Remark on transversals

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1 Transversal

In [1], the following was shown: For every partition of $X \subseteq [0, 1]$ into countable sets, there exists a transversal $Y$ such that $\mu^*(Y) = \mu^*(X)$. A similar result holds for category: For every partition of $X \subseteq [0, 1]$ into countable sets, there exists a transversal $Y$ which is everywhere non meager in $X$. Bill Weiss asked what happens if we consider partitions into sets of size $\aleph_1$. Note that under the continuum hypothesis the result continues to hold. Here, we show that, in the case of category, it can fail too.

**Theorem 1.1.** The following is consistent: There exists a non meager set $X \subseteq [0, 1]$ and a partition of $X$ into meager sets of size $\aleph_1$ such that every transversal is meager.

**Proof:** Define $I = \{ A \subseteq \omega_1 \times \omega_1 : (\exists i_0 < \omega_1)(\forall i > i_0)(|A_i| \leq \aleph_0) \}$ where $A_i = \{ j : (i, j) \in A \}$. Note that $I$ is a sigma ideal on $\omega_1 \times \omega_1$. Let $P$ be the forcing for adding $\aleph_1$ Cohen reals $\langle c_{i,j} : i, j < \omega_1 \rangle$. In $V^P$, let $Q$ be the finite support product $\prod\{ Q_A : A \in I \}$ where $Q_A$ is defined as follows: $p \in Q_A$ iff $p = (F_p, N_p, \bar{n}_p, \bar{\sigma}_p) = (F, N, \bar{n}, \bar{\sigma})$ where

- $F$ is a finite subset of $A$
- $N < \omega$
- $\bar{n} = \langle n_k : k \leq N \rangle$ is a strictly increasing sequence of integers with $n_0 = 0$
- $\bar{\sigma} = \langle \sigma_k : k < N \rangle$ where each $\sigma_k \in [n_k, n_{k+1})$

For $p, q \in Q_A$, $p \leq q$ iff $F_q \subseteq F_p$, $N_q \leq N_p$, $\bar{n}_q \leq \bar{n}_p$, $\bar{\sigma}_q \leq \bar{\sigma}_p$ and for every $N_q \leq k < N_p$, for every $(i, j) \in F_q$, $\sigma_{p,k} \neq c_{i,j} \upharpoonright [n_{p,k}, n_{p,k+1})$. Note that $Q_A$ is a sigma centered forcing making $\{ c_{i,j} : (i, j) \in A \}$ meager. The set of conditions $(p, q) \in P \star Q$ where $p \in P$ and for each $A \in \text{dom}(q)$, $p$ forces an actual value to $q(A)$ is dense in $P \star Q$.

In $V^{P \star Q}$, let $X = \{ c_{i,j} : i, j < \omega_1 \}$, $X_i = \{ c_{i,j} : j < \omega_1 \}$ for $i < \omega_1$. It is clear that each $X_i$ is meager. The next two claims finish the proof.

**Claim 1.2.** $X$ is non meager in $V^{P \star Q}$. 

Proof: Let $B$ be a meager $F_{\sigma}$-set coded in $V^{P\star Q}$. Since $P\star Q$ is ccc we can find a countable $F \subseteq I$ such that $B$ is coded in $V[\langle c_{i,j} : (i,j) \in \bigcup F \rangle][\prod \{G_{Q_{A}} : A \in F \}]$. Note that for each each $A \in F$, $Q_{A} \in V[\langle c_{i,j} : (i,j) \in \bigcup F \rangle]$. Choose $(i,j) \in (\omega_{1} \times \omega_{1}) \setminus \bigcup F$. Then $c_{i,j}$ is Cohen over $V[\langle c_{i,j} : (i,j) \in \bigcup F \rangle][\prod \{G_{Q_{A}} : A \in F \}]$ and hence does not belong to $B$. Therefore $X$ is non meager in $V^{P\star Q}$.

Claim 1.3. In $V^{P\star Q}$, every transversal of $\langle X_{i} : i < \omega_{1} \rangle$ is meager.

Proof: Suppose $\check{Y} \in V^{P\star Q}$ is a transversal of $\langle X_{i} : i < \omega_{1} \rangle$. For each $i < \omega_{1}$, choose a countable $A_{i} \subseteq \omega_{1}$ such that $\Vdash_{P\star Q} \check{Y} \cap \{c_{i,j} : j < \omega_{1} \} \subseteq \{c_{i,j} : j \in A_{i} \}$. Let $A = \{(i,j) : i < \omega_{1}, j \in A_{i} \}$. Then $A \in I$ and hence $Q_{A}$ makes $\check{Y}$ meager. \hfill \Box

References