On a theorem of Gitik and Shelah

Ashutosh Kumar*

January 2016

1 Higher products

For a cardinal $\kappa \geq 1$, let Cohen_κ (resp. Random_κ) denote the forcing for adding κ Cohen (resp. random) reals. In [1], Gitik and Shelah proved that forcing with a sigma ideal cannot be isomorphic to $\mathsf{Random}_1 \times \mathsf{Cohen}_1$. In [2] (Section 546), Fremlin asked if the result generalizes to a product that adds more of these reals. We show that this is the case.

Theorem 1.1. *Let $\theta, \lambda > 0$. Let $\mathcal{I}$ be a sigma ideal on X. Then forcing with $\mathcal{P}(X)/\mathcal{I}$ is not isomorphic to $\mathbb{P} = \mathsf{Cohen}_\theta \times \mathsf{Random}_\lambda$.*

Proof: Suppose not. Since $\mathbb{P}$ is homogeneous, by restricting $\mathcal{I}$, we can assume that for some $\kappa \geq \aleph_1$, for every $A \in \mathcal{I}^+$, the additivity of $\mathcal{I} \restriction A$ is κ. Let G be $\mathcal{P}(X)/\mathcal{I}$-generic over V. In $V[G]$, let M be the transitive collapse of the well founded G-ultrapower of V and $j : V \to M \subseteq V[G]$ the generic ultrapower embedding with critical point κ. Note that $V[G] \models \mathsf{Cov(Null)} = \aleph_1$ because adding a Cohen real makes the covering of the null ideal $\aleph_1$ - See Theorem 3.3.14 in [3]. Since M is closed under κ-sequences in $V[G]$, M and therefore V satisfy $\mathsf{Cov(Null)} = \aleph_1$. Let $\langle N_i : i < \omega_1 \rangle$ be a sequence of null G_δ sets coded in V such that $V \models \bigcup_{i<\omega_1} N_i = \mathbb{R}$. As j is elementary, $M \models \bigcup_{i<\omega_1} N_i = \mathbb{R}$. But this is impossible as M contains a random real over V. $\square$

References

[1] M. Gitik and S. Shelah, More on real-valued measurable cardinals and forcing with ideals, Israel J. Math 124 2001, 221-242

[2] D. H. Fremlin, Measure Theory Vol. 5: Set-theoretic measure theory, Part II

[3] T. Bartoszynski and H. Judah, Set theory: On the structure of the real line, A. K. Peters Ltd. (1995)

*I'd like to thank Professor Fremlin for discussing these matters with me.