Remark on transversals

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October 2015

1 Transversal

In [1], the following was shown: For every partition of $X \subseteq [0,1]$ into countable sets, there exists a transversal Y such that $\mu^\star(Y) = \mu^\star(X)$. A similar result holds for category: For every partition of $X \subseteq [0,1]$ into countable sets, there exists a transversal Y which is everywhere non meager in X. Bill Weiss asked what happens if we consider partitions into sets of size $\aleph_1$. Note that under the continuum hypothesis the result continues to hold. Here, we show that, in the case of category, it can fail too.

Theorem 1.1. *The following is consistent: There exists a non meager set $X \subseteq [0,1]$ and a partition of X into meager sets of size $\aleph_1$ such that every transversal is meager.*

Proof: Define $\mathcal{I} = \{A \subseteq \omega_1 \times \omega_1 : (\exists i_0 < \omega_1)(\forall i > i_0)(|A_i| \leq \aleph_0)\}$ where $A_i = \{j : (i,j) \in A\}$. Note that $\mathcal{I}$ is a sigma ideal on $\omega_1 \times \omega_1$. Let $\mathbb{P}$ be the forcing for adding $\aleph_1$ Cohen reals $\langle c_{i,j} : i,j < \omega_1 \rangle$. In $V^{\mathbb{P}}$, let $\mathbb{Q}$ be the finite support product $\prod\{\mathbb{Q}_A : A \in \mathcal{I}\}$ where $\mathbb{Q}_A$ is defined as follows: $p \in \mathbb{Q}_A$ iff $p = (F_p, N_p, \bar{n}_p, \bar{\sigma}_p) = (F, N, \bar{n}, \bar{\sigma})$ where

- F is a finite subset of A
- $N < \omega$
- $\bar{n} = \langle n_k : k \leq N \rangle$ is a strictly increasing sequence of integers with $n_0 = 0$
- $\bar{\sigma} = \langle \sigma_k : k < N \rangle$ where each $\sigma_k \in {}^{[n_k, n_{k+1})}2$

For $p, q \in \mathbb{Q}_A$, $p \leq q$ iff $F_q \subseteq F_p$, $N_q \leq N_p$, $\bar{n}_q \preceq \bar{n}_p$, $\bar{\sigma}_q \preceq \bar{\sigma}_p$ and for every $N_q \leq k < N_p$, for every $(i,j) \in F_q$, $\sigma_{p,k} \neq c_{i,j} \restriction [n_{p,k}, n_{p,k+1})$. Note that $\mathbb{Q}_A$ is a sigma centered forcing making $\{c_{i,j} : (i,j) \in A\}$ meager. The set of conditions $(p,q) \in \mathbb{P} \star \mathbb{Q}$ where $p \in \mathbb{P}$ and for each $A \in \mathrm{dom}(q)$, p forces an actual value to $q(A)$ is dense in $\mathbb{P} \star \mathbb{Q}$.

In $V^{\mathbb{P} \star \mathbb{Q}}$, let $X = \{c_{i,j} : i,j < \omega_1\}$, $X_i = \{c_{i,j} : j < \omega_1\}$ for $i < \omega_1$. It is clear that each X_i is meager. The next two claims finish the proof.

Claim 1.2. *X is non meager in $V^{\mathbb{P} \star \mathbb{Q}}$.*

Proof: Let B be a meager F_σ-set coded in $V^{\mathbb{P}\star\mathbb{Q}}$. Since $\mathbb{P}\star\mathbb{Q}$ is ccc we can find a countable $\mathcal{F} \subseteq \mathcal{I}$ such that B is coded in $V[\langle c_{i,j} : (i,j) \in \bigcup \mathcal{F}\rangle][\prod\{G_{\mathbb{Q}_A} : A \in \mathcal{F}\}]$. Note that for each each $A \in \mathcal{F}$, $\mathbb{Q}_A \in V[\langle c_{i,j} : (i,j) \in \bigcup \mathcal{F}\rangle]$. Choose $(i,j) \in (\omega_1 \times \omega_1) \setminus \bigcup \mathcal{F}$. Then $c_{i,j}$ is Cohen over $V[\langle c_{i,j} : (i,j) \in \bigcup \mathcal{F}\rangle][\prod\{G_{\mathbb{Q}_A} : A \in \mathcal{F}\}]$ and hence does not belong to B. Therefore X is non meager in $V^{\mathbb{P}\star\mathbb{Q}}$.

Claim 1.3. *In $V^{\mathbb{P}\star\mathbb{Q}}$, every transversal of $\langle X_i : i < \omega_1\rangle$ is meager.*

Proof: Suppose $\mathring{Y} \in V^{\mathbb{P}\star\mathbb{Q}}$ is a transversal of $\langle X_i : i < \omega_1\rangle$. For each $i < \omega_1$, choose a countable $A_i \subseteq \omega_1$ such that $\Vdash_{\mathbb{P}\star\mathbb{Q}} \mathring{Y} \cap \{c_{i,j} : j < \omega_1\} \subseteq \{c_{i,j} : j \in A_i\}$. Let $A = \{(i,j) : i < \omega_1, j \in A_i\}$. Then $A \in \mathcal{I}$ and hence $\mathbb{Q}_A$ makes $\mathring{Y}$ meager. $\square$

References

[1] A. Kumar, S. Shelah, A transversal of full outer measure, Preprint