κ into m-null sets. Since m is κ-complete, we must have $\kappa \leq \mathfrak{c}$.

Next, suppose m is not atomless and fix $A \subseteq \kappa$ such that $m(A) > 0$ and for every $B \subseteq A$, either $m(B) = 0$ or $m(B) = m(A)$. Define $m' : \mathcal{P}(\kappa) \to \{0, 1\}$ by

$$m'(E) = \frac{m(E \cap A)}{m(A)}$$

Then m' is a 2-valued total diffused κ-complete probability measure on κ. So we only need to show that $\kappa > \mathfrak{c}$. Suppose not. Then we can fix an injective $f : \kappa \to 2^\omega$. For each $\sigma \in 2^{<\omega}$, define $X_\sigma = \{\alpha \in \kappa : \sigma \subseteq f(\alpha)\}$. Then $\langle X_\sigma : \sigma \in 2^{<\omega} \rangle$ satisfies the following: $X_{\langle\rangle} = \kappa$ and for every $\sigma \in 2^{<\omega}$, $X_\sigma = X_{\sigma 0} \sqcup X_{\sigma 1}$. Choose $y \in 2^\omega$ such that for every $n < \omega$, $m(X_{y \restriction n}) = 1$. Since m is countably additive, we get $m(\bigcap\{X_{y \restriction n} : n < \omega\}) = 1$. But $\bigcap\{X_{y \restriction n} : n < \omega\} \subseteq f^{-1}[\{y\}]$ has size ≤ 1 since f is injective. Contradiction. $\square$

Claim 9.12. *Let $m : \mathcal{P}(\kappa) \to \{0, 1\}$ be a 2-valued κ-complete diffused probability measure on κ. Then κ is strongly inaccessible.*

Proof. Since m is κ complete, κ must be regular. Towards a contradiction, fix $\theta < \kappa$ such that $2^\theta \geq \kappa$. Let $f : \kappa \to 2^\theta$ be an injection. For each $\sigma \in 2^{<\theta}$, define $X_\sigma = \{\alpha \in \kappa : \sigma \subseteq f(\alpha)\}$. Then $\langle X_\sigma : \sigma \in 2^{<\theta} \rangle$ satisfies the following.

- $X_{\langle\rangle} = \kappa$.
- For every limit ordinal $\gamma < \theta$ and $\sigma \in 2^\gamma$, $X_\sigma = \bigcap\{X_{\sigma \restriction \alpha} : \alpha < \gamma\}$.
- For every $\sigma \in 2^{<\theta}$, $X_\sigma = X_{\sigma 0} \sqcup X_{\sigma 1}$.

Choose $y \in 2^\theta$ such that for every $\alpha < \theta$, $m(X_{y \restriction \alpha}) = 1$. Since m is κ-complete and $\theta < \kappa$, we get $m(\bigcap\{X_{y \restriction \alpha} : \alpha < \theta\}) = 1$. But $\bigcap\{X_{y \restriction \alpha} : \alpha < \theta\} \subseteq f^{-1}[\{y\}]$ has size ≤ 1 since f is injective. Contradiction $\square$

It is easy to check that Claims 9.11 and 9.12 give us the required dichotomy. $\square$

Corollary 9.13. *Suppose there is a total diffused probability measure on the unit interval $[0, 1]$. Then there is a weakly inaccessible cardinal $\leq \mathfrak{c}$. In particular $\mathfrak{c} > \omega_n$ for every $n < \omega$.*

10 Well founded relations

Axiom 10.1 (Foundation)**.** $(\forall x)[x \neq 0 \implies (\exists y \in x)(y \cap x = 0)]$

The axiom of foundation says that every nonempty set x has an $\in$-minimal member y.

Lemma 10.2. *x is an ordinal iff x is transitive and linearly ordered under $\in$.*

Definition 10.3 (von Neumann hierarchy)**.** *By transfinite induction, define $\langle V_\alpha : \alpha \in \mathbf{ORD}\rangle$ as follows.*

- $V_0 = 0$.
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
- *If γ is limit,* $V_\gamma = \bigcup\{V_\alpha : \alpha < \gamma\}$.

The following is easy to check.

Lemma 10.4. *For every α, V_α is transitive. If $\beta \leq \alpha$, then $V_\beta \subseteq V_\alpha$.*

Definition 10.5 (Transitive closure)**.** *By recursion on $n < \omega$, define $\bigcup^0 x = x$ and $\bigcup^{n+1} x = \bigcup(\bigcup^n x)$. Define the transitive closure of x,*

$$\mathit{trcl}(x) = \bigcup\{\cup^n x : n < \omega\}$$

The next lemma says that $\mathsf{trcl}(x)$ is the intersection of all transitive sets y such that $x \subseteq y$.

Lemma 10.6. *For every x, $x \subseteq \mathit{trcl}(x)$ and $\mathit{trcl}(x)$ is transitive. If y is transitive and $x \subseteq y$, then $\mathit{trcl}(x) \subseteq y$.*

Lemma 10.7. $(\forall x)(\exists \alpha)(x \in V_\alpha)$.

Proof. Put $\mathbf{W} = \bigcup\{V_\alpha : \alpha \in \mathbf{ORD}\}$. Note that it suffices to show that for every x, $x \subseteq \mathbf{W}$. Since then, by replacement, for some α, $x \subseteq V_\alpha$ and so $x \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$. Towards a contradiction, suppose $x \nsubseteq \mathbf{W}$. Put $y = \mathsf{trcl}(x)$. Since $x \subseteq y$, $y \nsubseteq \mathbf{W}$. Let z be an $\in$-minimal member of $\{w \in y : w \notin \mathbf{W}\}$. Then since y is transitive, $z \subseteq y$. So for every $w \in z$, $w \in \mathbf{W}$. So $z \subseteq \mathbf{W}$. Hence $z \in \mathbf{W}$ which is a contradiction. □ □

Definition 10.8 (Rank)**.**

$$\mathit{rank}(x) = \min\{\alpha : x \in V_{\alpha+1}\}$$

The following lemma is left to the reader.

Lemma 10.9. *The following hold.*

- $\mathit{rank}(\alpha) = \alpha$
- $x \in y \implies \mathit{rank}(x) < \mathit{rank}(y)$
- $\mathit{rank}(x) = \sup(\{\mathit{rank}(y) + 1 : y \in x\})$

Definition 10.10 (Well founded relation)**.** *Suppose R is a binary relation on a nonempty set X.*

(a) For $Y \subseteq X$ and $y \in Y$, we say that y is an R-minimal member of Y iff $(\forall x \in Y)(\neg(xRy))$.

(b) (X, R) is well founded iff every nonempty subset of X has an R-minimal member.

Note that if $\prec$ is a well ordering on X, then $(X, \prec)$ is well founded. The axiom of foundation says that the membership relation "$\in$" is well founded on every set.

Lemma 10.11. *Suppose (X, R) is well founded. Then there exists a unique $F : X \to \mathbf{ORD}$ satisfying for every $x \in X$,*

$$F(x) = \sup(\{F(y) + 1 : y \in X \wedge yRx\}$$

Proof. Call $D \subseteq X$, R-closed iff $(\forall x \in D)(\forall y \in X)(yRx \implies y \in D)$. Define

$$F = \bigcup \{f : f \text{ is a function} \wedge \mathrm{dom}(f) \subseteq X \text{ is R-closed} \wedge (\forall x \in \mathrm{dom}(f))[f(x) = \{f(y)+1 : yRx\}]\}$$

Use the fact that (X, R) is well founded to check that $F : X \to \mathbf{ORD}$ and for all $x \in X$, $F(x) = \sup(\{F(y)+1 : y \in X \wedge yRx\})$.

To see uniqueness, suppose F, F' are two such functions on X and $F \neq F'$. Let $x \in X$ be R-minimal such that $F(x) \neq F'(x)$. Then $F(x) = \sup(\{F(y)+1 : y \in X \wedge yRx\}) = \sup(\{F'(y)+1 : y \in X \wedge yRx\}) = F'(y)$ which is impossible. □

Definition 10.12 (Rank function)**.** *Suppose (X, R) is well founded. Define $\mathsf{rank}_{X,R} : X \to \boldsymbol{ORD}$ by*

$$\mathsf{rank}_{X,R}(x) = \sup(\{\mathsf{rank}_{X,R}(y)+1 : y \in X \wedge yRx\})$$

As an exercise, check that if $(X, \prec)$ is a well ordering, then $\mathsf{type}(X, \prec) = \mathrm{range}(\mathsf{rank}_{X,\prec})$.

Definition 10.13 (Extensional)**.** *Suppose R is a binary relation on X. We say that (X, R) is extensional iff*

$$(\forall x, y \in X)(\{v \in X : vRx\} = \{v \in X : vRy\} \implies x = y)$$

Note that the axiom of extensionality implies that for every transitive X, $(X, \in)$ is extensional. The next theorem says that every well founded and extensional relation is isomorphic to $(Y, \in)$ for some transitive set Y.

Theorem 10.14 (Mostowski collapse)**.** *Suppose (X, R) is well founded and extensional. Then there exists a transitive set Y and an isomorphism $F : (X, R) \to (Y, \in)$. Moreover, Y and F are unique.*

Proof. By induction on $\mathsf{rank}_{X,R}(x)$, define $F(x) = \{F(y) : y \in X \wedge yRx\}$ and put $Y = \mathrm{range}(F)$. That Y is transitive is clear. Since (X, R) is extensional, F is one-one. So $F : X \to Y$ is a bijection. It is also clear that xRy iff $F(x) \in F(y)$. So $F : (X, R) \to (Y, \in)$ is an isomorphism. Uniqueness also follows from the well foundedness of (X, R). □

11 Inner models and relative consistency proofs

Definition 11.1. *Suppose $\mathbf{M} = \{x : \phi\}$ is a class and ψ is a formula. The relativization of ψ to $\mathbf{M}$, denoted $\psi^{\mathbf{M}}$ is defined as follows.*

(i) *If ψ is quantifier-free, $\psi^{\mathbf{M}} \equiv \psi$*

(ii) $(\psi_1 \wedge \psi_2)^{\mathbf{M}} \equiv \psi_1^{\mathbf{M}} \wedge \psi_2^{\mathbf{M}}$.

(iii) $(\psi_1 \vee \psi_2)^{\mathbf{M}} \equiv \psi_1^{\mathbf{M}} \vee \psi_{\mathbf{M}}^2$.

(iv) $(\neg\psi)^{\mathbf{M}} \equiv \neg(\psi^{\mathbf{M}})$.

(v) $(\exists v(\psi))^{\mathbf{M}} \equiv (\exists v \in \mathbf{M})\psi^{\mathbf{M}}$.

(vi) $(\forall v(\psi))^{\mathbf{M}} \equiv (\forall v \in \mathbf{M})\psi^{\mathbf{M}}$.

We sometimes write $\mathbf{M} \models \psi$ (read "$\mathbf{M}$ models ψ" or "ψ holds in $\mathbf{M}$") for $\psi^{\mathbf{M}}$. Fact 11.2 below says that relativizations of logically valid sentences are logically valid. Informally, this is clear, since if ψ is logically valid, it holds in every $\in$-model $(N, \in^N)$. Now $\psi^{\mathbf{M}}$ is just the result of relativizing all quantifiers of ψ to $\mathbf{M}$ so $(N, \in^N) \models \psi^{\mathbf{M}}$ iff $(\{v \in N : (N, \in^N) \models \phi(v/x)\}, \in^N) \models \psi$ where $\mathbf{M} = \{x : \phi\}$. Of course, a formal proof of Fact 11.2 can be given by induction on the length of the formal deduction of ψ from the axioms of first order logic.

Fact 11.2. *Suppose* $\mathbf{M}$ *is a nonempty class and ψ is logically valid. Then $\psi^{\mathbf{M}}$ is logically valid.*

Theorem 11.3. *Suppose T is a (recursive) subtheory of ZFC and $\mathbf{M}$ is a nonempty class. Let χ be a sentence. Suppose for every $\psi \in T \cup \{\chi\}$, $T \vdash \psi^{\mathbf{M}}$ (i.e., T proves $\psi^{\mathbf{M}}$). Then if T is consistent, so is $T \cup \{\chi\}$.*

Proof. Suppose $T \cup \{\chi\}$ is inconsistent. Then for some $\psi_1, \psi_2, \ldots, \psi_n$ in T,

$$(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n) \implies \neg\chi$$

is logically valid. Hence

$$(\psi_1^{\mathbf{M}} \wedge \psi_2^{\mathbf{M}} \wedge \cdots \wedge \psi_n^{\mathbf{M}}) \implies \neg\chi^{\mathbf{M}}$$

is also logically valid. By assumption, $T \vdash (\psi_1^{\mathbf{M}} \wedge \psi_2^{\mathbf{M}} \wedge \cdots \wedge \psi_n^{\mathbf{M}}) \wedge \chi^{\mathbf{M}}$. Hence $T \vdash \chi^{\mathbf{M}} \wedge \neg\chi^{\mathbf{M}}$. So T is inconsistent. □

Definition 11.4 (Inner models)**.** *An inner model of ZF is a transitive class $\mathbf{M}$ such that $\boldsymbol{ORD} \subseteq \mathbf{M}$ and for every axiom ψ of ZF, $\mathbf{M} \models \psi$.*

Our next goal is to define an inner model $\mathbf{L}$ (called Gödel's constructible universe) of ZF and, working in ZF, show that $\mathbf{L} \models AC \wedge GCH$. This will establish the relative consistency of AC and GCH over ZF. Before introducing $\mathbf{L}$, we need to discuss absoluteness and the reflection theorem.

12 Absoluteness

The details of some tedious proofs in this section have been omitted. The reader is strongly advised to also look at Sections 3 and 5 in Chapter IV of Kunen's book.

Definition 12.1. *Suppose $\mathbf{M} \subseteq \mathbf{N}$ are classes and ϕ is a formula with free variables $x_1, x_2, \ldots, x_n$. We say that ψ is absolutes between $\mathbf{M}, \mathbf{N}$ iff*

$$(\forall x_1, x_2 \ldots x_n \in \mathbf{M})(\phi^{\mathbf{M}} \iff \phi^{\mathbf{N}})$$

We say that ϕ is absolute for $\mathbf{M}$ iff ϕ is absolute between $\mathbf{M}, \mathbf{V}$; equivalently,

$$(\forall x_1, x_2 \ldots x_n \in \mathbf{M})(\phi^{\mathbf{M}} \iff \phi)$$

Suppose $\mathbf{M} \subseteq \mathbf{N}$ and ϕ is quantifier-free. Then ϕ is absolute between $\mathbf{M}, \mathbf{N}$. If ϕ is absolute for $\mathbf{M}$, ϕ is absolute for $\mathbf{N}$ and $\mathbf{M} \subseteq \mathbf{N}$, then ϕ is absolute between $\mathbf{M}, \mathbf{N}$.

Definition 12.2 (Δ_0-formulas)**.** *The Δ_0-formulas are inductively defined as follows.*

(1) Every quantifier-free formula is a Δ_0-formula.

(2) If ϕ, ψ are Δ_0-formulas then so are the following.

(a) $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$

(b) $(\forall x \in y)(\phi)$, $(\exists x \in y)(\phi)$

We sometimes also refer to Δ_0-formulas as formulas with bounded quantifiers.

Lemma 12.3. *Suppose $\mathbf{M} \subseteq \mathbf{N}$ are transitive classes and ϕ is absolute between $\mathbf{M}, \mathbf{N}$. Then $(\exists x \in y)(\phi)$ is also absolute between $\mathbf{M}, \mathbf{N}$. Hence every Δ_0-formula is absolute between $\mathbf{M}, \mathbf{N}$.*

Proof. Let $\phi \equiv \phi(x, y, z_1, \ldots, z_n)$ where every free variables in ϕ occurs among $z, y, z_1, \ldots, z_n$. Now for any $y, z_1, \ldots z_n \in \mathbf{M}$,

$$(\exists x \in \mathbf{M})(x \in y \wedge \phi^{\mathbf{M}}) \iff (\exists x)(x \in y \wedge \phi^{\mathbf{M}}) \iff (\exists x)(x \in y \wedge \phi^{\mathbf{N}}) \iff (\exists x \in \mathbf{N})(x \in y \wedge \phi^{\mathbf{N}})$$

where the first $\iff$ used $y \subseteq \mathbf{M}$ (as $\mathbf{M}$ is transitive), the second $\iff$ used the absoluteness of ϕ between $\mathbf{M}, \mathbf{N}$ and the last $\iff$ used $y \subseteq \mathbf{N}$ (as $\mathbf{N}$ is also transitive).

□

Corollary 12.4. *Suppose* $\mathbf{M}$ *is a transitive class. Then* $\mathbf{M} \models$ Extensionality.

Although, the language of set theory does not allow terms like "$x \cup y$", we have introduced many terms as abbreviations. We can extend the notion of absoluteness to such terms as follows.

Definition 12.5. *Suppose* $\mathbf{M} \subseteq \mathbf{N}$ *and* $F(x_1, \ldots, x_n)$ *is a term. We say that* F *is absolute between* $\mathbf{M}, \mathbf{N}$ *iff the formula* $y = F(x_1, \ldots, x_n)$ *is absolute between* $\mathbf{M}, \mathbf{N}$.

Suppose $F(x_1, \ldots, x_n)$ is a term. Then, for some formula $\phi(y, x_1, \ldots, x_n)$, $F(x_1, \ldots, x_n)$ was introduced via

$$(\forall x_1, \ldots x_n)(y = F(x_1, \ldots, x_n) \iff (\exists! y)\phi(y, x_1, \ldots, x_n))$$

Now, there is no reason for the statement $\psi \equiv (\forall x_1, \ldots, x_n)(\exists! y)\phi(y, x_1, \ldots, x_n)$ to hold in $\mathbf{M}, \mathbf{N}$. In view of this, we'll discuss the absoluteness of $F(x_1, \ldots, x_n)$ between $\mathbf{M}, \mathbf{N}$ only when both $\psi^{\mathbf{M}}$ and $\psi^{\mathbf{N}}$ hold. It follows that, in this case, $F(x_1, \ldots, x_n)$ is absolute between $\mathbf{M}, \mathbf{N}$ iff $\phi(y, x_1, \ldots, y_n)$ is absolute between $\mathbf{M}, \mathbf{N}$.

Definition 12.6. *The theory BST (Basic set theory) consists of the following axioms:* Extensionality, Comprehension, Pairing, Union, Replacement.

Lemma 12.7. *Each of the following relations and terms was defined in BST, using a formula provably equivalent to a* Δ_0*-formula in BST. Hence they are all absolute for transitive models* $\mathbf{M}$ *of BST.*

$x \subseteq y$	$\{x, y\}$	$\{x\}$	$\langle x, y\rangle$
0	$x \cup y$	$x \cap y$	$x \setminus y$
$S(x)$	x *is transitive*	$\bigcup X$	$\bigcap X$
$A \times B$	R *is a relation*	*dom*(R)	*range*(R)
$(X, \prec)$ *is a linear ordering*	f *is a function*	f *is an injective function*	$f(x)$

We omit the simple but tedious verification. For example, to check that $y = A \times B$ is equivalent to a Δ_0-formula, note that $y = A \times B$ is equivalent to

$$(\forall z \in y)(\exists a \in A)(\exists b \in B)(z = \langle a, b\rangle) \wedge (\forall a \in A)(\forall b \in B)(\exists z \in y)(z = \langle a, b\rangle)$$

Next note that $z = \langle x, y\rangle$ is equivalent to

$$(\forall v \in z)(v = \{x\} \vee z = \{x, y\}) \wedge (\exists v \in z)(v = \{x\}) \wedge (\exists v \in z)(v = \{x, y\})$$

And finally, $z = \{x, y\}$ is equivalent to

$$(x \in z) \wedge (y \in z) \wedge (\forall v \in z)(v = x \vee v = y)$$

which is Δ_0-formula. By substituting back, $z = \langle x, y\rangle$ and $y = A \times B$ are also equivalent to Δ_0-formulas.

By referring back to the places where these terms were first introduced, the readers can convince themselves that the theory BST is enough for proving these equivalences. The next lemma says that absolute notions are closed under compositions. It can be used to greatly enlarge the collection of absolute terms and formulas.

Lemma 12.8 (Composition of absolute terms)**.** *Suppose* $\mathbf{M} \subseteq \mathbf{N}$ *and* $\phi(x_1, \ldots, x_n)$, $F(y_1, \ldots, y_n)$, $G_k(z_1, \ldots, z_m)$, *for* $1 \leq k \leq n$, *are absolute between* $\mathbf{M}, \mathbf{N}$. *Then*

$$\phi(G_1(z_1, \ldots, z_m), \ldots, G_n(z_1, \ldots, z_m))$$

and

$$F(G_1(z_1, \ldots, z_m), \ldots, G_n(z_1, \ldots z_m))$$

are also absolute between $\mathbf{M}, \mathbf{N}$.

Note that the axiom of infinity cannot be proven from BST. In fact, it is easy to check that V_ω is a transitive model of BST in which the axiom of infinity is false. The next lemma says for the axiom of infinity to hold in a transitive BST model $\mathbf{M}$, it is enough that $\omega \in \mathbf{M}$.

Lemma 12.9. *Suppose* $\mathbf{M}$ *is a transitive model of BST and* $\omega \in \mathbf{M}$*. Then the axiom of infinity holds in* $\mathbf{M}$*.*

Proof. The axiom of infinity relativized to $\mathbf{M}$ is equivalent to

$$(\exists x \in \mathbf{M})(0 \in x \wedge (\forall y \in x)(S(y) \in x))^{\mathbf{M}}$$

which, by absoluteness of the terms $0, S(y)$ for $\mathbf{M}$ is equivalent to

$$(\exists x \in \mathbf{M})(0 \in x \wedge (\forall y \in x)(S(y) \in x))$$

It is clear that $x = \omega \in \mathbf{M}$ witnesses this. □

Lemma 12.10. *Suppose* $\mathbf{M}$ *is a transitive model of BST,* $(A, \prec) \in \mathbf{M}$ *and* $(A, \prec)$ *is a well ordering. Then* $\mathbf{M} \models (A, \prec)$ *is a well ordering.*

Proof. Since "$(A, \prec)$ is a linear ordering" is Δ_0-formula over BST, we only need to check that

$$\mathbf{M} \models (\forall X)[(X \subseteq A \wedge X \neq 0) \implies (\exists x \in X)(\forall y \in X)(x \preceq y)]$$

So fix $X \in \mathbf{M}$. Since $(A, \prec)$ is a well order,

$$(X \subseteq A \wedge X \neq 0) \implies (\exists x \in X)(\forall y \in X)(y \preceq x)$$

which is a Δ_0-formula over BST, so by absoluteness, it holds in $\mathbf{M}$. □

Note that the argument of the proof shows that for every Δ_0-formula ϕ and transitive BST model $\mathbf{M}$,

$$\phi \implies ((\forall x)\phi)^{\mathbf{M}}$$

We conclude out introduction to absoluteness, by listing some absolute notions for transitive model of "BST + Infinity + Foundation".

Lemma 12.11. *The following terms and relations are absolute for transitive models* $\mathbf{M}$ *of "BST + Infinity + Foundation".*

x is an ordinal	*x is a successor ordinal*	*x is a finite ordinal*
ω	*x is finite*	A^n
$A^{<\omega}$	*$(A, \prec)$ is a well ordering*	$\mathsf{type}(A, \prec)$
$\alpha + \beta, \alpha \cdot \beta$	$\mathsf{rank}(x)$	$\mathsf{trcl}(x)$

Proof. For example, "BST + Infinity + Foundation" proves the following

(i) x is an ordinal iff x is transitive and $(x, \in)$ is a linear ordering.

(ii) $(A, \prec)$ is a well ordering iff there exist α, f such that α is an ordinal and $f : (A, \prec) \to (\alpha, \in)$ is an isomorphism.

We refer the reader to Kunen's Chapter IV, Section 5 for the rest. □

Let ZF be the theory ZFC without the axiom of choice (So ZF = "BST + Infinity + Foundation + Power Set"). Although, $\mathcal{P}(x)$ and V_α are in general not absolute even for transitive models of ZF, the following lemma says that they are somewhat well behaved.

Lemma 12.12. *Suppose* $\mathbf{M}$ *is a transitive model of ZF. Then*

(a) For every $x \in \mathbf{M}$*,* $(\mathcal{P}(x))^{\mathbf{M}} = \mathcal{P}(x) \cap \mathbf{M}$*.*

(b) For every $\alpha \in \mathbf{M}$*,* $(V_\alpha)^{\mathbf{M}} = V_\alpha \cap \mathbf{M}$*.*

Proof. (a) For every $x, y \in \mathbf{M}$, $(y = \mathcal{P}(x))^{\mathbf{M}}$ iff $(\forall z \in \mathbf{M})(z \in y \iff z \subseteq x)$. So $y = \mathcal{P}(x) \cap \mathbf{M}$. (b) For every $x, \alpha \in \mathbf{M}$, $(x \in V_\alpha)^{\mathbf{M}}$ iff $(\mathsf{rank}(x) < \alpha)^{\mathbf{M}}$ iff $\mathsf{rank}(x) < \alpha$ (by absoluteness of $\mathsf{rank}(x)$ for $\mathbf{M}$). □

13 Reflection theorems

Recall that every first order formula is logically equivalent to a formula that only uses $\neg, \wedge$ and $\exists$. For example, $\phi \implies \psi$ and $(\forall x)\phi$ are logically equivalent to $\neg(\phi \wedge \neg(\psi))$ and $\neg((\exists x)(\neg\phi))$ respectively.

Lemma 13.1 (Tarski-Vaught Criterion)**.** *Suppose* $\mathbf{M} \subseteq \mathbf{N}$ *are classes. Let* $\mathcal{F}$ *be a finite set of formulas such that for every pair of formulas* ϕ *and* ψ*, if* ψ *is a subformula of* ϕ *and* $\phi \in \mathcal{F}$*, then* $\psi \in \mathcal{F}$*. Then the following are equivalent.*

(a) Every formula in $\mathcal{F}$ *is absolute between* $\mathbf{M}, \mathbf{N}$*.*

(b) For every $\phi \in \mathcal{F}$*, if* $\phi \equiv (\exists x)(\psi(x, y_1, \dots, y_n))$ *(with free variables of* ψ *as displayed), then*

$$(\forall y_1, \dots y_n \in \mathbf{M})[(\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n)) \implies (\exists x \in \mathbf{M})((\psi^{\mathbf{N}}(x, y_1, \dots, y_n)))]$$

Proof. We can assume that the formulas in $\mathcal{F}$ only use $\wedge, \neg$ and $\exists$. First assume (a). Let $\phi \in \mathcal{F}$ be such that $\phi \equiv (\exists x)(\psi(x, y_1, \dots, y_n))$. Since ψ is a subformula of ϕ, we have $\psi \in \mathcal{F}$. Assume $y_1, \dots, y_n \in \mathbf{M}$ and suppose $(\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$ holds. By the absoluteness of ϕ between $\mathbf{M}, \mathbf{N}$, we also have $(\exists x \in \mathbf{M})(\psi^{\mathbf{M}}(x, y_1, \dots, y_n))$. Fix a witness $x \in \mathbf{M}$. Now use the absoluteness of ψ between $\mathbf{M}, \mathbf{N}$ to get $\psi^{\mathbf{N}}(x, y_1, \dots, y_n)$. It follows that $(\exists x \in \mathbf{M})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$.

Now assume (b) and we'll prove (a) by induction on the length of $\phi \in \mathcal{F}$. If ϕ is either quantifier free or of the form $\psi_1 \wedge \psi_2$, $\neg(\psi)$, then this is clear. So assume $\phi \equiv (\exists x)(\psi(x, y_1, \dots, y_n))$. Let $y_1, \dots, y_n \in \mathbf{M}$. We'll show that $\phi^{\mathbf{M}}(y_1, \dots, y_n) \iff \phi^{\mathbf{N}}(y_1, \dots, y_n)$. Equivalently, we have to show

$$(\exists x \in \mathbf{M})(\psi^{\mathbf{M}}(x, y_1, \dots, y_n)) \iff (\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$$

First assume $(\exists x \in \mathbf{M})(\psi^{\mathbf{M}}(x, y_1, \dots, y_n))$. Fix a witness $x \in \mathbf{M}$. Since ψ has smaller length than ϕ, by inductive hypothesis, it is absolute between $\mathbf{M}, \mathbf{N}$. Hence $\psi^{\mathbf{N}}(x, y_1, \dots, y_n)$. It follows that $(\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$.

Next suppose $(\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$. Using (b), we get $(\exists x \in \mathbf{M})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$. Fix a witness $x \in \mathbf{M}$. Since ψ has smaller length than ϕ, by inductive hypothesis, it is absolute between $\mathbf{M}, \mathbf{N}$, we get $\psi^{\mathbf{M}}(x, y_1, \dots, y_n)$. Hence $(\exists x \in \mathbf{N})(\psi^{\mathbf{N}}(x, y_1, \dots, y_n))$. So we are done. ☐

Theorem 13.2 (Reflection Theorem I)**.** *For any formula* $\chi \equiv \chi(z_1, \dots, z_k)$*, the following is a theorem in ZF*

$$(\forall \alpha)(\exists \beta > \alpha)(\forall z_1, \dots, z_k \in V_\beta)(\chi(z_1, \dots, z_n) \iff \chi^{V_\beta}(z_1, \dots, z_k))$$

Proof. We can assume that χ only uses $\wedge, \neg$ and $\exists$. Let $\mathcal{F}$ be the finite set of all subformulas of χ. For each formula $\phi \in \mathcal{F}$, define $F_\phi : \mathbf{V}^n \to \mathbf{ORD}$ as follows. If $\phi \equiv (\exists x)(\psi(x, y_1, \dots, y_n))$, $a_1, \dots, a_n \in \mathbf{V}$ and $(\exists x)(\psi(x, a_1, \dots, a_n))$, then $F_\phi(a_1, \dots, a_n)$ is the least α such that $(\exists x \in V_\alpha)(\psi(x, a_1, \dots, a_n))$. Otherwise, define $F_\phi(a_1, \dots, a_n) = 0$.

Next, for $\phi \in \mathcal{F}$, define $G_\phi : \mathbf{ORD} \to \mathbf{ORD}$ by $G_\phi(\alpha) = \sup(\{F_\phi(y_1, \dots, y_n) : y_1, \dots, y_n \in V_\alpha\})$ where $F_\phi : \mathbf{V}^n \to \mathbf{ORD}$.

Now let α be arbitrary and we'll find $\beta > \alpha$ such that for every $\phi \in \mathcal{F}$, ϕ is absolute between V_α and $\mathbf{V}$. This suffices since $\chi \in \mathcal{F}$. Define $\langle \beta_n : n < \omega \rangle$ as follows. $\beta_0 = \alpha + 1$ and $\beta_{n+1} = \sup(\{G_\phi(\beta_n) : \phi \in \mathcal{F}\})$. Put $\beta = \sup(\{\beta_n : n < \omega\})$. We claim that $\phi \iff \phi^{V_\beta}$. By the Tarski-Vaught criterion (Lemma 13.1), it suffices to show that for every $\phi \in \mathcal{F}$, if $\phi \equiv (\exists x)(\psi(x, y_1, \dots, y_n))$, then for every $y_1, \dots, y_n \in V_\beta$,

$$(\exists x)(\psi(x, y_1, \dots, y_n)) \implies (\exists x \in V_\beta)(\psi(x, y_1, \dots, y_n))$$

So fix $y_1, \dots, y_n \in V_\beta$. Since $\beta = \sup(\{\beta_n : n < \omega\})$, choose $m < \omega$ least such that $y_1, \dots, y_n \in V_{\beta_m}$. Since $\beta \geq G_\psi(\beta_m)$, it follows that $(\exists x \in V_\beta)(\psi(x, y_1, \dots, y_n))$. This completes the proof. ☐

The following more general version of Reflection theorem applies to many hierarchies other than von Neumann hierarchy. Its proof is identical to that of Theorem 13.2.

Theorem 13.3 (Reflection Theorem II)**.** *Suppose* $\langle A_\alpha : \alpha \in \boldsymbol{ORD} \rangle$ *satisfies the following.*

(a) $(\forall \alpha < \beta)(A_\alpha \subseteq A_\beta)$

(b) *For all limit* α, $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$

Let $\mathbf{A} = \bigcup\{A_\alpha : \alpha \in \boldsymbol{ORD}\}$. *Let* $\chi(z_1, \ldots, z_k)$ *be a formula. Then*

$$(\forall \alpha)(\exists \beta > \alpha)(\forall z_1, \ldots, z_n \in A_\beta)(\chi^{A_\beta}(z_1, \ldots, z_n) \iff \chi^{\mathbf{A}}(z_1, \ldots, z_n))$$

14 The constructible universe

Informally, $\mathsf{Def}(A, n)$ is the set of all n-ary relations on A which can be defined by a formula with n free variables ranging over A. The following definitions are intended to capture this notion.

Definition 14.1. *For* $n < \omega$ *and* $i, j < n$, *define the following.*

(a) $\mathsf{Proj}(A, R, n) = \{s \in A^n : \exists t \in R(t \restriction n = s)\}$.

(b) $\mathsf{Diag}_\in(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}$.

(c) $\mathsf{Diag}_=(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}$.

Definition 14.2 (Definable relations)**.** *By recursion on* $k < \omega$, *define* $D(k, A, n)$ *as follows.*

(i) $D(0, A, n) = \{\mathsf{Diag}_\in(A, n, i, j) : i, j < n\} \cup \{\mathsf{Diag}_=(A, n, i, j) : i, j < n\}$

(ii) $D(k+1, A, n) = D(k, A, n) \cup \{A^n \setminus R : R \in D(k, A, n)\} \cup \{R \cap S : R, S \in D(k, A, n)\} \cup \{\mathsf{Proj}(A, R, n) : R \in D(k, A, n+1)\}$

Define $\mathsf{Def}(A, n) = \bigcup\{D(k, A, n) : k < \omega\}$

We refer to $\mathsf{Def}(A, n)$ as the set of all definable n-ary relations on A. The following should be clear.

Lemma 14.3. *In ZF,* $(\forall k < \omega)(|D(k, A, n)| \leq \omega)$ *and* $|\mathsf{Def}(A, n)| \leq \omega$.

Lemma 14.4. *Let* $\phi(x_0, \ldots, x_{n-1})$ *be a formula with free variables* $x_0, x_1, \ldots, x_{n-1}$. *Then for every* A,

$$\{s \in A^n : \phi^A(s(0), s(1), \ldots, s(n))\} \in \mathsf{Def}(A, n)$$

Proof. By induction on the length of the formula ϕ. □

Using the methods of Section 12, one can verify the following.

Lemma 14.5. $\mathsf{Def}(A, n)$ *is absolute for transitive models of "BST + Infinity + Foundation".*

Informally, the definable power set of A, denoted $\mathcal{D}(A)$, is the set of all subsets of A which can be defined by a formula with parameters from A. Formally, we define

Definition 14.6 (Definable power set)**.**

$$\mathcal{D}(A) = \{X \subseteq A : (\exists n < \omega)(\exists s \in A^n)(\exists R \in \mathsf{Def}(A, n+1))[X = \{x \in A : s \cup \{(n, x)\} \in R\}]\}$$

$R \in \mathsf{Def}(A, n+1)$ is an $(n+1)$-ary "formula", $s(0), s(1), \ldots, s(n-1)$ are parameters in A and X is the set of all $x \in A$ such that $R(s(0), s(1), \ldots, s(n-1), x)$ holds.

Lemma 14.7. *Let* $\phi(v_0, v_1, \ldots, v_n, x)$ *be a formula with free variables as shown. Then for every* A *and* $s_0, s_1, \ldots, s_n \in A$,

$$\{x \in A : \phi^A(s_0, s_1, \ldots, s_n, x)\} \in \mathcal{D}(A)$$

Proof. By Lemma 14.4. □

Lemma 14.8. *For any A, the following hold.*

(a) $\mathcal{D}(A) \subseteq \mathcal{P}(A)$.

(b) If A is transitive, then $A \subseteq \mathcal{D}(A)$.

(c) $(\forall X \subseteq A)(|X| < \omega \implies X \in \mathcal{D}(A))$.

(d) (ZF) If A can be well ordered and $|A| = \kappa \geq \omega$, then $|\mathcal{D}(A)| \leq \kappa$.

Proof. (a) and (b) are left to the reader. For (c), use induction on $|X|$. For (d), use Lemma 14.3 and $(\forall n < \omega)(|A^n| = \kappa)$. □

Lemma 14.9. *$\mathcal{D}(A)$ is absolute for transitive models of BST + Infinity + Foundation.*

Proof. Using the absoluteness of $\mathsf{Def}(A, n)$. □

Definition 14.10 (The constructible hierarchy)**.** *By recursion on α, define $\langle L_\alpha : \alpha \in \boldsymbol{ORD} \rangle$*

(1) $L_0 = 0$.

(2) $L_{\alpha+1} = \mathcal{D}(L_\alpha)$.

(3) If α is limit, $L_\alpha = \bigcup\{L_\beta : \beta < \alpha\}$.

Define

$$\mathbf{L} = \bigcup\{L_\alpha : \alpha \in \boldsymbol{ORD}\}$$

Note the difference with the von Neumann hierarchy: $L_{\alpha+1} = \mathcal{D}(L_\alpha)$ is not the full power set of L_α.

Lemma 14.11. *For every α, L_α is transitive, $L_\alpha \cap \boldsymbol{ORD} = \alpha$ and if $\beta < \alpha$, then $L_\beta \subseteq L_\alpha$.*

Proof. By induction on α. If $\alpha = 0$ or a limit ordinal, all three statements are clear. Suppose $\alpha = \beta + 1$. Since L_α is transitive, by Lemma 14.8, $L_\alpha \subseteq \mathcal{D}(L_\alpha)$. Hence $L_{\alpha+1}$ is also transitive and $L_\alpha \subseteq L_{\alpha+1}$. Since $\alpha = \{\beta \in L_\alpha : (\beta \in \mathbf{ORD})^{L_\alpha}\}$ (by absoluteness of $\beta \in \mathbf{ORD}$), therefore $\alpha \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$. Also note that $\alpha + 1$ is not a subset of L_α since $\alpha \notin L_\alpha$. So $\alpha + 1 \notin L_{\alpha+1}$. Hence $L_{\alpha+1} \cap \mathbf{ORD} = \alpha + 1$. □

We will now check that $\mathbf{L}$ is a model of ZF.

(1) $\mathsf{Extensionality}$: Since $\mathbf{L}$ is transitive, $\mathsf{Extensionality}$ holds in $\mathbf{L}$.

(2) $\mathsf{Comprehension}$: Let $\phi(x, z, v_1, \ldots, v_n)$ be a formula with free variables as shown. Assume $z, v_1, \ldots, v_n \in \mathbf{L}$. It is enough to show

$$\{x \in z : \phi^{\mathbf{L}}(x, z, v_1, \ldots, v_n)\} \in \mathbf{L}$$

Let α be large enough so that $z, v_1, \ldots, v_n \in L_\alpha$. By the Reflection Theorem (Theorem 13.3), we can find $\beta > \alpha$ such that ϕ is absolute between L_β and $\mathbf{L}$. Hence

$$\phi^{L_\alpha}(x, z, v_1, \ldots, v_n) \iff \phi^{\mathbf{L}}(x, z, v_1, \ldots, v_n)$$

Now since $z \in L_\beta$,

$$\{x \in z : \phi^{L_\beta}(x, z, v_1, \ldots, v_n)\} = \{x \in L_\beta : (x \in z \wedge \phi(x, z, v_1, \ldots, v_n))^{L_\beta}\} \in \mathcal{D}(L_\beta) = L_{\beta+1}$$

so we are done.

(3) Pairing, Union: If $x, y \in L_\alpha$, then

$$\{x, y\} = \{z \in L_\alpha : (z = x \vee z = y)^{L_\alpha}\} \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$$

So Pairing holds in $\mathbf{L}$. If $X \in L_\alpha$, then

$$\bigcup X = \{y \in L_\alpha : ((\exists z \in X)(y \in z))^{L_\alpha}\} \in L_{\alpha+1}$$

so Union holds in $\mathbf{L}$.

(4) Replacement: Suppose $\phi(x, y, A, w_1, \ldots, w_n)$ is a formula with free variables as shown and let $A, w_1, \ldots, w_n \in \mathbf{L}$. Suppose

$$(\forall x \in A)(\exists! y \in \mathbf{L})(\phi^{\mathbf{L}}(x, y, A, w_1, \ldots, w_n))$$

It is enough to find $B \in \mathbf{L}$ such that

$$\{y : (\exists x \in A)(\phi^{\mathbf{L}}(x, y, A, z_1, \ldots, z_n))\} \subseteq B$$

Using replacement (in $\mathbf{V}$), we can find $\alpha \in \mathbf{ORD}$ such that for every $x \in A$, there exists $y \in L_\alpha$ such that $\phi^{\mathbf{L}}(x, y, A, w_1, \ldots, w_n)$ holds. So $B = L_\alpha$ is as required.

(5) Infinity: Follows from Lemma 12.9 the fact that $\omega \in \mathbf{L}$.

(6) Power Set: Suppose $x \in \mathbf{L}$. We need to find $y \in \mathbf{L}$ such that

$$(\forall z \in \mathbf{L})(z \in y \iff z \subseteq x)$$

Let $x \in L_\alpha$. Using replacement, find $\beta > \alpha$ such that for every $z \in \mathcal{P}(x) \cap \mathbf{L}$, $z \in L_\beta$. Define $y = \{z \in L_\beta : (z \subseteq x)^{\mathbf{L}_\beta}\}$. Then $y \in \mathcal{D}(L_\beta) = L_{\beta+1}$ is as required.

(7) Foundation: We need to show $(\forall x \in \mathbf{L})[x \neq \emptyset \implies (\exists y \in \mathbf{L})(y \in x \wedge y \cap x = 0)]$. Suppose $x \in \mathbf{L}$ is nonempty. Using foundation, choose $y \in x$ be such that $y \cap x = 0$. As $\mathbf{L}$ is transitive, $y \in \mathbf{L}$. So we are done.

This concludes the proof of the fact that $\mathbf{L}$ is a model of ZF. Before we proceed to check that AC and GCH hold in $\mathbf{L}$. It will be convenient to introduce the following.

Definition 14.12 (Axiom of constructibility)**.** *The Axiom of constructibility is the statement* $\mathbf{V} = \mathbf{L}$. *Equivalently,*

$$(\forall x)(\exists \alpha)(x \in L_\alpha)$$

Lemma 14.13. *$x \in L_\alpha$ is absolute for transitive models of BST + Infinity + Foundation.*

Proof. Using the facts that $\mathcal{D}(X)$ is absolute and $L_{\alpha+1} = \mathcal{D}(L_\alpha)$. See Theorem 5.6 in Chapter IV of Kunen's book for details. □

Corollary 14.14. *$ZF \vdash (\mathbf{V} = \mathbf{L})^{\mathbf{L}}$.*

Proof. We have to check that $(\forall x \in \mathbf{L})(\exists \alpha \in \mathbf{L})((x \in L_\alpha)^{\mathbf{L}})$. By the absoluteness of $x \in L_\alpha$, this is equivalent to $(\forall x \in \mathbf{L})(\exists \alpha \in \mathbf{L})(x \in L_\alpha)$ which clearly holds.

□

Corollary 14.15. *If ZF is consistent, so is ZF +* $\mathbf{V} = \mathbf{L}$. *For every sentence ϕ, if ZF proves* $(\mathbf{V} = \mathbf{L} \implies \phi)$*, then ZF proves* $\phi^{\mathbf{L}}$.

Proof. The first statement follows from Theorem 11.3 and Corollary 14.14. Next suppose ZF proves $(\mathbf{V} = \mathbf{L} \implies \phi)$. Then there is a finite conjunction ψ of axioms of ZF such that $\psi \wedge \mathbf{V} = \mathbf{L} \implies \phi$ is logically valid. By Fact 11.2, $\psi^{\mathbf{L}} \wedge (\mathbf{V} = \mathbf{L})^{\mathbf{L}} \implies \phi^{\mathbf{L}}$ is also logically valid. Now $\psi^{\mathbf{L}}$ and $(\mathbf{V} = \mathbf{L})^{\mathbf{L}}$ are theorems of ZF. Hence ZF proves $\phi^{\mathbf{L}}$. □

15 AC and GCH in L

Recall that the set of definable n-ary relations on A, $\mathsf{Def}(A,n)$ is countable. The following definition gives an enumeration of the members of $\mathsf{Def}(A,n)$ in a definable way.

Definition 15.1. *By induction on $m < \omega$, define $\mathsf{En}(m,A,n)$ as follows.*

(a) If $m = 2^i 3^j$ and $i,j < n$, then $\mathsf{En}(m,A,n) = \mathsf{Diag}_{\in}(A,n,i,j)$.

(b) If $m = 2^i 3^j 5$ and $i,j < n$, then $\mathsf{En}(m,A,n) = \mathsf{Diag}_{=}(A,n,i,j)$.

(c) If $m = 2^i 3^j 5^2$, then $\mathsf{En}(m,A,n) = A^n \setminus \mathsf{En}(i,A,n)$.

(d) If $m = 2^i 3^j 5^3$, then $\mathsf{En}(m,A,n) = \mathsf{En}(i,A,n) \cap \mathsf{En}(j,A,n)$.

(e) If $m = 2^i 3^j 5^4$, then $\mathsf{En}(m,A,n) = \mathsf{Proj}(A, \mathsf{En}(i,A,n+1), n)$.

(f) If m is not of the form specified in one of (a)-(e), then $\mathsf{En}(m,A,n) = 0$.

The following is easy to verify using the definition of $\mathsf{Def}(A,n)$.

Lemma 15.2. *$\mathsf{Def}(A,n) = \{\mathsf{En}(m,A,n) : m < \omega\}$.*

Definition 15.3. *Suppose $A \neq 0$. For $n < \omega$, $s \in A^n$ and $m < \omega$ define,*

$$\mathcal{D}(A,n,m,s) = \{x \in A : s \cup \{(n,x)\} \in \mathsf{En}(m,A,n+1)\}$$

Using the definition of $\mathcal{D}(A)$, the following is easy to verify.

Lemma 15.4. *$\mathcal{D}(A) = \{\mathcal{D}(A,n,m,s) : m,n < \omega \wedge s \in A^n\}$.*

Suppose $A \neq 0$ and R is a well order of A. Let R_n be the induced lexicographic well order on A^n. Let $\lhd$ be the well ordering on $\omega \times \omega \times A^{<\omega}$ defined by $(n,m,s) \lhd (n',m',s')$ iff

(i) $n < n'$ or

(ii) $n = n'$ and $m < m'$ or

(iii) $(n,m) = (n',m')$ and $sR_n s'$.

For each $X \in \mathcal{D}(A)$, let $(n(X), m(X), s(X))$ be $\lhd$-least $(n,m,s) \in \omega \times \omega \times A^{<\omega}$ such that $X = \mathcal{D}(A,n,m,s)$. For $X, Y \in \mathcal{D}(A)$, define $X \prec Y$ iff $(n(X), m(X), s(X)) \lhd (n(Y), m(Y), s(Y))$. Note that $\prec$ is a well order on $\mathcal{D}(A)$. We say that $(\mathcal{D}(A), \prec)$ is the well ordering induced by (A,R).

We can now define a well order on $\mathbf{L}$ as follows. By transfinite recursion, define $\langle <_\alpha : \alpha \in \mathbf{ORD} \rangle$ as follows.

(1) $<_0 = 0$.

(2) Suppose $<_\alpha$ has been been defined such that $(L_\alpha, <_\alpha)$ is a well ordering. Let $(\mathcal{D}(L_\alpha), \prec)$ be the well ordering induced by $(L_\alpha, <_\alpha)$. Define $x <_{\alpha+1} y$ iff
   - (a) $x, y \in L_\alpha$ and $x <_\alpha y$ or
   - (b) $x \in L_\alpha$ and $y \in L_{\alpha+1} \setminus L_\alpha$ or
   - (c) $x, y \in L_{\alpha+1} \setminus L_\alpha$ and $x \prec y$.

(3) If α is limit, define $<_\alpha = \bigcup\{<_\beta : \beta < \alpha\}$.

It is easy to see that each $(L_\alpha, <_\alpha)$ is a well ordering and if $\alpha \geq \omega$, then $\mathsf{type}(L_\alpha, <_\alpha) < |\alpha|^+$. Hence $\mathsf{type}(L_\kappa, <_\kappa) = \kappa$ for every infinite cardinal κ. So we have proved the following.

Theorem 15.5. *In ZF,* $\mathbf{V} = \mathbf{L} \implies$ *AC. Moreover,* $<_{\mathbf{L}} = \bigcup\{<_\alpha : \alpha \in \boldsymbol{ORD}\}$ *is a definable well-ordering of* $\mathbf{L}$.

We now proceed to show that $\mathbf{V} = \mathbf{L}$ implies GCH. The key fact here is that every constructible subset of L_α is a member of $L_{|\alpha|^+}$. The proof of this fact uses a Lowenheim Skolem argument.

Definition 15.6 (Height of a transitive set)**.** *For a transitive set M, define the height of M,*

$$o(M) = M \cap \boldsymbol{ORD}$$

Note that $o(M)$ is the least ordinal not in M.

Lemma 15.7. *There is a finite conjunction ψ of axioms of ZF - Powerset such that for every transitive M, $\psi^M \implies L_{o(M)} = \mathbf{L}^M \subseteq M$.*

Proof. Let ψ be a finite conjunction of sufficiently many axioms of ZF - Powerset such that ψ proves that the notions of ordinal, rank and L_α are absolute for transitive models of ψ and ψ implies that there is no largest ordinal. Let us show that ψ is as required. Suppose M is transitive and assume ψ^M holds. Then $o(M) = M \cap \mathbf{ORD}$ is a limit ordinal. For each $\alpha < o(M)$, $[(\exists y)(y = L_\alpha)]^M$ holds. By absoluteness of L_α for M, we get $(\forall \alpha < o(M))(L_\alpha^M = L_\alpha)$. So $L_{o(M)} \subseteq M$ and $\mathbf{L}^M = \bigcup\{L_\alpha^M : \alpha \in M \cap \mathbf{ORD}\} = L_{o(M)}$. □

The following is immediate.

Corollary 15.8. *There is a finite conjunction ψ of axioms of ZF - Powerset such that for every transitive M,*

$$\psi^M \wedge (\mathbf{V} = \mathbf{L})^M \implies M = L_{o(M)}$$

Lemma 15.9 (Downward Löwenheim-Skolem)**.** *Suppose $\chi(x_1, \ldots, x_n)$ is a formula with free variables $x_1, \ldots, x_n$. Then the following is a theorem of ZFC.*

$$(\forall X)(\exists Y)[X \subseteq Y \wedge |Y| = \max(|X|, \omega) \wedge (\forall x_1, \ldots, x_n \in Y)(\chi(x_1, \ldots, x_n) \iff \chi^Y(x_1, \ldots, x_n))]$$

Proof. We can assume that χ only uses $\neg, \wedge, \exists$. Using the Reflection Theorem 13.2, choose β such that $X \in V_\beta$ and χ is absolute for V_β. Using AC, fix a well ordering $\prec$ of V_β. We start repeating the proof of Theorem 13.2 with Skolem functions modified as follows. Let $\mathcal{F}$ be the set of all subformulas of χ. For each $\phi \in \mathcal{F}$ define the Skolem function F_ϕ as follows. If $\phi \equiv (\exists x)(\psi(x, y_1, \ldots, y_n))$, $a_1, \ldots, a_n \in V_\beta$ and $(\exists x \in V_\beta)(\psi(x, a_1, \ldots, a_n))$, then $F_\phi(a_1, \ldots, a_n)$ is the $\prec$-least such $x \in V_\beta$. Otherwise, define $F_\phi(a_1, \ldots, a_n) = 0$. Using Theorem 5.17, choose $Y \subseteq V_\beta$ such that $X \subseteq Y$, $|Y| = \max(|X|, \omega)$ and Y is closed under F_ϕ for each $\phi \in \mathcal{F}$. By the Tarski-Vaught criterion 13.1, χ is absolute between Y and V_β. Since χ is also absolute for V_β, it follows that χ is absolute for Y. □

Theorem 15.10. *In ZF,* $\mathbf{V} = \mathbf{L} \implies$ *GCH.*

Proof. Assume $\mathbf{V} = \mathbf{L}$. By Theorem 15.5, we can freely use the axiom of choice. Let κ be an infinite cardinal. We'll show $|\mathcal{P}(\kappa)| \leq \kappa^+$. Since $|L_{\kappa^+}| = \kappa^+$, it suffices to show that $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$. Fix $X \subseteq \kappa$. Let ψ be as in Corollary 15.8. By Lemma 15.9, choose Y such that $L_\kappa \cup \{X\} \subseteq Y$, $|Y| = |L_\kappa| = \kappa$ and $\psi^Y \wedge (\mathbf{V} = \mathbf{L})^Y$ holds. Since $(Y, \in)$ is well founded, by Theorem 10.14 fix M, F unique such that M is transitive and $F : (Y, \in) \to (M, \in)$ is an isomorphism. Then $\psi^M \wedge (\mathbf{V} = \mathbf{L})^M$ holds as well. Note that $F(X) = X$ since $\kappa \subseteq L_\kappa \subseteq Y$ and $F \restriction \kappa$ is the identity on κ. So $X \in M$. Put $o(M) = M \cap \mathbf{ORD} = \gamma$. By Corollary 15.8, $M = L_\gamma$. Since $|L_\gamma| = |M| = |Y| = |L_\kappa| = \kappa$, it follows that $\gamma < \kappa^+$. Therefore $X \in L_\gamma \subseteq L_{\kappa^+}$. Hence $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$ and we are done. □

Corollary 15.11. *If ZF is consistent, so is ZFC + GCH.*

Proof. By Corollary 14.15 and Theorems 15.5 and 15.10.

□

16 Countable transitive models

Let $\mathcal{L} = \{\in\}$. Recall that an $\mathcal{L}$-model is a pair (M, E) where M is a non empty set and E is a binary relation on M that interprets $\in$. If E is the actual membership relation on M, then we say that the model is **standard**. Every standard model (or just well founded model) is isomorphic to a standard transitive model, namely, its Mostowski collapse (Theorem 10.14). So, among the standard models, it is enough to analyse the transitive ones. The aim of this section is to sketch some of the necessary properties of a possible standard model of ZFC plus $\neg\mathsf{CH}$. We do not consider non well founded models at all.

The following lemma says that a proper transitive class cannot be used to produce a model of $\mathbf{V} \neq \mathbf{L}$ and therefore of $\neg\mathsf{CH}$.

Lemma 16.1. *Let* $\mathbf{M}$ *be a transitive class with* $\boldsymbol{ORD} \subseteq \mathbf{M}$*. Suppose for each ZFC axiom* ψ*, ZFC proves* $\psi^{\mathbf{M}}$*. Suppose also that ZFC proves* $(\mathbf{V} \neq \mathbf{L})^{\mathbf{M}}$*. Then ZFC is inconsistent.*

Proof. Since $\mathbf{ORD} \subseteq \mathbf{M}$, by absoluteness of L_α for $\mathbf{M}$, we must have $\mathbf{L} \subseteq \mathbf{M}$. Since $(\mathbf{V} \neq \mathbf{L})^{\mathbf{M}}$ and $(\mathbf{V} = \mathbf{L})^{\mathbf{L}}$ (by Corollary 14.14), we must have $\mathbf{M} \neq \mathbf{L}$. Thus $\mathbf{M} \setminus \mathbf{L} \neq 0$. So ZFC proves $\mathbf{V} \neq \mathbf{L}$ and therefore, by Corollary 14.15, it is inconsistent. □

Note that the argument strongly uses the fact that $\mathbf{ORD} \subseteq \mathbf{M}$. It $\mathbf{M}$ is a transitive set, then the proof only shows that $\mathbf{M} \setminus L_{o(\mathbf{M})} \neq 0$ which doesn't necessarily imply $\mathbf{V} \neq \mathbf{L}$ since the sets in $\mathbf{M} \setminus L_{o(\mathbf{M})}$ may appear in L_α for some $\alpha > o(\mathbf{M})$. In fact, a slight refinement of the above argument gives the following.

Lemma 16.2. *Let T be any theory which is consistent with ZFC plus* $\mathbf{V} = \mathbf{L}$*. Then T cannot prove the existence of an uncountable transitive set model of ZFC plus the negation of CH.*

Proof. Suppose not. We can assume that $\mathbf{V} = \mathbf{L}$ is an axiom of T. Work in T. Fix an uncountable transitive set M that models ZFC plus $\neg\mathsf{CH}$.

We claim that $o(M) \geq \omega_1$. Suppose not. Note that by the absoluteness of the rank function, $M = \bigcup\{(V_\alpha)^M : \alpha < o(M)\}$. Since $o(M)$ is countable, for some $\alpha < o(M)$, $(V_\alpha)^M$ is uncountable. So M contains an uncountable set A. Since AC holds in M, it also contains the order type of a well ordering of A. So $\omega_1 \in M$. Hence $o(M) > \omega_1$: Contradiction.

Since $\mathbf{V} = \mathbf{L}$ holds in T, $\mathcal{P}(\omega) = L_{\omega_1} \cap \mathcal{P}(\omega)$. It follows that every $X \in \mathcal{P}(\omega) \cap M$, there exists $\alpha < \omega_1 \leq o(M)$ such that $X \in L_\alpha$. By absoluteness, it follows that $(X \in L_\alpha)^M$. Hence M has enough ordinals to know that every set of integers is in $\mathbf{L}$. It follows that CH holds in M which is a contradiction. □

So if we have any hope of finding a standard model of $\neg\mathsf{CH}$, we must look at countable transitive models. Because of Gödel's second incompleteness theorem, we cannot prove the existence of such models in ZFC. But there is an easy way to get around this difficulty as follows.

Adjoin a constant symbol $\mathbf{M}$ to the language of ZFC (which has just one non logical symbol, namely, $\in$). Let T be the theory whose axioms are the axioms of ZFC, "$\mathbf{M}$ is countable transitive" and the relativizations $\phi^{\mathbf{M}}$ for each axiom ϕ of ZFC.

Lemma 16.3. *T is a conservative extension of ZFC. This means that for every sentence* ψ *in the language of ZFC, if T proves* ψ*, then ZFC proves* ψ*. In particular, if ZFC is consistent, so is T.*

Proof. Suppose ψ is a sentence in the language of ZFC and T proves ψ. Let ϕ be a finite conjunction of axioms of ZFC such that

$$(\phi \wedge \mathbf{M} \text{ is countable transitive} \wedge \phi^{\mathbf{M}}) \implies \psi$$

is logically valid. It suffices to show that

$$(\exists N)(N \text{ is countable transitive} \wedge \phi^{N})$$

is a theorem in ZFC. But this easily follows from Lemma 15.9: Just set $\chi = \phi$ and $X = 0$ to get a countable Y as there and take N to be the Mostowski collapse of Y. □

From now on, when we say "Fix a ctm (countable transitive model) $\mathbf{M}$ of ZFC", we mean work in the theory T as above. Since $\mathbf{M}$ could already satisfy $\mathbf{V} = \mathbf{L}$, we should try to construct a ctm $\mathbf{N}$ of ZFC plus $\neg\mathsf{CH}$ by enlarging $\mathbf{M}$. The next lemma says that, starting with $\mathbf{M}$, we can only hope to produce ctm's $\mathbf{N}$ of ZFC with $o(\mathbf{N}) = o(\mathbf{M})$.

Lemma 16.4. *It is consistent with ZFC plus "there is a ctm of ZFC" to assume that any two ctm's of ZFC have the same height.*

Proof. Note that if ZFC plus "there is a ctm of ZFC" is consistent, then ZFC plus "there is a ctm of ZFC" plus $\mathbf{V} = \mathbf{L}$ is also consistent (as witnessed by $\mathbf{L}$). Work in the theory ZFC plus "there is a ctm of ZFC" plus $\mathbf{V} = \mathbf{L}$. Fix a ctm $\mathbf{N}$ of ZFC of least height. If all ctm's of ZFC have the same height as $\mathbf{N}$, we are done. Otherwise, let $\alpha > o(\mathbf{N})$ be least such that there is a ctm $\mathbf{N}'$ of ZFC with $o(\mathbf{N}') = \alpha$. Since we are assuming $\mathbf{V} = \mathbf{L}$, $\mathbf{N} = L_{o(\mathbf{N})}$ and $\mathbf{N}' = L_\alpha$. Now $\mathbf{N}'$ thinks that there is a ctm of ZFC (namely $L_{o(\mathbf{N})}$) and all ctm's of ZFC have the same height (namely $o(\mathbf{N})$) so we are done. $\square$

Suppose $\mathbf{M}$ is a ctm of ZFC with height $o(M) = \gamma < \omega_1$. Since $\mathbf{M}$ is countable, most subsets of ω are not in $\mathbf{M}$. Let $(\omega_2)^{\mathbf{M}} = \theta$. Although θ is really a countable ordinal, $\mathbf{M}$ thinks it is the second uncountable cardinal. We would like to find a one-one sequence $\overline{X} = \langle X_\alpha : \alpha < \theta \rangle$ of subsets of ω and another ctm $\mathbf{N}$ of ZFC such that $\mathbf{M} \subseteq \mathbf{N}$, $\overline{X} \in \mathbf{N}$, $o(\mathbf{M}) = o(\mathbf{N})$ (see Lemma 16.4) and $(\omega_2)^{\mathbf{N}} = \theta$. Then $\mathbf{N}$ would satisfy $\neg$CH as witnessed by $\overline{X} \in \mathbf{N}$.

To simplify matters, let us first try to find one subset $X \subseteq \omega$ such that $X \notin \mathbf{M}$ and there is a ctm $\mathbf{M}[X]$ of ZFC such that $X \in \mathbf{M}[X]$, $\mathbf{M} \subseteq \mathbf{M}[X]$ and $o(\mathbf{M}) = o(\mathbf{M}[X])$. The requirement $o(\mathbf{M}) = o(\mathbf{M}[X])$ already rules out several candidates X. For example, X should not code a well ordering of ω of order type $\gamma > o(\mathbf{M})$ otherwise $\mathbf{M}[X]$ can decode γ from X. To solve this difficulty, Paul Cohen came up with the notion of what he called a "generic" subset $X \subseteq \omega$ over $\mathbf{M}$ and showed that they can always be adjoined to $\mathbf{M}$ to get a smallest ctm of ZFC extending $\mathbf{M}$ that contains X. Soon after Cohen's proof of consistency of $\neg\mathsf{CH}$, Robert Solovay generalized his construction to a more general setting of posets which we now discuss.

17 Forcing posets

Definition 17.1. *A partial ordering/poset/forcing notion is a triplet $(\mathbb{P}, \leq, 1_\mathbb{P})$ such that $1_\mathbb{P} \in \mathbb{P}$ and $\leq$ is a binary relation on $\mathbb{P}$ which satisfies*

(a) For all $p \in \mathbb{P}$, $p \leq p$.

(b) For all $p, q, r \in \mathbb{P}$, if $p \leq q$ and $q \leq r$, then $q \leq r$.

(c) For all $p \in \mathbb{P}$, $p \leq 1_\mathbb{P}$.

Note that we do not require anti-symmetry $p \leq q \wedge q \leq p \implies p = q$. We sometimes refer to members of $\mathbb{P}$ as conditions, call $1_\mathbb{P}$ "the trivial condition" and read $p \leq q$ by "the condition p extends q" or "p is a stronger condition than q". Some examples of forcing notions follow.

Example 17.2. *(1) $\mathbb{P} = \{f : f$ is a finite function $\wedge$ dom$(f) \subseteq \omega \wedge$ range$(f) \subseteq 2\}$, $p \leq q$ iff $q \subseteq p$ and $1_\mathbb{P} = 0$. We denote this poset by $Fn(\omega, 2)$.*

(2) $\mathbb{P}$ is the set of all finite partial functions from κ to 2, $p \leq q$ iff $q \subseteq p$ and $1_\mathbb{P} = 0$. We denote this poset by $Fn(\kappa, 2)$.

(3) $\mathbb{P}$ is the set of all compact subsets of $[0, 1]$ of positive Lebesgue measure, $p \leq q$ iff $p \setminus q$ is Lebesgue null and $1_\mathbb{P} = [0, 1]$. We call this poset random forcing.

Following standard abuses of notation, we'll sometimes write "$\mathbb{P}$ is a poset" instead of "$(\mathbb{P}, \leq, 1_\mathbb{P})$ is poset" when the ordering and the largest member are clear from the context. Note that "$\mathbb{P}$ is a poset" is a Δ_0-formula and therefore absolute for transitive models of ZF.

Definition 17.3 (Filter on a poset)**.** *Let $\mathbb{P}$ be a poset. A filter on $\mathbb{P}$ is a subset $G \subseteq \mathbb{P}$ satisfying the following.*

(i) $1_{\mathbb{P}} \in G$

(ii) *For every* $p, q \in G$*, there exists* $r \in G$ *such that* $r \leq p$ *and* $r \leq q$*.*

(iii) *For every* $p \in G$ *and* $q \in P$*, if* $p \leq q$*, then* $q \in G$*.*

Definition 17.4 (Compatible, Antichain, Dense)**.** *Let* $\mathbb{P}$ *be a poset.*

(a) *We say that* $p, q \in \mathbb{P}$ *are compatible, iff* p, q *have a common extension in* P*; i.e., there exists* $r \in \mathbb{P}$ *such that* $r \leq p$ *and* $r \leq q$*. We say that* p, q *are incompatible, denoted* $p \perp q$*, iff they are not compatible.*

(b) *A subset* $A \subseteq \mathbb{P}$ *is called an antichain in* $\mathbb{P}$ *iff* A *has pairwise incompatible members.*

(c) *A subset* $D \subseteq \mathbb{P}$ *is dense in* $\mathbb{P}$ *iff for every* $p \in \mathbb{P}$*, there exists* $q \in D$ *such that* $q \leq p$*.*

It is easily checked that all of these notions can be expressed as Δ_0*-formulas and therefore are absolute for transitive models of ZF.*

Lemma 17.5. *Suppose* $\mathbb{P}$ *is poset and* $\mathcal{F}$ *is a countable family of dense subsets of* $\mathbb{P}$*. Then there is a filter* G *on* $\mathbb{P}$ *such that for every* $D \in \mathcal{F}$*,* $G \cap D \neq 0$*.*

Proof. Let $\langle D_n : n < \omega \rangle$ enumerate $\mathcal{F}$. Inductively construct $\langle p_n : n < \omega \rangle$ as follows. $p_0 \in D_0$ is arbitrary. Suppose $p_0 \geq p_1 \geq \cdots \geq p_n$ have been chosen such that for every $k \leq n$, $p_k \in D_k$. Since D_{n+1} is dense in $\mathbb{P}$, we can find $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$.

Define $G = \{q \in \mathbb{P} : (\exists n < \omega)(p_n \leq q)\}$. Then it is easy to check that G is a filter on $\mathbb{P}$ that meets every D_n. □

Definition 17.6 (Generic filter)**.** *Suppose* $\mathbb{P}$ *is a poset and* $\boldsymbol{M}$ *is a ctm of ZFC. A filter* G *on* $\mathbb{P}$ *is called a* $\mathbb{P}$*-generic filter over* $\boldsymbol{M}$ *iff for every dense* $D \subseteq \mathbb{P}$*, if* $D \in \boldsymbol{M}$*, then* $G \cap D \neq 0$*.*

By Lemma 17.5, $\mathbb{P}$-generic filters over $\mathbf{M}$ always exist for every ctm $\mathbf{M}$ of ZFC.

The following is one of the fundamental facts about generic extensions. Although it is not difficult to state, its proof will require the idea of "forcing" which we will introduce later.

Theorem 17.7 (Generic extensions)**.** *Suppose* $\boldsymbol{M}$ *is a ctm of ZFC. Let* $\mathbb{P}$ *be a poset such that* $\mathbb{P}, \leq$ *and* $1_{\mathbb{P}}$ *are in* $\boldsymbol{M}$*. Let* G *be a* $\mathbb{P}$*-generic filter over* $\boldsymbol{M}$*. Then there is a ctm* $\mathbf{N}$ *of ZFC such that* $\boldsymbol{M} \subseteq \mathbf{N}$ *and* $G \in \mathbf{N}$*. Furthermore, for every ctm* $\mathbf{N}'$ *of ZFC, if* $\boldsymbol{M} \subseteq \mathbf{N}'$ *and* $G \in \mathbf{N}'$*, then* $\mathbf{N}' \subseteq \mathbf{N}$*.*

Let $\mathbf{M}, \mathbb{P}, G, \mathbf{N}$ be as in Theorem 17.7. We say that $\mathbf{N}$ is the generic extension of $\mathbf{M}$ obtained by adjoining G to $\mathbf{M}$ and write $\mathbf{M}[G]$ for $\mathbf{N}$. We'll explicitly describe $\mathbf{M}[G]$ in the next section.

Let us discuss the ideas introduced in this section via a concrete poset. Let $\mathbb{P} = Fn(\omega, 2)$. We can think of a condition $p \in \mathbb{P}$ as a finite approximation to some $f : \omega \to 2$. Compatibility of two conditions in $\mathbb{P}$ just means that the conditions agree on their common domain so that their union is a common extension. Every filter $G \subseteq \mathbb{P}$ gives rise to a partial function $f_G = \bigcup G$ from ω to 2. For each $n < \omega$, let $D_n = \{p \in \mathbb{P} : n \in \text{dom}(p)\}$. Then D_n is a dense subset of $\mathbb{P}$. Suppose G is a filter on $\mathbb{P}$ such that for every $n < \omega$, $G \cap D_n \neq 0$. Then $f_G : \omega \to 2$.

Now fix a ctm $\mathbf{M}$ of ZFC. Note that $\mathbb{P} \in \mathbf{M}$ and for every $n < \omega$, $D_n \in \mathbf{M}$. Let G be $\mathbb{P}$-generic over $\mathbf{M}$ (Such filters exist by Lemma 17.5). Since $G \cap D_n \neq 0$ for every $n < \omega$, it follows that $f_G : \omega \to 2$. Observe that $G = \{p \in \mathbb{P} : p \subseteq f_g\}$. So f_G and G can be "computed" from each other. We leave the following as an exercise for the reader: For every $f : \omega \to 2$, if $f \in \mathbf{M}$, then $f \neq f_G$. In particular $\mathbf{M}[G]$ is a proper extension of $\mathbf{M}$. The following says this is always the case if $\mathbb{P}$ is non-trivial.

Lemma 17.8. *Let* $\mathbb{P}$ *be a poset in which each condition has two incompatible extensions. Let* $\boldsymbol{M}$ *be a ctm of ZFC and* $\mathbb{P} \in \boldsymbol{M}$*. Let* G *be* $\mathbb{P}$*-generic over* $\boldsymbol{M}$*. Then* $G \notin \boldsymbol{M}$*.*

Proof. Suppose not. Then $D = \mathbb{P} \setminus G \in \mathbf{M}$. Suppose $p \in \mathbb{P}$. Choose $q, r \leq p$ such that $q \perp r$. Then one of q, r is not in G. So D is dense in $\mathbb{P}$. But $G \cap D = 0$ which is impossible as G meets every dense subset of $\mathbb{P}$ in $\mathbf{M}$. □

18 Generic extensions

Definition 18.1 ($\mathbb{P}$-name)**.** *Let $\mathbb{P}$ be a poset. We say that τ is a $\mathbb{P}$-name iff τ is a relation and for every $(\sigma, p) \in \tau$, $p \in \mathbb{P}$ and σ is a $\mathbb{P}$-name. We denote the class of $\mathbb{P}$-names by $\mathbf{V}^{\mathbb{P}}$.*

Note that this is a definition by transfinite recursion on the rank of τ: To check whether τ is a $\mathbb{P}$-name, it suffices to know if σ is a $\mathbb{P}$-name for all σ such that $\mathsf{rank}(\sigma) < \mathsf{rank}(\tau)$. As an exercie, check that "τ is a $\mathbb{P}$-name" is absolute for transitive models of ZF.

Definition 18.2. *If $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC and $\mathbb{P} \in \boldsymbol{M}$, then $\boldsymbol{M}^{\mathbb{P}}$ denotes the set of all $\mathbb{P}$-names in $\boldsymbol{M}$. By absoluteness, $\boldsymbol{M}^{\mathbb{P}} = V^{\mathbb{P}} \cap \boldsymbol{M}$.*

Definition 18.3 (Evaluating $\mathbb{P}$-names)**.** *Suppose $\mathbb{P}$ is a poset, τ is a $\mathbb{P}$-name and $G \subseteq \mathbb{P}$. Define*

$$\mathit{val}(\tau, G) = \{\mathit{val}(\sigma, G) : (\exists p \in G)((\sigma, p) \in \tau)\}$$

We sometimes also write $\tau[G]$ and $\mathit{val}_G(\tau)$ for $\mathit{val}(\tau, G)$.

Once again $\mathsf{val}(\tau, G)$ is being defined by transfinite recursion on the rank of τ. Also, $\mathsf{val}(\tau, G)$ is absolute for transitive models of ZF.

Definition 18.4. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$ and $G \subseteq \mathbb{P}$. Then*

$$\boldsymbol{M}[G] = \{\mathit{val}(\tau, G) : \tau \in \boldsymbol{M}^{\mathbb{P}}\}$$

Lemma 18.5. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$ and $G \subseteq \mathbb{P}$. Suppose $\mathbf{N}$ is a ctm of ZFC such that $\boldsymbol{M} \subseteq \mathbf{N}$ and $G \in \mathbf{N}$. Then $\boldsymbol{M}[G] \subseteq \mathbf{N}$.*

Proof. Suppose $\tau \in \mathbf{M}^{\mathbb{P}}$, then since $G \in \mathbf{N}$, by the absoluteness of $\mathsf{val}(\tau, G)$, $\mathsf{val}(\tau, G)^{\mathbf{N}} = \mathsf{val}(\tau, G) \in \mathbf{N}$. $\square$

Definition 18.6. *Suppose $\mathbb{P}$ is a poset. By transfinite recursion on the rank of x, define*

$$\check{x} = \{(\check{y}, 1_{\mathbb{P}}) : y \in x\}$$

Lemma 18.7. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$, $G \subseteq \mathbb{P}$ and $1_{\mathbb{P}} \in G$. Then for all $x \in \boldsymbol{M}$, $\check{x} \in \boldsymbol{M}^{\mathbb{P}}$ and $\mathit{val}(\check{x}, G) = x$. Hence $\boldsymbol{M} \subseteq \boldsymbol{M}[G]$.*

Proof. That $(\forall x \in \mathbf{M})(\check{x} \in \mathbf{M})$ follows from the absoluteness of $\check{x}$ for transitive models of ZF. $\mathsf{val}(\check{x}, G) = x$ can be proved by induction on the $\mathsf{rank}(x)$. $\square$

Lemma 18.8. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$, $G \subseteq \mathbb{P}$ and $1_{\mathbb{P}} \in G$. Then $G \in \boldsymbol{M}[G]$.*

Proof. Let $\tau = \{(\check{p}, p) : p \in \mathbb{P}\}$. Then $\tau \in \mathbf{M}$. It is easy to see that $\mathsf{val}(\tau, G) = G$. Hence $G \in \mathbf{M}[G]$. $\square$

We leave the next lemma as an exercise for the reader.

Lemma 18.9. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$, $G \subseteq \mathbb{P}$ and $1_{\mathbb{P}} \in G$. Then the following hold.*

(1) $\boldsymbol{M}[G]$ is transitive and hence it satisfies the axiom of extensionality.

(2) $o(\boldsymbol{M}[G]) = o(\boldsymbol{M})$.

(3) $\boldsymbol{M}[G]$ satisfies the axioms of pairing, union and foundation.

Definition 18.10. *Suppose $\mathbb{P}$ is a poset, $p \in \mathbb{P}$ and $D \subseteq \mathbb{P}$. We say that D is dense below p iff for every $q \leq p$, there exists $r \in D$ such that $r \leq q$.*

The next lemma will be useful in checking that the other axioms of ZFC hold in $\mathbf{M}[G]$.

Lemma 18.11. *Suppose $\mathbb{P}$ is a poset, $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P} \in \boldsymbol{M}$ and G is a $\mathbb{P}$-generic filter over $\boldsymbol{M}$. Let $E \subseteq \mathbb{P}$ and $E \in \boldsymbol{M}$.*

(i) Either $G \cap E \neq 0$ of there exists $q \in G$ such that q is incompatible with every member of E.

(ii) If $p \in G$ and E is dense below p, then $G \cap E \neq 0$.

Proof. (i) Let $D = \{p \in \mathbb{P} : (\exists r \in E)(p \leq r)\} \cup \{q \in \mathbb{P} : (\forall r \in E)(r \perp q)\}$. Note that $D \in \mathbf{M}$ as $E \in \mathbf{M}$. We claim that D is dense in $\mathbb{P}$. To see this, fix $p \in \mathbb{P}$ and we'll find an extension of p in D. If some extension $t \leq p$ is below a condition of E, then this is clear. So assume that this is not the case. Then p must be incompatible with every condition in E. Otherwise choose $r \in E$ and a common extension t of r, p. Then t is an extension of p which is below $r \in E$ which is impossible.

(ii) Let $D = \{q \in \mathbb{P} : (q \perp p) \vee (q \leq p)\}$. Then $D \in \mathbf{M}$ and it is easy to check that D is dense in $\mathbb{P}$. Let $q \in D \cap G$. Then since $p \in G$, $q \perp p$ is impossible. So we must have $q \leq p$. $\square$

19 Truth in generic extension

Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P}$ is a poset in $\mathbf{M}$. Let $\phi(x_1, x_2 \ldots, x_n)$ be a formula in the language of ZFC with free variables $x_1, x_2, \ldots, x_n$. Suppose $\tau_1, \tau_2, \ldots \tau_n$ are $\mathbb{P}$-names in $\mathbf{M}$. Suppose G is a $\mathbb{P}$-generic filter over $\mathbf{M}$ such that $\mathbf{M}[G] \models \phi(\mathsf{val}(\tau_1, G), \mathsf{val}(\tau_2, G), \ldots, \mathsf{val}(\tau_n, G))$. A fundamental fact about forcing (sometimes called "Truth is forced" which we'll show later, see Theorem 19.6(2)) is that there must exist some $p \in G$ such that for every $\mathbb{P}$-generic filter H over $\mathbf{M}$, if $p \in H$, then $\mathbf{M}[H] \models \phi(\mathsf{val}(\tau_1, H), \mathsf{val}(\tau_2, H), \ldots, \mathsf{val}(\tau_n, H))$. This motivates the following definition.

Definition 19.1 (Forcing)**.** *Suppose $\boldsymbol{M}$ is a ctm of ZFC, $\mathbb{P}$ is a poset in $\boldsymbol{M}$. Let $\phi(x_1, x_2 \ldots, x_n)$ be a formula in the language of ZFC with free variables $x_1, x_2, \ldots, x_n$. Suppose $\tau_1, \tau_2, \ldots \tau_n$ are $\mathbb{P}$-names in $\boldsymbol{M}$. Let $p \in \mathbb{P}$. We say that p forces $\phi(\tau_1, \tau_2, \ldots, \tau_n)$ and write $p \Vdash_{\mathbb{P}, \boldsymbol{M}} \phi(\tau_1, \tau_2, \ldots, \tau_n)$ iff for every for every $\mathbb{P}$-generic filter G over $\boldsymbol{M}$, if $p \in G$, then $\boldsymbol{M}[G] \models \phi(\mathit{val}(\tau_1, G), \mathit{val}(\tau_2, G), \ldots, \mathit{val}(\tau_n, G))$.*

When $\mathbf{M}, \mathbb{P}$ are clear from the context, we drop them from $\Vdash_{\mathbb{P}, \mathbf{M}}$ and just write $\Vdash$. The following is easy to verify.

Lemma 19.2. *The following hold.*

(a) If $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ and $q \leq p$, then $q \Vdash \phi(\tau_1, \ldots, \tau_n)$.

(b) $p \Vdash (\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\sigma_1, \ldots, \sigma_m))$ iff $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ and $p \Vdash \psi(\sigma_1, \ldots, \sigma_m)$.

Another fundamental fact about forcing (sometimes called "Forcing is definable" see Theorem 19.6(1)) is that the relation $p \Vdash \phi(\tau_1, \tau_2, \ldots, \tau_n)$ is definable in $\mathbf{M}$. The aim of this section is to establish these facts and then use them to show that all the axioms of ZFC hold in $\mathbf{M}[G]$.

To show that $\{(p, \tau_1, \ldots, \tau_n) : p \in \mathbb{P} \wedge \tau_1, \ldots \tau_n \in \mathbf{M} \wedge p \Vdash_{\mathbb{P}, \mathbf{M}} \phi(\tau_1, \ldots, \tau_n)\}$ is definable in $\mathbf{M}$, we'll introduce another relation $p \Vdash^\star \phi(\tau_1, \ldots, \tau_n)$ and show that, $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ iff $(p \Vdash^\star \phi(\tau_1, \ldots, \tau_n))^{\mathbf{M}}$. The definition of $p \Vdash^\star \phi(\tau_1, \ldots, \tau_n)$ is by induction on the complexity of ϕ.

Definition 19.3 (Star-forcing)**.** *Let $\mathbb{P}$ be a poset and $p \in \mathbb{P}$. For a formula $\phi(x_1, \ldots, x_n)$ with free variable as shown and $\tau_1, \ldots, \tau_n \in \mathbf{V}^{\mathbb{P}}$, define $p \Vdash^\star \phi(\tau_1, \ldots, \tau_n)$ by recursion on length of ϕ as follows.*

(a) $p \Vdash^\star \tau_1 = \tau_2$ iff for every $(\pi_1, s_1) \in \tau_1$, the set

$$\{q \leq p : q \leq s_1 \implies (\exists(\pi_2, s_2) \in \tau_2)(q \leq s_2 \wedge q \Vdash^\star \pi_1 = \pi_2)\}$$

is dense below p and for every $(\pi_2, s_2) \in \tau_2$, the set

$$\{q \leq p : q \leq s_2 \implies (\exists(\pi_1, s_1) \in \tau_1)(q \leq s_1 \wedge q \Vdash^\star \pi_1 = \pi_2)\}$$

is dense below p.

(b) $p \Vdash^\star \tau_1 \in \tau_2$ iff

$$\{q : (\exists(\pi, s) \in \tau_2)(q \leq s \wedge q \Vdash^\star \pi = \tau_1)\}$$

is dense below p.

(c) $p \Vdash^{\star} (\phi(\tau_1, \dots, \tau_n) \wedge \psi(\sigma_1, \dots, \sigma_m))$ iff $p \Vdash^{\star} \phi(\tau_1, \dots, \tau_n)$ and $p \Vdash^{\star} \psi(\sigma_1, \dots, \sigma_m)$.

(d) $p \Vdash^{\star} \neg\phi(\tau_1, \dots, \tau_n)$ iff there is no $q \leq p$ such that $q \Vdash^{\star} \phi(\tau_1, \dots, \tau_n)$.

(e) $p \Vdash^{\star} (\exists x)(\phi(x, \tau_1, \dots, \tau_n))$ iff the set

$$\{r : (\exists \sigma \in \mathbf{V}^{\mathbb{P}})(r \Vdash^{\star} \phi(\sigma, \tau_1, \dots, \tau_n))$$

is dense below p.

Note that the definition in Clause (a) is by recursion on $\max(\mathsf{rank}(\tau_1), \mathsf{rank}(\tau_2))$. The following lemma is easy to verify using Definition 19.3.

Lemma 19.4. *The following are equivalent.*

(1) $p \Vdash^{\star} \phi(\tau_1, \dots, \tau_n)$.

(2) $(\forall q \leq p)(q \Vdash^{\star} \phi(\tau_1, \dots, \tau_n))$.

(3) $\{q : q \Vdash^{\star} \phi(\tau_1, \dots, \tau_n)\}$ *is dense below* p.

Theorem 19.5. *Suppose* $\boldsymbol{M}$ *is a ctm of ZFC,* $\mathbb{P}$ *is a poset in* $\boldsymbol{M}$*. Let* $\phi(x_1, x_2 \dots, x_n)$ *be a formula in the language of ZFC with free variables* $x_1, x_2, \dots, x_n$*. Suppose* $\tau_1, \tau_2, \dots \tau_n$ *are* $\mathbb{P}$*-names in* $\boldsymbol{M}$*. Let* G *be a* $\mathbb{P}$*-generic filter over* $\boldsymbol{M}$*. Then the following hold.*

(1) *If* $p \in G$ *and* $(p \Vdash^{\star} \phi(\tau_1, \dots, \tau_n))^{\boldsymbol{M}}$*, then* $\boldsymbol{M}[G] \models \phi(\mathsf{val}(\tau_1, G), \dots, \mathsf{val}(\tau_n, G))$.

(2) *If* $\boldsymbol{M}[G] \models \phi(\mathsf{val}(\tau_1, G), \dots, \mathsf{val}(\tau_n, G))$*, then there exists* $p \in G$ *such that* $(p \Vdash^{\star} \phi(\tau_1, \dots, \tau_n))^{\boldsymbol{M}}$.

Proof. See Theorem 3.5 in Chapter VII of Kunen's book. The proof will be covered in the lecture. $\square$

Theorem 19.6. *Suppose* $\boldsymbol{M}$ *is a ctm of ZFC,* $\mathbb{P}$ *is a poset in* $\boldsymbol{M}$*. Let* $\phi(x_1, x_2 \dots, x_n)$ *be a formula in the language of ZFC with free variables* $x_1, x_2, \dots, x_n$*. Suppose* $\tau_1, \tau_2, \dots \tau_n$ *are* $\mathbb{P}$*-names in* $\boldsymbol{M}$*.*

(1) *For all* $p \in \mathbb{P}$,

$$p \Vdash_{\mathbb{P}, \boldsymbol{M}} \phi(\tau_1, \dots, \tau_n) \iff (p \Vdash^{\star} \phi(\tau_1, \dots, \tau_n))^{\boldsymbol{M}}$$

(2) *For every* $\mathbb{P}$*-generic filter* G *over* $\boldsymbol{M}$*,* $\boldsymbol{M}[G] \models \phi(\tau_1[G], \dots \tau_n[G])$ *iff* $(\exists p \in G)(p \Vdash_{\mathbb{P}, \boldsymbol{M}} \phi(\tau_1, \dots, \tau_n))$.

Proof. (1) The right to left implication follows from Theorem 19.5(1). For the other direction, assume $p \Vdash \phi(\tau_1, \dots, \tau_n)$. By Lemma 19.4, it suffices to show that $D = \{q \leq p : (q \Vdash^{\star} \phi(\tau_1, \dots, \tau_n))^{\mathbf{M}}\}$ is dense below p. Suppose not and let $q \leq p$ be such that $(\forall r \leq q)(r \notin D)$. Then $(q \Vdash^{\star} \neg\phi(\tau_1, \dots, \tau_n))^{\mathbf{M}}$. By the right to left implication of (1), $q \Vdash \phi(\tau_1, \dots, \tau_n)$ which is impossible since $p \Vdash \phi(\tau_1, \dots, \tau_n)$ and $q \leq p$.

(2) The right to left implication is immediate from the definition of $\Vdash$. The left to right implication follows from Theorem 19.5(2) and Clause (1) above. $\square$

Corollary 19.7. *Suppose* $\boldsymbol{M}$ *is a ctm of ZFC,* $\mathbb{P}$ *is a poset in* $\boldsymbol{M}$ *and* $\sigma, \tau_1, \dots, \tau_n \in \boldsymbol{M}^{\mathbb{P}}$.

(a) $\{p \in \mathbb{P} : p \Vdash \phi(\tau_1, \dots, \tau_n)\} \cup \{q \in \mathbb{P} : q \Vdash \neg\phi(\tau_1, \dots, \tau_n)\}$ *is dense in* $\mathbb{P}$.

(b) $p \Vdash \neg\phi(\tau_1, \dots, \tau_n)$ *iff there does not exist* $q \leq p$ *such that* $q \Vdash \phi(\tau_1, \dots, \tau_n)$.

(c) $p \Vdash (\exists x)(\psi(x, \tau_1, \dots, \tau_n))$ *iff* $\{r \leq p : (\exists \pi \in \boldsymbol{M}^{\mathbb{P}})(r \Vdash \psi(\pi, \tau_1, \dots, \tau_n))\}$ *is dense below* p.

(d) *If* $p \Vdash (\exists x)(x \in \sigma \wedge \psi(x, \tau_1, \dots, \tau_n))$*, then there exists* $q \leq p$ *and* $\pi \in dom(\sigma)$ *such that* $q \Vdash \psi(\pi, \tau_1, \dots, \tau_n)$.

Proof. (a), (b) and (c) follow from the definition of $\Vdash^{\star}$ and Theorem 19.6. To see (d), fix a a $\mathbb{P}$-generic filter G over $\mathbf{M}$ with $p \in G$. Choose $(\pi, s) \in \sigma$ such that $s \in G$, and $\mathbf{M}[G] \models \psi(\pi[G], \tau_1[G], \dots, \tau_n[G])$. By Theorem 19.6(2), we can choose $r \in G$ such that $r \Vdash \psi(\pi, \tau_1, \dots, \tau_n)$. Let $q \in G$ be a common extension of p, s in G. Then q, π are as required. $\square$

20 ZFC in $\mathbf{M}[G]$

Theorem 20.1. *Suppose $\boldsymbol{M}$ is a ctm of ZFC, $(\mathbb{P}, \leq, 1_{\mathbb{P}})$ is a poset in $\boldsymbol{M}$ and G is a $\mathbb{P}$-generic filter over $\boldsymbol{M}$. Then $\boldsymbol{M}[G] \models$ ZFC.*

Proof. That the axioms of Extensionaltiy, Pairing, Union and Foundation hold in $\mathbf{M}[G]$ follows from Lemma 18.9. Let us check the other axioms.

(a) Comprehension: Suppose $\phi(x, v, y_1, \ldots, y_n)$ is a formula and $\sigma, \tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$. We must show that

$$Y = \{a \in \sigma[G] : \phi^{\mathbf{M}[G]}(a, \sigma[G], \tau_1[G], \ldots, \tau_n[G])\} \in \mathbf{M}[G]$$

Let

$$\rho = \{(\pi, p) \in \mathrm{dom}(\sigma) \times \mathbb{P} : p \Vdash (\pi \in \sigma \wedge \phi(\pi, \sigma, \tau_1, \ldots, \tau_n))\}$$

Then $\rho \in \mathbf{M}^{\mathbb{P}}$ by definability of forcing (Theorem 19.6(1)). We claim that $\rho[G] = Y$. That $\rho[G] \subseteq Y$ is easily verified. Next, suppose $a \in Y$. Then $a = \pi[G]$ for some $\pi \in \mathrm{dom}(\sigma)$ and $\mathbf{M}[G] \models \pi[G] \in \sigma[G] \wedge \phi(\pi[G], \sigma[G], \tau_1[G], \ldots, \tau_n[G])$. Since truth in $\mathbf{M}[G]$ is forced by some condition in G (Theorem 19.6(2)), for some $p \in G$, $p \Vdash \pi \in \sigma \wedge \phi(\pi, \sigma, \tau_1, \ldots, \tau_n)$. It follows that $(\pi, p) \in \rho$. Hence $a = \pi[G] \in \rho[G]$.

(b) Replacement: Let $\phi(x, y, A, z_1, \ldots, z_n)$ be a formula, $\sigma, \tau_1, \ldots, \tau_n \in \mathbf{M}^{\mathbb{P}}$ and suppose

$$\mathbf{M}[G] \models (\forall x \in \sigma[G])(\exists! y)(\phi(x, y, \sigma[G], \tau_1[G], \ldots, \tau_n[G]))$$

Since replacement holds in $\mathbf{M}$ and forcing is definable in $\mathbf{M}$, there exists $S \in \mathbf{M}$ such that $S \subseteq \mathbf{M}^{\mathbb{P}}$ and for every $\pi \in \mathrm{dom}(\sigma)$ and $p \in \mathbb{P}$,

$$(\exists \mu \in \mathbf{M}^{\mathbb{P}})(p \Vdash \phi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n)) \implies (\exists \mu \in S)(p \Vdash \phi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))$$

Let $\rho = S \times \{1_{\mathbb{P}}\}$. Then $\rho \in \mathbf{M}^{\mathbb{P}}$. It is easy to check that for every $x \in \sigma[G]$, there exists $y \in \rho[G]$ such that $\mathbf{M}[G] \models \phi(x, y, \sigma[G], \tau_1, \ldots, \tau_n[G])$.

(c) Infinity: Since $\omega \in \mathbf{M} \subseteq \mathbf{M}[G]$ and $\mathbf{M}[G]$ satisfies BST, by Lemma 12.9, the axiom of infinity holds in $\mathbf{M}[G]$.

(d) Power Set: Let $\sigma \in \mathbf{M}^{\mathbb{P}}$. We'll find $\rho \in \mathbf{M}^{\mathbb{P}}$ such that $(\forall x \in \mathbf{M}[G])(x \subseteq \sigma[G] \implies x \in \rho[G])$. Put $\rho = S \times \{1_{\mathbb{P}}\}$ where

$$S = \{\tau \in \mathbf{M}^{\mathbb{P}} : \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\sigma)\}$$

Note that S and ρ are in $\mathbf{M}$. Let $\mu \in \mathbf{M}^{\mathbb{P}}$ such that $\mu[G] \subseteq \sigma[G]$. Let $\tau = \{(\pi, p) : \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \mu\}$. Then $\tau \in S$ and so $\tau[G] \in \rho[G]$. So it suffices to show that $\tau[G] = \mu[G]$. That $\tau[G] \subseteq \mu[G]$ is clear. For the converse, suppose $\pi \in \mathrm{dom}(\mu)$ and $\pi[G] \in \mu[G]$. As $\mu[G] \subseteq \sigma[G]$, we can choose $\pi' \in \mathrm{dom}(\sigma)$ such that $\pi[G] = \pi'[G]$. Choose $p \in G$ such that $p \Vdash \pi' \in \mu$. Then $(\pi', p) \in \tau$. Hence $\pi[G] = \pi'[G] \in \tau[G]$.

(e) Choice: It suffices to show that

$$\mathbf{M}[G] \models (\forall X)(\exists \gamma)(\exists f)(f \text{ is a function } \wedge \mathrm{dom}(f) = \gamma \wedge X \subseteq \mathrm{range}(f))$$

for then we can well order X as follows: $x \prec y$ iff $\min(f^{-1}[\{x\}]) < \min(f^{-1}[\{y\}])$. Let $\sigma \in \mathbf{M}^{\mathbb{P}}$. Since $\mathbf{M}$ satisfies AC, we can well order $\mathrm{dom}(\sigma) = \langle \pi_\alpha : \alpha < \gamma \rangle$. Let $\tau = \{op(\check{\alpha}, \pi_\alpha) : \alpha < \gamma\} \times \{1_{\mathbb{P}}\}$ where $op(\sigma, \tau)[G] = (\sigma[G], \tau[G])$. Then $\tau \in \mathbf{M}^{\mathbb{P}}$ and $\tau[G]$ is a function with $\mathrm{dom}(\tau[G]) = \gamma$ and $X \subseteq \mathrm{range}(\tau[G])$.

□

Corollary 20.2. *If ZFC is consistent, then ZFC $+\mathbf{V} \neq \mathbf{L}$ is consistent.*

Proof. Let $\mathbf{M}$ be a ctm of ZFC, $\mathbb{P} = Fn(\omega, 2) \in \mathbf{M}$ and suppose G is $\mathbb{P}$-generic over $\mathbf{M}$. Put $\mathbf{N} = \mathbf{M}[G]$. By Lemma 17.8, $\mathbf{M} \neq \mathbf{N}$. Towards a contradiction, suppose $\mathbf{N} \models \mathbf{V} = \mathbf{L}$. Then by Corollary 15.8, $\mathbf{N} = L_{o(\mathbf{N})} = L_{o(\mathbf{M})} = \mathbf{L}^{\mathbf{M}} \subseteq \mathbf{M}$ which is impossible. □

21 Consistency of ¬ CH

Definition 21.1. *$Fn(X,Y)$ is the poset whose conditions are finite partial functions from X to Y ordered by $p \leq q$ iff $q \subseteq p$.*

Observe that if $\mathbf{M}$ is a ctm of ZFC and $X, Y \in \mathbf{M}$, then $\mathsf{Fn}(X,Y) \in \mathbf{M}$.

Lemma 21.2. *Let G be $\mathsf{Fn}(X,Y)$-generic filter over $\boldsymbol{M}$. Define $f_G = \bigcup G$. Then $f_G : X \to Y$ and if X is infinite, then range$(f_G) = Y$.*

Proof. Since G is a filter, for any two functions $f, g \in G$, there is a function $h \in G$ that extends both f, g. It follows that $f_G = \bigcup G$ is a function. Next, note that for every $x \in X$, $D_x = \{p \in \mathsf{Fn}(X,Y) : x \in \mathrm{dom}(p)\}$ is dense in $\mathbb{P}$ and, by comprehension in $\mathbf{M}$, $D_x \in \mathbf{M}$. Since X is infinite, for every $y \in Y$, $E_y = \{p \in \mathsf{Fn}(X,Y) : y \in \mathrm{range}(p)\}$ is a dense subset of $\mathbb{P}$ in $\mathbf{M}$. It follows that $G \cap D_x \neq 0$ and $G \cap E_y \neq 0$ for every $x \in X$ and $y \in Y$. Hence $\mathrm{dom}(f_G) = X$ and $\mathrm{range}(f_g) = Y$. □

Let $(\kappa = \omega_2)^{\mathbf{M}}$ (So κ is really a countable ordinal but $\mathbf{M}$ thinks that it's the second uncountable cardinal). Let G be $\mathsf{Fn}(\kappa \times \omega, 2)$-generic filter over $\mathbf{M}$. We'll show that $\mathbf{M}[G] \models \neg\mathrm{CH}$. Note that $F_G = \bigcup G : \kappa \times \omega \to 2$. For each $\alpha < \kappa$, let $f_\alpha : \omega \to 2$ be defined by $f_\alpha(n) = F_G(\alpha, n)$. Then it is easy to check that $\langle f_\alpha : \alpha < \kappa \rangle$ has pairwise distinct functions from ω to 2 (Consider, for $\alpha < \beta < \kappa$, the dense sets $D_{\alpha,\beta} = \{p \in \mathsf{Fn}(\kappa \times \omega, 2) : (\exists n < \omega)(p(\alpha, n) \neq p(\beta, n))\}$). So we'll be done **if** we can show that $(\kappa = \omega_2)^{\mathbf{M}[G]}$. Note that this is not true for all forcings since, for example, if H is $\mathsf{Fn}(\omega, \kappa)$-generic over $\mathbf{M}$, then by Lemma 21.2, $(|\kappa| = \omega)^{\mathbf{M}[H]}$. The key combinatorial idea behind the proof of $(\kappa = \omega_2)^{\mathbf{M}[G]}$ is the fact that $\mathsf{Fn}(X, \omega)$ satisfies the countable chain condition.

Definition 21.3 (Chain conditions)**.** *Suppose $\mathbb{P}$ is a poset and κ is a cardinal. We say that $\mathbb{P}$ satisfies the κ-chain condition (abbreviated κ-c.c.) iff for every antichain $A \subseteq \mathbb{P}$, $|A| < \kappa$. $\mathbb{P}$ satisfies the countable chain condition (abbreviated c.c.c.) iff $\mathbb{P}$ satisfies the ω_1-c.c.*

Fact 21.4 (Δ-system lemma)**.** *Suppose κ is an uncountable cardinal and $\langle A_i : i < \kappa \rangle$ is a sequence of finite sets. Then there exist $X \in [\kappa]^\kappa$ and R such that $(\forall i, j \in X)(i \neq j \implies A_i \cap A_j = R)$.*

Proof. By induction on n. □

The following is an easy consequence of Fact 21.4.

Lemma 21.5. *$Fn(X,Y)$ satisfies the c.c.c. iff Y is countable.*

Lemma 21.6. *Suppose $\mathbb{P}$ is a poset in $\boldsymbol{M}$ and ($\mathbb{P}$ satisfies c.c.c.)$^{\boldsymbol{M}}$. Let $A, B \in \boldsymbol{M}$. Let G be $\mathbb{P}$-generic filter over $\boldsymbol{M}$ and suppose $f \in \boldsymbol{M}[G]$ and $f : A \to B$. Then there is an $F : A \to \mathcal{P}(B)$ such that $F \in \boldsymbol{M}$ and for every $a \in A$, $f(a) \in F(a)$ and $(|F(a)| \leq \omega)^{\boldsymbol{M}}$.*

Proof. Fix $\tau \in \mathbf{M}^{\mathbb{P}}$ such that $\tau[G] = f$. Since $\mathbf{M}[G] \models \tau[G] : A \to B$, we can find $p \in \mathbb{P}$ such that $p \Vdash \tau : \check{A} \to \check{B}$.

For each $a \in A$, let $F(a) = \{b \in B : (\exists q \leq p)(q \Vdash \tau(\check{a} = \check{b}))\}$. By the definability of forcing in $\mathbf{M}$, $F \in \mathbf{M}$. Note that if $b = f(a) = \tau[G](a)$, then for some $q \leq p$, $q \in G$ and $q \Vdash \tau(\check{a}) = \check{b}$. So $b = f(a) \in F(a)$ for every $a \in A$.

Now fix $a \in A$ and we'll show that $(|F(a)| \leq \omega)^{\mathbf{M}}$. Using AC in $\mathbf{M}$, choose a function $b \mapsto q_b$ such that for every $b \in F(a)$, $q_b \leq p$ and $q_b \Vdash \tau(\check{a}) = \check{b}$. Then $\{q_b : b \in F(a)\} \in \mathbf{M}$ is an antichain in $\mathbb{P}$ since if $b \neq c$, then q_b, q_c force contradictory statements. So it is countable in $\mathbf{M}$. It follows that $F(a)$ is also countable in $\mathbf{M}$. □

Definition 21.7 (Cardinal preservation)**.** *Suppose $\mathbb{P}$ is a poset in $\boldsymbol{M}$. We say that $\mathbb{P}$ preserves cardinals iff for every $\mathbb{P}$-generic filter G over $\boldsymbol{M}$,*

$$(\forall \alpha < o(\boldsymbol{M})) \left[(\alpha \text{ is a cardinal})^{\boldsymbol{M}} \iff (\alpha \text{ is a cardinal})^{\boldsymbol{M}[G]} \right]$$

Corollary 21.8. *Suppose $\mathbb{P}$ is a poset in $\boldsymbol{M}$ and ($\mathbb{P}$ satisfies c.c.c.)$^{\boldsymbol{M}}$. Let (κ is a cardinal)$^{\boldsymbol{M}}$. Let G be $\mathbb{P}$-generic over $\boldsymbol{M}$. Then (κ is a cardinal)$^{\boldsymbol{M}[G]}$.*

Proof. We can assume that $(\kappa \geq \omega_1)^{\mathbf{M}}$. Towards a contradiction, suppose $\kappa \in \mathbf{M}$ such that $(\kappa \text{ is a cardinal})^{\mathbf{M}}$ and $(\kappa \text{ is not a cardinal})^{\mathbf{M}[G]}$. Then for some $\alpha < \kappa$, $(|\kappa| = \alpha)^{\mathbf{M}[G]}$. Hence there is a function $f : \alpha \to \kappa$ such that $f \in \mathbf{M}[G]$ and $\text{range}(f) = \kappa$. By Lemma 21.6, choose $F : \alpha \to \mathcal{P}(\kappa)$ such that $F \in \mathbf{M}$ and for every $\beta < \alpha$, $(|F(\beta)| \leq \omega)^{\mathbf{M}}$. Let $W = \bigcup \text{range}(F)$. Then $W \in \mathbf{M}$ and $(|W| \leq |\alpha \times \omega| = |\alpha|)^{\mathbf{M}}$. But $\kappa = \text{range}(f) \subseteq W$. So $(|\kappa| \leq |\alpha|)^{\mathbf{M}}$ which contradicts the fact that κ is a cardinal in $\mathbf{M}$. $\square$

Corollary 21.9. *Let $\mathbf{M}$ be a ctm of ZFC, $(\kappa = \omega_2)^{\mathbf{M}}$ and $\mathbb{P} = \mathsf{Fn}(\kappa \times \omega, 2)$. Let G be $\mathbb{P}$-generic over $\mathbf{M}$. Then $(2^\omega \geq \omega_2)^{\mathbf{M}[G]}$.*

Proof. Since $(\mathbb{P} \text{ satisfies c.c.c.})^{\mathbf{M}}$, $\mathbb{P}$ preserves cardinals. So $(\omega_1)^{\mathbf{M}} = (\omega_1)^{\mathbf{M}[G]}$ and $\kappa = (\omega_2)^{\mathbf{M}} = (\omega_2)^{\mathbf{M}[G]}$. As noted earlier in this section, $(2^\omega \geq \kappa)^{\mathbf{M}[G]}$. So $(2^\omega \geq \omega_2)^{\mathbf{M}[G]}$. $\square$

22 Countably closed forcing and $\diamondsuit$

Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P} \in \mathbf{M}$ is a poset, G is $\mathbb{P}$-generic over $\mathbf{M}$ and $X \in \mathbf{M}[G]$. Then for some $\tau \in \mathbf{M}^{\mathbb{P}}$, $X = \tau[G]$. From now on, we'll use $\mathring{X}$ to denote such a $\mathbb{P}$-name (in $\mathbf{M}$) for $X \in \mathbf{M}[G]$. So $\mathring{X}[G] = X$. We'll sometimes also drop the superscript from $\check{A}$ when it is clear that $A \in \mathbf{M}$.

Definition 22.1 (Countably closed forcing)**.** *Let $\mathbb{P}$ be a poset. We say that $\mathbb{P}$ is countably closed iff for every sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{P}$, if $(\forall n < \omega)(p_{n+1} \leq p_n)$, then there exists $p \in \mathbb{P}$ such that $(\forall n < \omega)(p \leq p_n)$.*

Lemma 22.2. *Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P} \in \mathbf{M}$ is a poset and $(\mathbb{P} \text{ is countably closed})^{\mathbf{M}}$. Let G be $\mathbb{P}$-generic over $\mathbf{M}$. Suppose $X \in \mathbf{M}$. Then ${}^{\omega}X \cap \mathbf{M} = {}^{\omega}X \cap \mathbf{M}[G]$.*

Proof. It suffices to show that if $f : \omega \to X$ and $f \in \mathbf{M}[G]$, then $f \in \mathbf{M}$. Let $F = {}^{\omega}X \cap \mathbf{M}$. Towards a contradiction, assume $f \notin F$. Choose $p \in G$ such that $p \Vdash \mathring{f} : \omega \to \check{X} \wedge \mathring{f} \notin \check{F}$. Working in $\mathbf{M}$, construct $\langle (p_n, x_n) : n < \omega \rangle$ such that the following hold.

(i) $p_0 \leq p$, $x_0 \in X$ and $p_0 \Vdash \mathring{f}(0) = \check{x}_0$

(ii) $p_{n+1} \leq p_n$, $x_{n+1} \in X$ and $p_{n+1} \leq p_n$ and $p_{n+1} \Vdash \mathring{f}(n+1) = \check{x}_{n+1}$

Since $(\mathbb{P} \text{ is countably closed})^{\mathbf{M}}$, we can choose $q \in \mathbb{P}$ such that $(\forall n < \omega)(q \leq p_n)$. Let $g : \omega \to X$ be defined by $g(n) = x_n$. Then $g \in \mathbf{M}$ and $q \Vdash (\forall n < \omega)(\mathring{f}(n) = \check{g}(n))$. It follows that $q \Vdash \mathring{f} = \check{g} \in \check{F}$ which is a contradiction since $q \leq p$ while $p \Vdash \mathring{f} \notin \check{F}$. $\square$

Corollary 22.3. *Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P} \in \mathbf{M}$ is a poset and $(\mathbb{P} \text{ is countably closed})^{\mathbf{M}}$. Let G be $\mathbb{P}$-generic over $\mathbf{M}$. Then $(\omega_1)^{\mathbf{M}} = (\omega_1)^{\mathbf{M}[G]}$.*

Proof. Let $(\kappa = \omega_1)^{\mathbf{M}}$ and suppose $(\kappa < \omega_1)^{\mathbf{M}[G]}$. Then κ is countable in $\mathbf{M}[G]$. It follows that there exists $f : \omega \to \kappa$ such that $f \in \mathbf{M}[G]$ and $\text{range}(f) = \kappa$. Since $(\mathbb{P} \text{ is countably closed})^{\mathbf{M}}$, by Lemma 22.2, $f \in \mathbf{M}$. But then κ is countable in $\mathbf{M}$ which is a contradiction. $\square$

Definition 22.4. *Let $\mathbb{S}$ be the poset defined as follows. $p \in \mathbb{S}$ iff p is a function, dom$(p) < \omega_1$ and for every $\xi \in$ dom(p), $p(\xi) \subseteq \xi$. For $p, q \in \mathbb{S}$, define $p \leq q$ iff $q \subseteq p$.*

Note that $\mathbb{S}$ is a countably closed poset. Fix a ctm $\mathbf{M}$ of ZFC. Put $(\kappa = \omega_1)^{\mathbf{M}}$. Consider the relativization $\mathbb{P}$ of $\mathbb{S}$ to $\mathbf{M}$, i.e., $\mathbb{P} = \mathbb{S}^{\mathbf{M}}$. So $\mathbb{P} \in \mathbf{M}$ consists of all functions $p \in \mathbf{M}$ such that $\text{dom}(p) < \kappa$ and for every $\xi < \text{dom}(p)$, $p(\xi) \subseteq \xi$. Clearly, $(\mathbb{P} \text{ is countably closed})^{\mathbf{M}}$. Let G be $\mathbb{P}$-generic over $\mathbf{M}$. By Corollary 22.3, $(\kappa = \omega_1)^{\mathbf{M}[G]}$.

Now, by an easy density argument, $\bigcup G = f$ is a function with domain κ such that for every $\xi < \kappa$, $f(\xi) \subseteq \xi$. Put $A_\xi = f(\xi)$. We'll show that $\langle A_\xi : \xi < \kappa \rangle$ witnesses that $\diamondsuit$ holds in $\mathbf{M}[G]$.

Suppose $A \in \mathbf{M}[G]$ and $A \subseteq \kappa$. Put $W = \{\xi < \kappa : A_\xi = A \cap \xi\}$. We must show that $(W \text{ is stationary})^{\mathbf{M}[G]}$. Towards a contradiction, suppose this fails and fix $C \subseteq \kappa$ such that $(C \text{ is a club in } \kappa)^{\mathbf{M}[G]}$ and $C \cap W = 0$. Let $h : \kappa \to 2$ be the characteristic function of A: So $(\forall \xi < \kappa)(\xi \in A \iff h(\xi) = 1)$. Note that since

($\mathbb{P}$ is countably closed)$^{\mathbf{M}}$, by Lemma 22.2, for each $\xi < \kappa$, $h \restriction \xi \in \mathbf{M}$. Choose $p \in G$ that forces all of this: $\mathring{h}$ is the characteristic function of $\mathring{A}$, $(\forall \xi < \kappa)(\mathring{h} \restriction \xi \in \mathbf{M})$, $\mathring{C}$ is a club in κ and $(\forall \xi \in \mathring{C})(\mathring{A}_\xi \neq \mathring{A} \cap \xi)$. We'll find $p_\star \leq p$ and $\xi_\star < \kappa$ such that $p_\star \Vdash (\xi_\star \in \mathring{C}) \wedge (\mathring{A}_{\xi_\star} = \mathring{A} \cap \xi_\star)$ which is a contradiction.

Working in $\mathbf{M}$, recursively construct $\langle (p_n, \xi_n, g_n) : n < \omega \rangle$ as follows.

(a) $\xi_0 = \omega$, $p_0 \leq p$, $g_0 : \omega \to 2$ and $p_0 \Vdash \check{g}_0 = \mathring{h} \restriction \omega$.

(b) Suppose p_n, ξ_n, g_n have been defined. Choose $\xi_{n+1} < \kappa$, $p_{n+1} \leq p_n$ and $g_n : \xi_{n+1} \to 2$ such that

   (i) $\xi_{n+1} > \max(\xi_n, \mathrm{dom}(p_n))$,

   (ii) $p_{n+1} \Vdash \xi_{n+1} \in \mathring{C}$,

   (iii) $p_{n+1} \Vdash \mathring{h} \restriction \xi_{n+1} = g_{n+1}$ and

   (iv) $\xi_n \in \mathrm{dom}(p_{n+1})$.

   Note that such p_{n+1}, g_n exist because $p \Vdash (\forall \xi < \kappa)(\mathring{h} \restriction \xi \in \mathbf{M})$.

Having completed the construction, put $\xi_\star = \sup(\{\xi_n : n < \omega\})$ and $q = \bigcup\{p_n : n < \omega\}$. Note that $q \in \mathbb{P}$, $\xi_\star$ is a countable limit ordinal in $\mathbf{M}$ and $\mathrm{dom}(q) = \xi_\star$. Since $q \Vdash (\forall n < \omega)(\mathring{h} \restriction \xi_n = g_n)$, it follows that $g_\star = \bigcup\{g_n : n < \omega\}$ is a function from $\xi_\star$ to 2. Since $q \Vdash (\forall n < \omega)(\xi_n \in \mathring{C})$ and $q \Vdash \mathring{C}$ is a club in κ, we get $q \Vdash \xi_\star \in \mathring{C}$. Let $A_\star = \{\xi < \xi_\star : g_\star(\xi) = 1\}$. Define $p_\star = q \cup \{(\xi_\star, A_\star)\}$. Then $p_\star \leq p$ and $p_\star \Vdash (\xi_\star \in \mathring{C}) \wedge (\mathring{A}_{\xi_\star} = A_\star = \mathring{A} \cap \xi_\star)$ which gives us the desired contradiction.

23 Martin's axiom and Suslin's hypothesis

Definition 23.1. $\mathfrak{c} = 2^\omega = |\mathcal{P}(\omega)|$ *denotes the cardinality of the continuum.*

Definition 23.2 (Martin's axiom at κ)**.** *For an infinite cardinal κ, MA_κ is the following statement: For every poset $\mathbb{P}$ which satisfies c.c.c., for every family $\mathcal{F}$ of dense subsets of $\mathbb{P}$, if $|\mathcal{F}| \leq \kappa$, then there is a filter G on $\mathbb{P}$ such that for every $D \in \mathcal{F}$, $G \cap D \neq 0$.*

Note that if $\kappa < \lambda$, then $\mathsf{MA}_\lambda \implies \mathsf{MA}_\kappa$. The following is a useful fact when applying MA_κ.

Lemma 23.3. *Assume MA_κ. Suppose $\mathbb{P}$ is a c.c.c. poset and let $\mathcal{F}$ be a family of dense subsets of $\mathbb{P}$ such that $|\mathcal{F}| \leq \kappa$. Then for every $p \in \mathbb{P}$, there exists a filter G on $\mathbb{P}$ such that $p \in G$ and $(\forall D \in \mathcal{F})(G \cap D \neq 0)$.*

Proof. Let $\mathbb{Q} = \{q \in \mathbb{P} : q \leq p\}$. Then $\mathbb{Q}$ is a c.c.c. subposet of $\mathbb{P}$. For each $D \in \mathcal{F}$, let $D' = D \cap \mathbb{Q}$. Then $\mathcal{F}' = \{D' : D \in \mathcal{F}\}$ is a family of dense subsets of $\mathbb{Q}$. Now apply MA_κ to $\mathbb{Q}$ and $\mathcal{F}'$. □

Lemma 23.4. *Let κ be the least infinite cardinal for which MA_κ fails. Then $\omega_1 \leq \kappa \leq \mathfrak{c}$.*

Proof. By Lemma 17.5, MA_ω holds. So $\kappa \geq \omega_1$. To show that $\kappa \leq \mathfrak{c}$, it is enough to show that $\mathsf{MA}_\mathfrak{c}$ fails.

Let $\mathbb{P} = \mathsf{Fn}(\omega, 2)$. $\mathbb{P}$ is countable and hence satisfies c.c.c. For each $f : \omega \to 2$, let $D_f = \{p \in \mathbb{P} : (\exists n \in \mathrm{dom}(p))(p(n) \neq f(n))\}$. For each $n < \omega$, let $E_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p)\}$. Put $\mathcal{F} = \{D_f : f \in 2^\omega\} \cup \{E_n : n < \omega\}$. Then $\mathcal{F}$ is a family of dense subsets of $\mathbb{P}$ and $|\mathcal{F}| \leq \mathfrak{c}$. Towards a contradiction, suppose G is a filter on $\mathbb{P}$ that meets every dense set in $\mathcal{F}$. Put $\bigcup G = g$. Then $\mathrm{dom}(g) = \omega$ since G meets every E_n. Hence $g : \omega \to 2$. So G meets D_g which means g disagrees with itself at some n which is impossible. □

Definition 23.5 (Martin's axiom)**.** *MA is the statement: For every $\kappa < \mathfrak{c}$, MA_κ holds.*

It is clear that $\mathfrak{c} = \omega_1$ (CH) implies MA. It is also possible that $\mathfrak{c} > \omega_1$ and MA holds. We skip the proof which uses what are called finite support iteration of c.c.c. forcings (see Chapter VIII, Section 6 in Kunen's book).

Fact 23.6. *If ZFC is consistent, then ZFC + MA + $\mathfrak{c} = \omega_{100}$ is consistent.*

We saw earlier that $\diamondsuit$ implies the existence of a Suslin line. We'll show that $\mathsf{MA} + \mathfrak{c} > \omega_1$ implies that there is no Suslin line.

Definition 23.7 (Product of posets). *The product of two posets $(\mathbb{P}, \leq_1, 1_{\mathbb{Q}})$, $(\mathbb{Q}, \leq_2, 1_{\mathbb{Q}})$ is defined to be the poset $(\mathbb{P} \times \mathbb{Q}, \leq, (1_{\mathbb{P}}, 1_{\mathbb{Q}}))$ where $(p, q) \leq (p', q')$ iff $p \leq_1 p'$ and $q \leq_2 q'$.*

Lemma 23.8. *Assume MA_{ω_1}. Let $\mathbb{P}$ be a c.c.c. poset and suppose $A \subseteq \mathbb{P}$ is uncountable. Then there exists a filter G on $\mathbb{P}$ such that $G \cap A$ is uncountable. It follows that A has an uncountable subset $B = G \cap A$ such that any finite set of members of B has a common extension.*

Proof. Let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be a sequence of pairwise distinct members of A.

First, we claim that

$$W = \{\alpha < \omega_1 : (\exists r \leq p_\alpha)(\exists \beta < \omega_1)(\forall \gamma < \omega_1)(\beta < \gamma \implies r \perp p_\gamma)\}$$

is countable. Suppose not. Then we can inductively construct $W' \subseteq W$ and $\langle r_\gamma : \gamma \in W' \rangle$ such that $|W'| = \omega_1$ and for every $\gamma < \gamma'$ in W', $r_\gamma \leq p_\gamma$, $r_{\gamma'} \leq p_{\gamma'}$ and $r_\gamma \perp r_{\gamma'}$. But then $\{r_\gamma : \gamma \in W'\}$ is an uncountable antichain in $\mathbb{P}$ which is impossible. So W is countable.

By throwing away p_α's for $\alpha \in W$, we can assume that

$$(\forall \alpha < \omega_1)(\forall r \leq p_\alpha)(|\{\gamma < \omega_1 : r, p_\gamma \text{ are compatible}\}| = \omega_1)$$

This means that the following sets are dense in $\mathbb{P}$ for every $\alpha < \beta < \omega_1$.

$$D_{\alpha,\beta} = \{r \in \mathbb{P} : (r \perp p_\alpha) \text{ or } (\exists \gamma < \omega_1)(\beta < \gamma \wedge r \leq p_\alpha \wedge r \leq p_\gamma)\}$$

Using MA_{ω_1}, choose a filter G on $\mathbb{P}$ such that $p_0 \in G$ and G meets $D_{0,\beta}$ for every $\beta < \omega_1$. We claim that $G \cap A$ is uncountable. To see this, fix $\beta < \omega_1$ and we'll find $\gamma > \beta$ such that $p_\gamma \in G$. Choose $r \in G \cap D_{0,\beta}$. Then $r \perp p_0$ is impossible since $p_0 \in G$ and G is a filter on $\mathbb{P}$. So for some $\gamma > \beta$, $r \leq p_\gamma$ and hence $p_\gamma \in G$. ☐

Theorem 23.9. *Assume MA_{ω_1}. Then the product of two c.c.c. posets is c.c.c.*

Proof. Let $\mathbb{P}, \mathbb{Q}$ be two c.c.c. posets with orderings $\leq_1$ and $\leq_2$ respectively. Towards a contradiction, suppose $A \subseteq \mathbb{P} \times \mathbb{Q}$ is an uncountable subset of $\mathbb{P} \times \mathbb{Q}$. Let $W = \{p : (\exists q \in \mathbb{Q})((p, q) \in A)\}$.

First suppose W is countable. Then we can find $p \in W$ such that $A_p = \{q : (p, q) \in A\}$ is uncountable. Since $\mathbb{Q}$ is c.c.c., we can find $q_1, q_2 \in A_p$ such that q_1, q_2 are compatible in $\mathbb{Q}$ with a common extension, say $r \in \mathbb{Q}$. Then (p, r) extends both (p, q_1) and (p, q_2). Hence A is not an antichain in $\mathbb{P} \times \mathbb{Q}$.

Now assume that W is uncountable and using Lemma 23.8, choose a filter G on $\mathbb{P}$ such that $B = G \cap W_1$ is uncountable. For each $p \in B$, choose $q_p \in \mathbb{Q}$ such that $(p, q_p) \in A$. Put $C = \{q_p : p \in B\} \subseteq \mathbb{Q}$. If C is countable, then for some $p \neq p'$ in B, $q_p = q_{p'} = q$ so that (p, q) and (p', q) are compatible members of A. If C is uncountable, then since $\mathbb{Q}$ is c.c.c., we can find distinct $q, q' \in C$ such that q, q' are compatible in $\mathbb{Q}$. Let $p, p' \in B$ be such that $q = q_p$ and $q' = q_{p'}$. Then (p, q) and (p', q') are compatible members of A. It follows that A is not an antichain in $\mathbb{P} \times \mathbb{Q}$.

☐

Theorem 23.10. *Assume MA_{ω_1}. Then there is no Suslin line.*

Proof. Suppose $(L, \prec)$ is a Suslin line. Define a poset $\mathbb{P}$ as follows. $p \in \mathbb{P}$ iff $p = (a, b) = \{x \in L : a \prec x \prec b\}$ for some $a, b \in L$ such that $a \prec b$. So $\mathbb{P}$ is the set of open intervals with end-points in L. Define the ordering on $\mathbb{P}$ by inclusion: $p \leq q$ iff $p \subseteq q$. Note that $\mathbb{P}$ satisfies c.c.c. because there is no uncountable family of pairwise disjoint intervals in $(L, \prec)$. We'll show that $\mathbb{P} \times \mathbb{P}$ does not satisfy c.c.c. which contradicts Theorem 23.9 and completes the proof.

Inductively choose $\langle a_\alpha, b_\alpha, c_\alpha : \alpha < \omega_1 \rangle$ in L such that the following hold.

(1) $a_\alpha \prec b_\alpha \prec c_\alpha$

(2) For every $\beta < \alpha$, b_β is not $\prec$-between a_α and c_α

To see that this can be done, suppose $\langle a_\beta, b_\beta, c_\beta : \beta < \alpha \rangle$ have been chosen. Since $B_\alpha = \{b_\beta : \beta < \alpha\}$ is a countable subset of L, it is not dense in $(L, \prec)$. So we can find an interval (a_α, c_α) in $(L, \prec)$ disjoint with B_α. Choose b_α to be any point in (a_α, c_α).

Let $A = \{\langle (a_\alpha, b_\alpha), (b_\alpha, c_\alpha) \rangle : \alpha < \omega_1\}$. We'll show that A is an uncountable antichain in $\mathbb{P} \times \mathbb{P}$. Fix $\alpha < \beta < \omega_1$ and put $U_\alpha = (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha)$ and $U_\beta = (a_\beta, b_\beta) \times (b_\beta, c_\beta)$. Since $b_\beta \notin (a_\alpha, c_\alpha)$, either $b_\beta \preceq a_\alpha$ or or $c_\alpha \preceq b_\beta$. In either case, $U_\alpha \cap U_\beta = 0$. It follows that A is an antichain in $\mathbb{P} \times \mathbb{P}$. ☐

24 Almost disjoint forcing

Definition 24.1. *For a cardinal κ, define $[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}$.*

Definition 24.2 (Almost disjoint family)**.** *Two sets are almost disjoint iff their intersection is finite. We say that $\mathcal{F}$ is an almost disjoint family iff $\mathcal{F} \subseteq [\omega]^\omega$ and*

$$(\forall A, B \in \mathcal{F})(A \neq B \implies |A \cap B| < \omega)$$

Definition 24.3 (MAD family)**.** *Let $\mathcal{F} \subseteq [\omega]^\omega$. We say that $\mathcal{F}$ is a MAD (maximal almost disjoint) family iff $\mathcal{F}$ is an infinite maximal almost disjoint family.*

So if $\mathcal{F}$ is a MAD family iff it is an infinite almost disjoint family and for every $Y \in [\omega]^\omega \setminus \mathcal{F}$, $\mathcal{F} \cup \{Y\}$ is not an almost disjoint family. We require $\mathcal{F}$ to be infinite to avoid finite maximal almost disjoint families like $\{X, \omega \setminus X\}$ where X is an infinite co-infinite subsets of ω.

Note that, by Zorn's lemma, every almost disjoint family is contained in a MAD family.

Lemma 24.4. *There is an almost disjoint family $\mathcal{F} \subseteq [\omega]^\omega$ such that $|\mathcal{F}| = \mathfrak{c}$.*

Proof. Let $h : 2^{<\omega} \to \omega$ be a one-one function. For each $f : \omega \to 2$, define $A_f = \{h(f \upharpoonright n) : n < \omega\}$. Then it is easy to see that $\mathcal{F} = \{A_f : f : \omega \to 2\}$ is as required. ☐

Lemma 24.5. *Suppose $\mathcal{F} \subseteq [\omega]^\omega$ is an almost disjoint family and $\mathcal{F} = \omega$. Then $\mathcal{F}$ is not a MAD family.*

Proof. Let $\langle A_n : n < \omega \rangle$ list $\mathcal{F}$. Note that the union of every finite subfamily of $\mathcal{F}$ is a coinfinite subset of ω. Construct a strictly increasing sequence $\langle k_n : n < \omega \rangle$ such that $k_n \in \omega \setminus \bigcup\{A_m : m \leq n\}$. Then $\{k_n : n < \omega\}$ is almost disjoint with every member of $\mathcal{F}$. ☐

It follows that the least cardinality of a MAD family is somewhere between ω_1 and $\mathfrak{c}$. We'll show that under MA_κ, there is no MAD family of cardinality κ.

Definition 24.6. *Suppose $\mathcal{A} \subseteq [\omega]^\omega$. Define the poset $\mathbb{P}_\mathcal{A}$ as follows.*

(1) $p \in \mathbb{P}_\mathcal{A}$ iff $p = (s_p, F_p)$ where $s_p \in [\omega]^{<\omega}$ and $F_p \in [\mathcal{A}]^{<\omega}$

(2) $p \leq q$ iff $s_q \subseteq s_p$ and $F_q \subseteq F_p$ and for every $A \in F_q$, $s_p \cap A \subseteq s_q$.

Lemma 24.7. *$\mathbb{P}_\mathcal{A}$ is c.c.c.*

Proof. Note that if $s_p = s_q$, then $(s_p \cup s_q, F_p \cup F_q)$ is a common extension of p, q. Since $[\omega]^{<\omega}$ is countable, it follows that $\mathbb{P}_\mathcal{A}$ is c.c.c. ☐

Theorem 24.8. *Assume MA_κ. Let $\mathcal{A}, \mathcal{C} \subseteq [\omega]^\omega$ where $|\mathcal{A}| \leq \kappa$, $|\mathcal{C}| \leq \kappa$ and for every $Y \in \mathcal{C}$ and $F \in [\mathcal{A}]^{<\omega}$, $|Y \setminus \bigcup F| = \omega$. Then there exists $X \subseteq \omega$ such that $(\forall A \in \mathcal{A})(|X \cap A| < \omega)$ and $(\forall Y \in \mathcal{C})(|X \cap Y| = \omega)$.*

Proof. Let $\mathbb{P} = \mathbb{P}_\mathcal{A}$ be as in Definition 24.6. For each $Y \in \mathcal{C}$ and $n < \omega$, let

$$E_{n,Y} = \{p \in \mathcal{P} : (\exists m > n)(m \in s_p \cap Y)\}$$

We claim that $E_{n,Y}$ is dense in $\mathbb{P}$. For suppose $p \in \mathbb{P}$. Note that, by assumption, $Y \setminus \bigcup F_p$ is infinite so we can choose $m \in (Y \setminus \bigcup F_p)$ such that $m > n$. Then $(s_p \cup \{m\}, F_p)$ is an extension of p in $E_{n,Y}$.

Next, observe that for each $A \in \mathcal{A}$, the set $D_A = \{p \in \mathbb{P} : A \in s_p\}$ is dense in $\mathbb{P}$. By MA_κ, there is a filter G on $\mathbb{P}$ that meets each of the dense sets in

$$\{E_{n,Y} : n < \omega, Y \in \mathcal{C}\} \cup \{D_A : A \in \mathcal{A}\}$$

Put $X = \bigcup\{s_p : p \in G\}$. We'll show that X is as required. Let $A \in \mathcal{A}$ and choose $p \in D_A \cap G$. So $A \in F_p$. Hence for every extension $q \leq p$ in G, $s_q \cap A \subseteq s_p$. It follows that $X \cap A \subseteq s_p$ and therefore $X \cap A$ is finite. Next suppose $Y \in \mathcal{C}$ and suppose $n < \omega$. Choose $p \in G \cap E_{n,Y}$. Let $m > n$ be such that $m \in s_p \cap Y$. Then $m \in X \cap Y$. It follows that $X \cap Y$ is infinite. □

Corollary 24.9. *Assume MA_κ. Let $\mathcal{A} \subseteq [\omega]^\omega$ be an infinite almost disjoint family of cardinality $\leq \kappa$. Then $\mathcal{A}$ is not MAD.*

Proof. Since $\mathcal{A}$ is infinite, for every $F \in [\mathcal{A}]^{<\omega}$, $|\omega \setminus \bigcup F| = \omega$. So we can apply Theorem 24.8 with $\mathcal{C} = \{\omega\}$ to get X as in the conclusion over there. Then X is infinite and $|X \cap A| < \omega$ for every $A \in \mathcal{A}$. It follows that $\mathcal{A}$ is not MAD. □

Corollary 24.10. *Assume MA_κ. Then $2^\kappa = \mathfrak{c}$.*

Proof. By Lemma 24.4, fix an almost disjoint family $\mathcal{B} \subseteq [\omega]^\omega$ of cardinality κ. For each $\mathcal{A} \subseteq \mathcal{B}$, note that the hypotheses of Theorem 24.8 are satisfied with $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$. So for each $\mathcal{A} \subseteq \mathcal{B}$, we can find $X_\mathcal{A} \subseteq \omega$ such that $(\forall A \in \mathcal{A})(|X_\mathcal{A} \cap A| < \omega)$ and $(\forall Y \in \mathcal{B} \setminus \mathcal{A})(|X_\mathcal{A} \cap Y| = \omega)$. It follows that the function $\mathcal{A} \mapsto X_\mathcal{A}$ is one-one for $\mathcal{A} \in \mathcal{P}(\mathcal{B})$. Hence $2^\kappa = |\mathcal{P}(\mathcal{B})| \leq |\mathcal{P}(\omega)| = \mathfrak{c}$. As κ is infinite, $2^\kappa \geq \mathfrak{c}$. So $2^\kappa = \mathfrak{c}$. □

Corollary 24.11. *Assume MA. Then $\mathfrak{c}$ is regular.*

Proof. It suffices to show that for every $\kappa < \mathfrak{c}$, $\mathrm{cf}(\mathfrak{c}) > \kappa$. So fix $\kappa < \mathfrak{c}$ infinite. Then, by Corollary 24.10, $2^\kappa = \mathfrak{c}$. By Corollary 6.12, $\mathrm{cf}(2^\kappa) > \kappa$. Hence $\mathrm{cf}(\mathfrak{c}) > \kappa$. □