MTH727: *Set theory and forcing*

Homework 1 Spring, 2022/23

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on Feb. 1, Wednesday in class.

1. In ZF, show that the following statement is equivalent to AC: For every nonempty set A, there exists a function $\circ : A \times A \to A$ such that $(A, \circ)$ is a group. [5 Points]

2. (Kunen Ex I.16) Assume CH. Show that $\omega_n^\omega = \omega_n$ for every $1 \leq n < \omega$. [5 Points]

3. Suppose (X, d) is a metric space in which every open ball is uncountable. Show that there exists an uncountable family $\mathcal{F}$ consisting of pairwise disjoint dense subsets of X. Recall that $D \subseteq X$ is dense in X iff for every $y \in X$ and $r > 0$, the open ball $B(y, r)$ contains at least one point from D. [5 Points]

4. Let κ be an infinite cardinal and λ be the least cardinal satisfying $\kappa^\lambda > \kappa$. Show that λ is regular. [5 Points]

5. (Kunen Ex I.19) Let κ be an infinite (possibly singular) cardinal and $<_1$ be a well-order on κ. Show that there exists $X \subseteq \kappa$ such that $|X| = \kappa$ and $(\forall \alpha, \beta \in X)(\alpha < \beta \iff \alpha <_1 \beta)$. [5 Points]

MTH727: *Set theory and forcing*

Homework 2 2022-23 II

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on Feb. 15, Wednesday in class.

1. Suppose $\mathcal{A}$ is an uncountable family of pairwise disjoint non-stationary subsets of ω_1. Show that there exists an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B}$ is non-stationary in ω_1. [5 Points]

2. Prove Lemmas 8.6 and 8.7 in Lecture notes. [2 + 3 Points]

3. (Kunen II.31) Let $(L, <)$ be a ccc linear ordering. Show that it has a dense subset of size $\leq \omega_1$. [5 Points]

4. Suppose $\overline{A} = \langle A_\alpha : \alpha \in \mathsf{Lim}(\omega_1) \rangle$ satisfies the following.

   (i) For every $\alpha \in \mathsf{Lim}(\omega_1)$, A_α is an unbounded subset of α of order type ω.

   (ii) For every $A \in [\omega_1]^{\omega_1}$, there exists $\alpha \in \mathsf{Lim}(\omega_1)$ such that $A_\alpha \subseteq A$.

   Show that for every $A \in [\omega_1]^{\omega_1}$, $\{\alpha \in \mathsf{Lim}(\omega_1) : A_\alpha \subseteq A\}$ is stationary in ω_1. [5 Points]

5. The principle $\clubsuit$ says that there exists $\overline{A}$ satisfying (i)+(ii) in Problem 4 above. Show that $\diamondsuit$ is equivalent to $\clubsuit + CH$. [5 Points]

MTH727: *Set theory and forcing*
Homework 3 2022-23 II

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on March 1, Wednesday in class.

1. Let $m : \mathcal{P}(X) \to [0,1]$ be an atomless probability measure. Show that range$(m) = [0,1]$. [5 Points]
2. Recall that $\mathbf{W} = \bigcup\{V_\alpha : \alpha \in \mathbf{ORD}\}$. Let T be the theory ZF without the axiom of foundation. Show that for every ϕ in $T \cup \{\mathsf{Foundation}\}$, $T \vdash \phi^{\mathbf{W}}$. Conclude that if $ZF - \{\mathsf{Foundation}\}$ is consistent, then ZF is also consistent. [5 Points]
3. Show that $V_{\omega+\omega}$ is a model of all the axioms of ZFC except the axiom of replacement. [5 Points]
4. For an infinite cardinal κ, define $H_\kappa = \{x : |\mathsf{trcl}(x)| < \kappa\}$. Show the following.
   (i) H_κ is transitive. [1 Point]
   (ii) $H_\kappa \cap \mathbf{ORD} = \kappa$. [1 Point]
   (iii) $H_\kappa \subseteq V_\kappa$. [1 Point]
   (iv) $H_\kappa = V_\kappa$ iff κ is strongly inaccessible. [2 Points]
5. Show that H_{ω_1} is a model of all the axioms of ZFC except the axiom of power set. [5 Points]

MTH727: *Set theory and forcing*
Homework 4 2022-23 II

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on March 29, Wednesday in class.

1. Assume $\mathbf{V} = \mathbf{L}$. Show that $L_\kappa = H_\kappa$ for every infinite cardinal κ. Recall that $H_\kappa = \{x : |\mathsf{trcl}(x)| < \kappa\}$. [**5 Points**]

2. Define $B \prec A$ (read $(B, \in)$ is an **elementary substructure** of $(A, \in)$) by $B \subseteq A$ and $(\forall m, n < \omega)(\mathsf{En}(m, A, n) \cap B^k = \mathsf{En}(m, B, n))$. Recall that $\mathsf{En}(m, A, n)$ is the mth n-ary definable relation on A (see Definition 15.1 in Lecture notes). Assume $B \prec A$. Show that every formula ϕ is absolute between B and A. [**5 Points**]

3. Let $Y \subseteq X$. Show that there exists Z such that $Y \subseteq Z \subseteq X$, $|Z| = \max(\omega, |X|)$ and $Z \prec X$. [**5 Points**]

4. Assume $\mathbf{V} = \mathbf{L}$. Let $<_{\omega_2}$ be the well-ordering on L_{ω_2} from page 27 in the Lecture notes. Observe that $<_{\omega_2}$ is definable in L_{ω_2}. By recursion on $\alpha < \omega_1$, define (A_α, C_α) as follows: $A_0 = C_0 = \emptyset$ and for each $1 \leq \alpha < \omega_1$, $(A_\alpha, C_\alpha) \in L_{\omega_2}$ is the $<_{\omega_2}$-least pair of subsets of α such that C_α is a closed unbounded subset of α and for every $\xi \in C_\alpha$, $A_\alpha \cap \xi \neq A_\xi$. If there is no such pair, define $A_\alpha = C_\alpha = 0$. Show that $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. [**10 Points**]

   Hint: See Theorem 13.21 on page 191 in Jech's 2003 set theory book.

MTH727: *Set theory and forcing*
Homework 5 2022-23 II

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on April 12, Wednesday in class.

1. Prove Lemma 19.4 from notes. [**5 Points**]
2. Suppose $\mathbf{M}$ is a ctm of ZFC and $\mathbb{P}$ is a poset in $\mathbf{M}$. Assume that eveyr condition in $\mathbb{P}$ has two incompatible extensions. Show the following.

   (a) If G is a $\mathbb{P}$-generic filter over $\mathbf{M}$, then $\mathbb{P} \setminus G$ is dense in $\mathbb{P}$. Conclude that $G \notin \mathbf{M}$. [**2 Points**]

   (b) There is a filter G on $\mathbb{P}$ such that $\mathbf{M}[G]$ is not a model of ZFC. [**3 Points**]
3. Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P}$ is a poset in $\mathbf{M}$ and G is a filter on $\mathbb{P}$. Show that G is $\mathbb{P}$-generic over $\mathbf{M}$ iff for every maximal antichain $A \subseteq \mathbb{P}$, if $A \in \mathbf{M}$, then $G \cap A \neq 0$. [**5 Points**]
4. Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P}$ is a poset in $\mathbf{M}$ and G, H are two $\mathbb{P}$-generic filters over $\mathbf{M}$. Show that either $G = H$ or there exist $p \in G$ and $q \in H$ such that $p \perp q$. [**5 Points**]
5. Let $\mathbb{P}$ be a poset in which every condition has two incompatible extensions. Define $\mathsf{cc}(\mathbb{P})$ to be the least κ such that every antichain in $\mathbb{P}$ has size $< \kappa$. Show that κ is a regular infinite cardinal. [**5 Points**]

MTH727: *Set theory and forcing*
Homework 6 2022-23 II

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers. You may work in groups but you must write your own solutions. This homework is due on April 21 Friday.

1. Let $\mathbf{M}$ be a ctm of ZFC and $f : \omega \to 2$. We say that f is a Cohen real over $\mathbf{M}$ iff $G_f = \{p \in \mathsf{Fn}(\omega, 2) : p \subseteq f\}$ is an $\mathsf{Fn}(\omega, 2)$-generic filter over $\mathbf{M}$. Suppose f is a Cohen real over $\mathbf{M}$, $g : \omega \to 2$ and $g \in \mathbf{M}$. Prove that $f + g$ is a Cohen real over $\mathbf{M}$ where $(f + g)(n) = f(n) + g(n)$ (modulo 2). [**5 Points**]

2. Let $\mathbf{M}$ be a ctm of ZFC and suppose $f : \omega \to 2$ is a Cohen real over $\mathbf{M}$. Let $X_f = \{n < \omega : f(n) = 1\}$. Suppose $Y \in \mathcal{P}(\omega) \cap \mathbf{M}$ and Y is infinite. Show that both $Y \cap X_f$ and $Y \setminus X_f$ are infinite. [**5 Points**]

3. Suppose $\mathbf{M}$ is a ctm of ZFC, $\mathbb{P} = \mathsf{Fn}(\omega, 2)$ and G is a $\mathbb{P}$-generic filter over $\mathbf{M}$. Show that for every $f \in \omega^\omega \cap \mathbf{M}[G]$, there exists $g \in \omega^\omega \cap \mathbf{M}$ such that $\{n < \omega : f(n) = g(n)\}$ is infinite. [**5 Points**]

4. Assume MA_{ω_1}. Let $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ be a c.c.c. poset and let κ be an infinite cardinal. Define $\mathbb{Q}$ to be the set of all $f : \kappa \to \mathbb{P}$ such that $\{\alpha < \kappa : f(\alpha) \neq 1_{\mathbb{P}}\}$ is finite. For $f, g \in \mathbb{Q}$, define $f \leq_{\mathbb{Q}} g$ iff $(\forall \alpha < \kappa)(f(\alpha) \leq_{\mathbb{P}} g(\alpha))$. Show that $\mathbb{Q}$ satisfies c.c.c. [**5 Points**]

5. Assume MA_{ω_1}. Let $\mathbb{P}$ be a c.c.c. poset and suppose $A \in [\mathbb{P}]^{\omega_1}$. Show that there exists $\langle G_n : n < \omega \rangle$ such that each G_n is a filter on $\mathbb{P}$ and $A \subseteq \bigcup\{G_n : n < \omega\}$. **Hint**: Use the previous problem with $\kappa = \omega$. [**5 Points**]