SUPERSATURATED IDEALS

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Dedicated to the memory of Ken Kunen

Abstract. An ideal $I$ on a set $X$ is supersaturated iff $\text{add}(I) \geq \omega_2$ and for every family $F$ of $I$-positive sets with $|F| < \text{add}(I)$, there exists a countable set that meets every set in $F$. We show that many well-known ccc forcings preserve supersaturation. We also show that the existence of supersaturated ideals is independent of ZFC plus “There exists an $\omega_1$-saturated $\sigma$-ideal”.

1. Introduction

Saturation properties of ideals are ubiquitous in modern set theory and there is a considerable body of work (for example, see [3, 5, 6, 7]) on the study of a large number of such properties. Throughout this paper, by an ideal $I$ on $X$, we mean an ideal $I$ on $X$ that contains every finite subset of $X$. Supersaturation is a strengthening of $\omega_1$-saturation defined as follows.

Definition 1.1. Suppose $I$ is an ideal on $X$ and $\lambda$ is a cardinal. We say that $I$ is $\lambda$-supersaturated iff $\text{add}(I) \geq \lambda^+$ and for every $A \subseteq I^+$, if $|A| < \text{add}(I)$, then there exists $W \in [X]^{<\lambda}$ such that for every $A \in A$, $A \cap W \neq \emptyset$. $I$ is supersaturated iff it is $\omega_1$-supersaturated.

Suppose $I$ is a supersaturated ideal on $X$. Since $\text{add}(I) \geq \omega_2$, it follows that $I^+$ cannot have an uncountable subfamily of pairwise disjoint sets because no countable set can meet all of them. So $I$ is $\omega_1$-saturated. Let $\mu = \text{add}(I)$. Ulam showed that either $\mu$ is a measurable cardinal or $\mu$ is a weakly inaccessible cardinal $\leq \mathfrak{c}$. Solovay showed that $\mu$ admits a normal $\omega_1$-saturated ideal $J$ and $\mu$ is a measurable cardinal in the inner model $L[J]$. For proofs of these facts, see [7].

Though closely related to some of the works of Fremlin, supersaturated ideals were formally introduced in [4] where it was shown that if $\kappa \leq \mathfrak{c}$ admits a normal supersaturated ideal then the order dimension of the Turing degrees is at least $\kappa$. An earlier motivation for investigating these ideals comes from the following question of Fremlin – See Problem EG(h) in [1].

Question 1.2 (Fremlin). Suppose $\kappa$ is real valued measurable and $m : \mathcal{P}(\kappa) \to [0, 1]$ is a witnessing normal measure. Let $F$ be a family of subsets of $\kappa$ such that $|F| < \kappa$ and for every $A \in F$, $m(A) > 0$. Must there exist a countable $N \subseteq \kappa$ such that for every $A \in F$, $N \cap A \neq \emptyset$?

So Question 1.2 is asking if the null ideal of every normal witnessing measure on a real valued measurable cardinal must be supersaturated. One of the standard ways of obtaining $\omega_1$-saturated ideals on cardinals below the continuum is to start

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with a measurable cardinal $\kappa$ and a witnessing normal prime ideal $\mathcal{I}$ on $\kappa$, and force with a ccc forcing $\mathbb{P}$ that adds $\geq \kappa$ reals. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^\mathbb{P}$. Then $\mathcal{J}$ is always an $\omega_1$-saturated normal ideal on $\kappa \leq \mathfrak{c}$. But whether or not $\mathcal{J}$ is supersaturated will depend on the choice of $\mathbb{P}$. This motivates the notion of supersaturation preserving forcings (Definition 2.1). In Section 2, we show that a large class of ccc forcings for adding new reals are supersaturation preserving. In particular, the following holds.

**Theorem 1.3.** Let $\text{Random}_\lambda$ denote the forcing for adding $\lambda$ random reals.

(1) Every $\sigma$-linked forcing is supersaturation preserving.

(2) $\text{Random}_\lambda$ is supersaturation preserving for every $\lambda$.

The question of whether every $\omega_1$-saturated ideal must be supersaturated was raised in [4]. Our main result shows that this is independent.

**Theorem 1.4.** Each of the following is consistent.

(1) There is an $\omega_1$-saturated ideal on a cardinal below the continuum and there are no supersaturated ideals.

(2) There is an $\omega_1$-saturated ideal on a cardinal below the continuum and every $\omega_1$-saturated ideal is supersaturated.

**Notation:** Let $\mathcal{I}$ be an ideal on $X$. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. $\text{add}(\mathcal{I})$ denotes the least cardinality of a subfamily of $\mathcal{I}$ whose union is in $\mathcal{I}^+$. For $A \subseteq X$, define $\mathcal{I} \upharpoonright A = \{Y \subseteq X : Y \cap A \in \mathcal{I}\}$. Suppose $V \subseteq W$ are transitive models of set theory, $X, \mathcal{I} \in V$ and $V \models \"\mathcal{I} is an ideal on $X\"$. Recall that the ideal generated by $\mathcal{I}$ in $W$ is $\mathcal{J} = \{A \in W : (\exists B \in \mathcal{I})(A \subseteq B)\}$.

For a set of ordinals $X$, $\otimes(X)$ denotes the order type of $X$. An ordinal $\delta$ is indecomposable iff for every $X \subseteq \delta$, either $\otimes(X) = \delta$ or $\otimes(\delta \setminus X) = \delta$. If $\mathbb{P}, \mathbb{Q}$ are forcing notions, we write $\mathbb{P} \leq \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ is also a maximal antichain in $\mathbb{Q}$. $\text{Cohen}_\lambda$ denotes the forcing for adding $\lambda$ Cohen reals. $\text{Random}_\lambda$ is the measure algebra on $2^\lambda$ equipped with the usual product measure denoted by $\mu_\lambda$. If $\lambda$ is clear from the context, then we drop it and just write $\mu$.

2. CCC FORCINGS AND SUPERSATURATION

**Definition 2.1.** A forcing $\mathbb{P}$ is $\kappa$-ssp (ssp = supersaturation preserving) iff for every normal supersaturated ideal $\mathcal{I}$ on $\kappa$, $V^\mathbb{P} \models \"the ideal generated by $\mathcal{I}$ is supersaturated\"$, $\mathbb{P}$ is ssp iff it is $\kappa$-ssp for every $\kappa$.

In [4], the following forcings were shown to be $\kappa$-ssp for every $\kappa$.

(a) $\text{Cohen}_\lambda$ for any $\lambda$.

(b) Any finite support iteration of ccc forcings of size $< \kappa$.

It was also shown that $\text{Random}_\lambda$ is $\kappa$-ssp for any measurable $\kappa$. The next theorem improves this to all $\kappa$.

**Theorem 2.2.** $\text{Random}_\lambda$ is $\kappa$-ssp for every $\kappa$ and $\lambda$.

**Proof.** Fix a normal supersaturated ideal $\mathcal{I}$ on $\kappa$. Put $\mathbb{B} = \text{Random}_\lambda$ and let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^\mathbb{B}$. Suppose $\theta < \kappa$ and $\models _\mathbb{B} \langle \dot{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets. It suffices to find $B \in [\kappa]^{\mathfrak{c}}_\theta$ such that $\models _\mathbb{B} (\forall i < \theta)(\dot{A}_i \cap B \neq \emptyset)$.

For $i < \theta$ and $\alpha < \kappa$, put $p_{i,\alpha} = [[\alpha \in \dot{A}_i]]_\mathbb{B}$. So each $p_{i,\alpha}$ is a Baire subset of $2^\lambda$. Put $T_i = \{\alpha < \kappa : p_{i,\alpha} \neq 0_\mathbb{B}\}$. 
Claim 2.3. For each $p \in B \setminus \{0_B\}$, $\{\alpha \in T_i : p_i,\alpha \cap p \neq 0_B\} \subseteq I^+$. 

Proof. Put $X_p = \{\alpha \in T_i : p_i,\alpha \cap p \neq 0_B\}$ and suppose $X_p \subseteq I$. Since the empty condition forces that $A_i \in \mathcal{F}^+$, it follows that for every $X \subseteq I$, $\{p_i,\alpha : \alpha \in T_i \setminus X\}$ is predense in $B$. But every condition in $\{p_i,\alpha : \alpha \in T_i \setminus X_p\}$ is incompatible with $p$ which is impossible. \hfill \Box

For a finite partial function $f$ from $\lambda$ to 2, define $[f] = \{x \in 2^\lambda : x \upharpoonright \text{dom}(f) = f\}$. For a clopen $K \subseteq 2^\lambda$, define $\text{supp}(K)$ to be the smallest finite set $S \subseteq \lambda$ such that $(\forall x,y \in 2^\lambda)(x \upharpoonright S = y \upharpoonright S \implies (x \in K \iff y \in K))$. If $\text{supp}(K) = S$, then there is finite list $\{f_{K,n} : n < n_*\}$ where $f_{K,n}$'s are pairwise distinct functions from $S$ to 2 and $K = \bigsqcup_{n < n_*} [f_{K,n}]$.

Definition 2.4. Suppose $C$ is a family of clopen sets in $2^\lambda$. We say that $C$ is a strong $\Delta$-system of width $(n_*,N_*)$ iff $n_*,N_* < \omega$ and the following hold.

(a) $\langle \text{supp}(K) : K \in C \rangle$ is a $\Delta$-system with root $R$.
(b) For every $K \in C$, $|\text{supp}(K) \setminus R| = n_*$.
(c) For every $K \in C$, $K = \bigsqcup_{n < N_*} [f_{K,n}]$ where each $f_{K,n} : \text{supp}(K) \to 2$ and $f_{K,n}$'s are pairwise distinct.
(d) For every $K_1,K_2 \in C$ and $n < N_*$, $\forall i,\alpha,\epsilon,\gamma,\delta \in [\lambda]^{n_*}$ such that the following hold.

(i) $f_{K_1,n} \upharpoonright R = f_{K_2,n} \upharpoonright R$ and $f_{K_1,n} \upharpoonright R = f_{K_2,n} \upharpoonright R$
(ii) if for $m \in \{1,2\}$, $\{\xi_m^n : j < |R| + n_*\}$ lists $\text{supp}(K_m)$ in increasing order, then $f_{K_1,j}(\xi_j^1) = f_{K_2,j}(\xi_j^2)$ for every $j < |R| + n_*$.

Lemma 2.5. Suppose $p \subseteq 2^\lambda$ is Baire and $C$ is an infinite strong $\Delta$-system of clopen sets in $2^\lambda$ of width $(n_*,N_*)$. Let $\epsilon > 0$ and assume that for infinitely many $K \in C$, $\mu(p \cap K) \geq \epsilon$. Then for all but finitely many $K \in C$, $\mu(p \cap K) \geq \epsilon/2$.

Proof. Let $R$ be the root of $\langle \text{supp}(K) : K \in C \rangle$. For each $K \in C$, fix $\{f_{K,n} : n < N_*\}$ such that $K = \bigsqcup_{n < N_*} [f_{K,n}]$. First suppose that $p$ is clopen. Let $C_p = \{K \in C : \text{supp}(K) \setminus R \cap \text{supp}(p) = \emptyset\}$. Then $C \setminus C_p$ is finite and for each $K \in C_p$, $\mu(p \cap K) = \sum_{n < N_*} \mu(p \cap [f_{K,n}]) = 2^{-n_*} \sum_{n < N_*} \mu(p \cap [f_{K,n} \upharpoonright R])$

which does not depend on $K \in C_p$. It follows that the result holds if $p$ is clopen. The general case follows by applying the previous case to a clopen $q \subseteq 2^\lambda$ satisfying $\mu(p \Delta q) < \epsilon/2$. \hfill \Box

For each $\alpha \in T_i$, fix $S_{i,\alpha} \subseteq [\lambda]^N_0$ such that $p_{i,\alpha}$ is supported in $S_{i,\alpha}$. For every $i < \theta$, $\alpha \in T_i$, and $\epsilon > 0$ rational, choose a clopen set $K_{i,\alpha,\epsilon} \subseteq 2^\lambda$ with $\text{supp}(K_{i,\alpha,\epsilon}) \subseteq S_{i,\alpha}$ such that $\frac{\mu(p_{i,\alpha} \Delta K_{i,\alpha,\epsilon})}{\mu(K_{i,\alpha,\epsilon})} < \epsilon$

Claim 2.6. For each $i < \theta$ and $\epsilon > 0$ rational, we can find $F_{i,\epsilon} \subseteq I^+$ and $\langle (n_{i,\epsilon,Y},N_{i,\epsilon,Y}) : Y \in F_{i,\epsilon} \rangle$ such that the following hold.

(1) $F_{i,\epsilon}$ is a countable family of pairwise disjoint sets and $T_i \setminus \bigsqcup F_{i,\epsilon} \subseteq I$.
(2) For each $Y \in F_{i,\epsilon}$, $\{K_{i,\alpha,\epsilon} : \alpha \in Y\}$ is a strong $\Delta$-system of width $(n_{i,\epsilon,Y},N_{i,\epsilon,Y})$. 


Proof. Fix $i < \theta$ and $\varepsilon > 0$ rational. To simplify notation, we write $K_\alpha$ instead of $K_{i,\alpha,\varepsilon}$. It suffices to show that for every $\mathcal{I}$-positive $X \subseteq T_i$, there exists $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and there exist $(n_Y, N_Y)$ such that $\{K_\alpha : \alpha \in Y\}$ is a strong $\Delta$-system of width $(n_Y, N_Y)$. Since then we can take $\mathcal{F}_{i,\varepsilon}$ to be a maximal disjoint family of such $Y$'s. That each $\mathcal{F}_{i,\varepsilon}$ is countable follows from the fact that $\mathcal{I}$ is $\omega_1$-saturated.

Fix a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $\alpha \in T_i \cap \gamma$, $\text{max}(\text{supp}(K_\alpha)) < \gamma$. Suppose $X \subseteq T_i \cap E$ and $X \in \mathcal{I}^+$. Since $\mathcal{I}$ is normal and the map $\alpha \mapsto \text{max}(\text{supp}(K_\alpha \cap \alpha))$ is regressive on $X$, we can find $R \subseteq \kappa$ finite and $Y_1 \subseteq X$ such that $Y_1 \in \mathcal{I}^+$, $(\forall \alpha \in Y_1)(\text{supp}(K_\alpha) \cap \alpha = R)$ and $\text{supp}(K_\alpha) \setminus R = n_\ast$ does not depend on $\alpha \in Y_1$. It also follows that $(\text{supp}(K_\alpha) : \alpha \in Y_1)$ forms a $\Delta$-system with root $R$. For each $\alpha \in Y_1$, let $K_\alpha = \bigcup_{n_\ast < N_\ast}[f_{\alpha, n}]$ where each $f_{\alpha, n} : \text{supp}(K_\alpha) \rightarrow 2$. Choose $Y_2 \subseteq Y_1$ such that $Y_2 \in \mathcal{I}^+$ and $N_\ast = N_\ast$ does not depend on $\alpha \in Y_2$. Finally, choose $Y \subseteq Y_2$ such that $Y \in \mathcal{I}^+$ and $\{K_\alpha : \alpha \in Y\}$ is a strong $\Delta$-system of width $(n_\ast, N_\ast)$. \hfill \Box

Since $\mathcal{I}$ is supersaturated, we can choose $B \in [\kappa]^{\omega_0}$ such that for every $i < \theta$, $\varepsilon > 0$ rational and $Y \in \mathcal{F}_{i,\varepsilon}$, we have $|B \cap Y| = N_0$. It suffices to show that for each $i < \theta$, $\{p_\alpha : \alpha \in B\}$ is predense in $\mathcal{P}$.

Suppose not. Fix $i < \theta$ and $p \subseteq 2^\lambda$ Baire such that $\mu(p) > 0$ and for every $\alpha \in B$, $\mu(p_\alpha \cap p) = 0$. Let $X = \{\alpha \in T_i : \mu(p_\alpha \cap p) > 0\}$. By Claim 2.3, $X \in \mathcal{I}^+$. Using the argument in the proof of Claim 2.3, we can choose $\varepsilon > 0$ rational, $X_\ast \subseteq X$ and $n_\ast, N_\ast < \omega$ such that

(a) $X_\ast \in \mathcal{I}^+$ and for each $\alpha \in X_\ast$, $\mu(p_\alpha \cap p) \geq 4\varepsilon$.

(b) $\{K_{i,\alpha,\varepsilon} : \alpha \in X_\ast\}$ is a strong $\Delta$-system of width $(n_\ast, N_\ast)$.

Choose $Y \in \mathcal{F}_{i,\varepsilon}$ such that $Y \cap X_\ast \in \mathcal{I}^+$. Since $|Y \cap X_\ast| \geq N_0$ and $|Y \cap B| = N_0$, by Lemma 2.5, we can choose $\alpha \in Y \cap B$ such that $\mu(p \cap K_{i,\alpha} \varepsilon) \geq 2\varepsilon$. But since $\mu(p_\alpha \Delta K_{i,\alpha,\varepsilon}) \leq \varepsilon \mu(K_{i,\alpha,\varepsilon}) \leq \varepsilon$, it follows that $\mu(p \cap p_\alpha) \geq \varepsilon > 0$: Contradiction. This completes the proof of Theorem 2.2. \hfill \Box

**Theorem 2.7.** Every $\sigma$-linked forcing is $\kappa$-ssp for every $\kappa$.

**Proof.** Let $\mathcal{I}$ be a normal supersaturated ideal on $\kappa$. Suppose $\mathcal{P}$ is a $\sigma$-linked forcing and $\mathcal{J}$ is the ideal generated by $\mathcal{I}$ in $V^\mathcal{P}$. Fix $\theta < \kappa$ and WLOG, assume that the trivial condition forces that $\{\dot{A}_i : i < \theta\}$ is a sequence of $\mathcal{J}$-positive sets. It suffices to construct $X \in [\kappa]^{\omega_0}$ such that $\Vdash_{\mathcal{P}} (\forall i < \theta)(X \cap \dot{A}_i \neq \emptyset)$.

Since $\mathcal{P}$ is $\sigma$-linked, we can write $\mathcal{P} = \bigcup\{L_n : n < \omega\}$ where each $L_n \subseteq \mathcal{P}$ has pairwise compatible members. For each $i < \theta$ and $n < \omega$, define

$$B_{i, n} = \{\alpha < \kappa : \exists p \in L_n)(p \vdash \alpha \in \dot{A}_i)\}$$

**Claim 2.8.** $W_i = \bigcup\{L_n : n < \omega, B_{i, n} \in \mathcal{I}^+\}$ is dense in $\mathcal{P}$.

**Proof.** Suppose not and fix $p \in \mathcal{P}$ such that no extension of $p$ lies in $W_i$. Put $C = \{\alpha < \kappa : (\exists q \leq p)(q \vdash \alpha \in \dot{A}_i)\}$. Since no extension of $p$ lies in $W_i$, it follows that $C \subseteq \bigcup\{B_{i, n} : n < \omega, B_{i, n} \in \mathcal{I}\}$ and hence $C \in \mathcal{I}$. It now follows that $p \vdash \dot{A}_i \in \mathcal{J}$ which is impossible. \hfill \Box

Since $\mathcal{I}$ is supersaturated, we can find a countable $X \subseteq \kappa$ such that for every $i < \theta$ and $n < \omega$, if $B_{i, n} \in \mathcal{I}^+$, then $X \cap B_{i, n} \neq \emptyset$. We claim that $\Vdash (\forall i < \theta)(X \cap \dot{A}_i \neq \emptyset)$.
Suppose not and fix \( p \in P \) and \( i < \theta \) such that \( p \Vdash X \cap \dot{A}_i = \emptyset \). Using Claim \[2.8\] choose \( n < \omega \) and \( p' \leq p \) such that \( p' \in L_n \) and \( B_{i,n} \in \mathcal{I}^+ \). Choose \( \alpha \in B_{i,n} \cap X \) and \( q \in L_n \) such that \( q \Vdash \alpha \in \dot{A}_i \). Since \( L_n \) is linked, we can find a common extension \( r \in P \) of \( p', q \). But \( r \Vdash \alpha \in X \cap \dot{A}_i \): Contradiction.

**Corollary 2.9.** Each of the following forcings is ssp: Cohen, random, Amoeba, Hechler, Eventually different real forcing.

We do not know if we can improve Theorem \[2.7\] to the class of \( \sigma \)-finite-cc forcings. For example, one can ask the following.

**Question 2.10.** Suppose \( \mathbb{B} \) is a boolean algebra and \( m : \mathbb{B} \to [0,1] \) is a strictly positive finitely additive measure on \( \mathbb{B} \). Must \( \mathbb{B} \) be supersaturation preserving?

The next two facts are well known.

**Fact 2.11.** Suppose \( P \) is a separative \( \sigma \)-linked forcing. Then \( |P| \leq \mathfrak{c} \).

**Fact 2.12.** Let \( \langle P_\xi, \dot{Q}_\xi \rangle : \xi < \lambda \) be a finite support iteration with limit \( P_\lambda \) where for every \( \xi < \lambda \), \( V^P_\xi \models \dot{Q}_\xi \) is \( \sigma \)-linked. Assume \( \lambda < \mathfrak{c}^+ \). Then \( P_\lambda \) is also \( \sigma \)-linked.

**Theorem 2.13.** Let \( \mathcal{I} \) be a normal supersaturated ideal on \( \kappa \) and let \( \lambda \leq \kappa^+ \). Suppose \( \langle \langle P_\xi, \dot{Q}_\xi \rangle : \xi < \lambda \rangle \) is a finite support iteration with limit \( P_\lambda \) where for every \( \xi < \lambda \), \( V^P_\xi \models \dot{Q}_\xi \) is \( \sigma \)-linked. Let \( \mathcal{J} \) be the ideal generated by \( \mathcal{I} \) in \( V^{P_\lambda} \). Then \( \mathcal{J} \) is supersaturated.

**Proof.** By induction on \( \lambda \). First suppose \( \kappa \leq \mathfrak{c} \). If \( \lambda < \kappa^+ \), then by Fact \[2.12\] \( P_\lambda \) is \( \sigma \)-linked and the claim holds by Theorem \[2.7\] So assume \( \lambda = \kappa^+ \) and fix any \( P_\lambda \)-generic filter \( G_\lambda \) over \( V \). Let \( \langle A_i : i < \theta \rangle \) be a sequence of \( \mathcal{J} \)-positive sets in \( V[G_\lambda] \) where \( \theta < \kappa \). Since \( P_\lambda \) is a finite support iteration of ccc forcings, there exists \( \eta < \lambda = \kappa^+ \) such that \( \langle A_i : i < \theta \rangle \in V[G_\eta] \) where \( G_\eta = P_\eta \cap G_\lambda \). Note that each \( A_i \) is \( J_\eta \)-positive where \( J_\eta \) is the ideal generated by \( \mathcal{I} \) in \( V[G_\eta] \). By inductive hypothesis, there is a countable set that meets \( A_i \) for every \( i < \theta \). Hence \( \mathcal{J} \) is supersaturated.

Next assume \( \kappa > \mathfrak{c} \). Then \( \kappa \) is measurable and \( \mathcal{I} \) is a normal prime ideal on \( \kappa \). First suppose \( \lambda \leq \kappa \). By Fact \[2.11\] \( |P_\xi| \leq |\xi \cdot \kappa| < \kappa \) for every \( \xi < \kappa \). Hence by Theorem 4.9 in [4], it follows that \( \mathcal{J} \) is supersaturated. Next suppose \( \kappa < \lambda \leq \kappa^+ \). Note that \( V^{P_\lambda} \models V \) \( \mathfrak{c} \geq \kappa \) since Cohen reals are added at each stage of cofinality \( \omega \). So we can work in \( V^{P_\lambda} \) and repeat the argument for the case \( \kappa \leq \mathfrak{c} \).

It is now natural to ask the following.

**Question 2.14 ([3]).** Suppose \( \kappa \) is measurable. Is every ccc forcing \( \kappa \)-ssp?

In Section [4] we’ll show that the answer is negative. We end this section with the following weaker positive result.

**Theorem 2.15.** Suppose \( \kappa \) is measurable and \( \mathcal{I} \) is a normal prime ideal on \( \kappa \). Let \( \mathbb{B} \) be a ccc complete boolean algebra. Then \( V^\mathbb{B} \models \text{“the ideal generated by } \mathcal{I} \text{ is } \omega_2 \text{-supersaturated.”} \)

**Proof.** It suffices to show that the following holds in \( V^\mathbb{B} \): For every \( A \subseteq \mathcal{J}^+ \), if \( |A| < \kappa \), then there exists \( X \in [\kappa]^\mathbb{B} \) such that \( X \) meets every member of \( A \).
Suppose $\theta < \kappa$ and $\|\mathcal{B}\{\mathcal{A}_i : i < \theta\} \subseteq \mathcal{I}^+$. Choose $Y \subseteq \kappa$ of $\mathcal{I}$-measure one such that for every $i < \theta$ and $\alpha \in Y$, $p_{i,\alpha} = [\{\alpha \in \mathcal{A}_i\}] > 0_\mathcal{B}$. Using the inaccessibility of $\kappa$, the following claim is easy to check.

**Claim 2.16.** There exists $(\mathcal{B}_\alpha : \alpha < \kappa)$ such that the following hold.

1. $\mathcal{B}_\alpha \ll \mathcal{B}$ and $|\mathcal{B}_\alpha| < \kappa$.
2. $\mathcal{B}_\alpha$'s are increasing and continuous at $\alpha$ when $cf(\alpha) > \kappa_0$.
3. $\{p_{i,\beta} : \beta < \alpha, i < \theta\} \subseteq \mathcal{B}_\alpha$.

Let $\pi_\alpha : \mathcal{B} \to \mathcal{B}_\alpha$ be a projection map witnessing $\mathcal{B}_\alpha \ll \mathcal{B}$. Choose $f : \kappa \to \kappa$ such that for every $i < \theta$ and $\alpha < \kappa$, we have $\alpha < f(\alpha)$ and $p_{i,\alpha} \in \mathcal{B}_f(\alpha)$. Choose $Y_1 \subseteq Y$ of measure one and $\alpha_* < \kappa$ such that for every $i < \theta$, $\pi_\alpha(p_{i,\alpha}) = p_{i,*} \in \mathcal{B}_{\alpha_*}$ does not depend on $\alpha \in Y_1$ and range$(f \upharpoonright \alpha) \subseteq \alpha$ for every $\alpha \in Y_1$. Note that $p_{i,*} = 1_\mathcal{B}$ since $\|\mathcal{B}\mathcal{A}_i \subseteq \mathcal{I}^+$. Let $X \subseteq Y \setminus \alpha_*$ be such that otp$(X) = \omega_1$ and for every $\alpha < \beta$ in $X$, $f(\alpha) < f(\beta)$.

**Claim 2.17.** For every $i < \theta$, $\{p_{i,\alpha} : \alpha \in X\}$ is predense in $\mathcal{B}$.

**Proof.** Let $\sup(X) = \gamma_*$. Then $cf(\gamma_*) = \aleph_1$ and hence $\mathcal{B}_{\gamma_*} = \bigcup\{\mathcal{B}_\gamma : \gamma \in X\}$. Fix $i < \theta$. Given $p \in \mathcal{B}$, choose $\gamma \in X$ such that $\pi_{\gamma_1}(p) \in \mathcal{B}_{\gamma}$. Now since

$$\mathcal{B} = \mathcal{B}_\gamma \ast \mathcal{B}_{\gamma_2}/\mathcal{B}_\gamma \ast \mathcal{B}/\mathcal{B}_{\gamma},$$

we can decompose $p = (\pi_{\gamma_1}(p), 1, x)$ and $p_{i,\gamma} = (1, y, 1)$. Hence $p, p_{i,\gamma}$ are compatible. \hfill \Box

It follows that $\mathcal{J}$ is $\omega_2$-supersaturated. \hfill \Box

3. **Consistently, there are $\omega_1$-saturated ideals on $\mathfrak{c}$ and all of them are supersaturated**

The aim of this section is to show that it is consistent that every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Theorem 3.1.** It is consistent that there is a normal supersaturated ideal on $\mathfrak{c}$ and every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Lemma 3.2.** Suppose that every $\sigma$-ideal $\mathcal{I}$ satisfying $(i)$-$(iv)$ below is supersaturated.

1. $\mathcal{I}$ is a uniform ideal on $\lambda$,
2. $\mu \leq \lambda$,
3. for every $X \in \mathcal{I}^+$, add$(\mathcal{I} \upharpoonright X) = \mu$ and $\mathcal{I}$ is $\omega_1$-saturated.

Then every $\omega_1$-saturated $\sigma$-ideal is supersaturated.

**Proof.** Suppose $\mathcal{J}$ is an $\omega_1$-saturated $\sigma$-ideal on $X$. Note that for every $A \in \mathcal{J}^+$, there exists $B \subseteq A$ such that $(\ast)_B$ holds where

$$(\ast)_B$$

says the following: $B \in \mathcal{J}^+, [B]^{<|B|} \subseteq \mathcal{J}$ and for every $C \subseteq B$, if $C \in \mathcal{J}^+$, then add$(\mathcal{J} \upharpoonright C) = \text{add}(\mathcal{J} \upharpoonright B)$.

Since $\mathcal{J}$ is $\omega_1$-saturated, we can find a countable partition $\mathcal{F}$ of $X$ such that for each $B \in \mathcal{F}$, $(\ast)_B$ holds. Now by assumption, each $\mathcal{J} \upharpoonright B$ is supersaturated. Hence $\mathcal{J}$ is also supersaturated. \hfill \Box
Lemma 3.3. Suppose $P$ is a ccc forcing, $\kappa > c$ and $V^P \models \mathcal{J}$ is a $\kappa$-complete $\omega_1$-saturated uniform ideal on $\lambda$. Let $\mathcal{I} = \{X \subseteq \kappa : 1_p \models X \in \mathcal{J}\}$. Then there is a countable partition $\mathcal{F}$ of $\lambda$ such that for every $A \in \mathcal{F}$, $\mathcal{I} \upharpoonright A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a $\kappa$-complete prime ideal on $\lambda$.

Proof. It is clear that $\mathcal{I}$ is a $\kappa$-complete uniform ideal on $\lambda$. Suppose $\mathcal{F} \subseteq \mathcal{I}^+$ is an uncountable family of pairwise disjoint sets. For each $A \in \mathcal{F}$, choose $p_A \in P$ such that $p_A \models A \not\in \mathcal{J}$. Since $P$ is ccc, some $p \in P$ forces uncountably many $p_A$’s into the $P$-generic filter. But this contradicts the fact that $\mathcal{J}$ is $\omega_1$-saturated in $V^P$. So $\mathcal{I}$ is $\omega_1$-saturated. Since $\mathcal{I}$ is $\kappa$-complete and $\kappa > c$, $\mathcal{I}$ is nowhere atomless. Hence there is a countable partition $\mathcal{F}$ of $\lambda$ such that for every $A \in \mathcal{F}$, $\mathcal{I} \upharpoonright A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a $\kappa$-complete prime ideal on $\lambda$. □

Lemma 3.4. Suppose $\kappa$ is an inaccessible cardinal and $U$ is a $\kappa$-complete uniform ultrafilter on $\lambda$. Let $P = \text{Cohen}_\kappa$. Let $\mathcal{J}$ be the ideal generated by the dual ideal of $U$ in $V^P$. Then for each $A \subseteq \mathcal{J}^+$, if $|A| < \kappa$, then there exists a countable set that meets every member of $A$.

Proof. We identify conditions $p \in P$ as members of the Baire algebra on $2^\kappa$ which is the $\sigma$-algebra generated by clopen subsets of $2^\kappa$. Note that for every Baire $p \subseteq 2^\kappa$ there is a countable $S \subseteq \kappa$ such that for every $x, y \in 2^\kappa$ satisfying $x \upharpoonright S = y \upharpoonright S$, we have $x \in p$ if and only if $y \in p$. We call such an $S$, a support of $p$. The ordering on Cohen is defined by $p \leq q$ if $p \setminus q$ is meager in $2^\kappa$. Recall that if $p \subseteq 2^\kappa$ is Baire and $S \in [\kappa]^{\omega_1}$ is a support of $p$ then there is a countable family $\mathcal{P}$ of clopen subsets of $2^\kappa$ each supported in $S$ such that the symmetric difference of $p$ and $\bigcup \mathcal{P}$ is meager. So $p$ is completely determined by the family $\mathcal{P}$.

It is clear that $\mathcal{J}$ is a $\kappa$-complete uniform ideal on $\lambda$. Suppose $\theta < \kappa$ and $\langle \dot{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets in $V^P$. WLOG, assume that the trivial condition forces this. For $i < \theta$ and $\alpha < \lambda$, let $p_{i,\alpha} = [\{\alpha \in A_i\}]_P$. Note that for each $i < \theta$, and $Z \in U$, $\{p_{i,\alpha} : \alpha \in Z\}$ is predense in $P$ since otherwise some condition will force $\dot{A}_i \in \mathcal{J}$. Since $U$ is $\kappa$-complete, we can choose $X \in U$ such that for every $i < \theta$ and $\alpha \in X$, $p_{i,\alpha} > 0_P$. Let $S_{i,\alpha} \in [\kappa]^{\omega_1}$ be a support of $p_{i,\alpha}$. Since $\kappa$ is inaccessible, we can choose $Y \subseteq X$ such that $Y \in U$ and for each $i < \theta$, the following hold.

(a) For every $\alpha, \beta \in Y$, $(S_{i,\alpha}, 2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (S_{i,\beta}, 2^{S_{i,\beta}}, p_{i,\beta})$. Put $\text{otp}(S_{i,\alpha}) = \gamma_i$. Let $h_{i,\alpha} : \gamma_i \rightarrow S_{i,\alpha}$ be the order isomorphism and define $H_{i,\alpha} : 2^{\gamma_i} \rightarrow 2^{S_{i,\alpha}}$ by $H_{i,\alpha}(x) = x \circ h_{i,\alpha}^{-1}$. Choose $p_i \subseteq 2^{\gamma_i}$ such that $H_{i,\alpha}[p_i] = p_{i,\alpha}$.

(b) For each $\gamma < \gamma_i$, either $|\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1$ or for every $Z \in U$, $|\{h_{i,\alpha}(\gamma) : \alpha \in Z \cap Y\}| \geq \kappa$. Put $\Gamma_i = \{\gamma < \gamma_i : |\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1\}$ and $h_{i,\alpha}[\Gamma_i] = R_i$.

Define

$$B_{i,\alpha} = \{x \in 2^{R_i} : \{y \upharpoonright (S_{i,\alpha} \setminus R_i) : y \in p_{i,\alpha} \land y \upharpoonright R_i = x\} \text{ is meager}\}.$$ 

Then $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y$ and $B_i$ is meager in $2^{R_i}$ since otherwise $\{p_{i,\alpha} : \alpha \in Y\}$ will not be predense in $P$.

Using (b), choose $B \in [Y]^{\omega_1}$ such that for every $i < \theta$ and $\alpha \neq \beta$ in $B$, $S_{i,\alpha} \cap S_{i,\beta} = R_i$. It follows now that for every $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in $P$. Hence $\models (\forall i < \theta)(B \cap \dot{A}_i \neq \emptyset)$. □
Proof of Theorem 3.1 Let $V \models "\kappa = \omega_1 \text{ and } \kappa \text{ is the least measurable cardinal}"$. Let $\mathbb{P} = \text{Cohen}_\kappa$. We already know that there is a normal supersaturated ideal on $\kappa = \kappa$ in $V^\mathbb{P}$. Let us check that, $V^\mathbb{P} \models "\text{Every } \omega_1\text{-saturated } \sigma\text{-ideal is supersaturated}"$. By Lemma 3.2, it suffices to consider ideals $\mathcal{J}$ that satisfy the following for some $\omega_1 \leq \mu \leq \lambda$.

(i) $\mathcal{J}$ is a uniform ideal on $\lambda$,

(ii) for every $X \in \mathcal{J}^+$, $\text{add}(\mathcal{J} \upharpoonright X) = \mu$ and

(ii) $\mathcal{J}$ is $\omega_1$-saturated.

Since $V^\mathbb{P} \models \kappa = \kappa$, we can assume that $\mu \leq \kappa$. Otherwise there is a countable partition $\mathcal{E}$ of $\lambda$ into $\mathcal{J}$-positive sets such that for each $X \in \mathcal{E}$, $\mathcal{J} \upharpoonright X$ is a $\mu$-complete prime ideal and it easily follows that $\mathcal{J}$ is supersaturated.

Towards a contradiction, suppose $\mu < \kappa$. Working in $V^\mathbb{P}$, define an ideal $\mathcal{K}$ on $\mu$ as follows. Since $\text{add}(\mathcal{J}) = \mu$, we can choose a family $\{A_i : i < \mu\} \subseteq \mathcal{J}$ of pairwise disjoint sets such that $\bigcup_{i < \mu} A_i \in \mathcal{J}^+$. Define

$$\mathcal{K} = \{\Gamma \subseteq \mu : \bigcup\{A_i : i \in \Gamma\} \in \mathcal{J}\}$$

It is easy to see that $\mathcal{K}$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. For simplicity, assume that $1_p \Vdash \mathcal{K}$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. Coming back to $V$, define $\mathcal{K}' = \{X \subseteq \mu : 1_p \Vdash X \in \mathcal{K}\}$. It is clear that $V \models \mathcal{K}'$ is a $\mu$-additive ideal on $\mu$. We claim that $V \models \mathcal{K}'$ is $\omega_1$-saturated. Suppose not and fix $\langle (A_\xi, p_\xi) : \xi < \omega_1 \rangle$ such that $A_\xi$’s are pairwise disjoint subsets of $\mu$ and for every $\xi < \omega_1$, $p_\xi \Vdash A_\xi \notin \mathcal{K}$. Since $\mathbb{P}$ is ccc, we can find some $p_\kappa \in \mathbb{P}$ that forces uncountable many $p_\xi$’s into the generic $G_\mathbb{P}$. But this means that $p_\kappa \Vdash \mathcal{K}$ is not $\omega_1$-saturated which is impossible. So $V \models \mathcal{K}'$ is $\omega_1$-saturated. So $\mu$ is weakly inaccessible in $V$. Since $V \models \mu > \omega_1 = \kappa$, it follows that $\mu$ must be measurable in $V$. But $\kappa$ is the least measurable cardinal in $V$. Hence $\mu \geq \kappa$: Contradiction.

So we must have $\mu = \kappa$. Let $\mathcal{I} = \{Y \subseteq \lambda : 1_p \Vdash X \in \mathcal{J}\}$. By Lemma 3.3 there is a countable partition $\mathcal{F}$ of $\lambda$ such that for each $X \in \mathcal{F}$, $\mathcal{I} \upharpoonright X$ is a $\kappa$-complete prime ideal on $\lambda$. For each $X \in \mathcal{F}$, let $\mathcal{I}_X$ be the ideal generated by $\mathcal{I} \upharpoonright X$ in $V^\mathbb{P}$. By Lemma 3.4 for every $\mathcal{A} \subseteq \mathcal{I}_X^+$, if $|\mathcal{A}| < \kappa$, then there is a countable set that meets every member of $\mathcal{A}$. Since $\mathcal{I}_\lambda \subseteq \mathcal{I} \upharpoonright A$ and $\text{add}(\mathcal{J} \upharpoonright A) = \kappa$, it follows that $\mathcal{J} \upharpoonright A$ is supersaturated for each $A \in \mathcal{F}$. Since $\mathcal{F}$ is a countable partition of $\lambda$, it follows that $\mathcal{J}$ is also supersaturated.

4. Killing supersaturated ideals

Definition 4.1. Suppose $\delta < \omega_1$ is indecomposable and $\kappa$ is an infinite cardinal. Let $Q_\delta^\kappa$ consist of all countable partial maps from $\kappa$ to $2$ such that

1. $\text{otp}(\text{dom}(p)) < \delta$ and
2. $\{\xi \in \text{dom}(p) : p(\xi) = 1\}$ is finite.

For $p, q \in Q_\delta^\kappa$ define $p \leq q$ iff $q \subseteq p$. Let $\mathbb{P}_\kappa$ be the finite support product of $\{Q_\delta^\kappa : \delta < \omega_1, \delta \text{ indecomposable}\}$.

Lemma 4.2. Let $\mathbb{P}_\kappa$ be as in Definition 4.1.

1. $\mathbb{P}_\kappa$ is ccc.
2. If $\kappa \geq \omega_1$, then $\mathbb{P}_\kappa$ is not $\sigma$-finite-cc.
Proof. (1) Towards a contradiction, suppose \( A = \{ p_i : i < \omega_1 \} \) is an uncountable antichain in \( \mathbb{P}_\kappa \). Put \( D_i = \text{dom}(p_i) \). By passing to an uncountable subset of \( A \), we can assume that \( D_i \)'s form a \( \Delta \)-system with root \( D \). For each \( \delta \in D \) and \( i < \omega_1 \), put \( s_{i,\delta} = \{ \gamma : p_i(\delta)(\gamma) = 1 \} \) and \( X_{i,\delta} = \{ \gamma : p_i(\delta)(\gamma) = 0 \} \). Note that \( \text{otp}(X_{i,\delta}) < \delta \).

Choose \( B \in [A]^{\omega_1} \) such that for each \( \delta \in D \), \( \langle s_{i,\delta} : i \in B \rangle \) is a \( \Delta \)-system with root \( s_\delta \) and for every \( i < j \) in \( B \), \( s_{j,\delta} \cap X_{i,\delta} = \emptyset \).

Choose \( j \in B \) and \( \delta \in D \) such that letting \( C = \{ i \in B \cap j : p_i(\delta) \perp Q, p_j(\delta) \} \), every transversal of \( \{ s_{i,\delta} \setminus s_\delta : i \in C \} \) has order type \( \geq \delta \). Now observe that \( X_{j,\delta} \) has to meet \( s_{i,\delta} \setminus s_\delta \) for every \( i \in C \). Hence \( \text{otp}(X_{j,\delta}) \geq \delta \): Contradiction.

(2) It is enough to show that \( Q = Q^{\omega_1}_\omega \) is not \( \sigma \)-finite-cc. Towards a contradiction, suppose \( Q = \bigcup_{n < \omega} W_n \) where no \( W_n \) has an infinite antichain. Choose \( \langle A_n : n < \omega \rangle \) as follows.

(a) \( A_0 \subseteq W_0 \) is a maximal antichain of conditions \( p \) such that \( \max(\text{dom}(p)) = \gamma_p \) exists and \( p(\gamma_p) = 1 \). Define \( \gamma_0 = \max(\{ \gamma_p : p \in A_0 \}) \).

(b) \( A_{n+1} \subseteq W_{n+1} \) is a maximal antichain of conditions \( p \in W_{n+1} \) such that \( \max(\text{dom}(p)) = \gamma_p \) exists, \( \gamma_p > \gamma_n \) and \( p(\gamma_p) = 1 \). If \( A_{n+1} \neq \emptyset \), define \( \gamma_{n+1} = \max(\{ \gamma_p : p \in A_{n+1} \}) \). Otherwise, \( \gamma_{n+1} = \gamma_n \).

Put \( A = \bigcup_{n < \omega} A_n \) and \( \gamma = \sup(\{ \gamma_n : n < \omega \}) \). Fix \( \gamma_\ast \in (\gamma, \omega_1) \). Let \( p_* \) be defined by \( \text{dom}(p_*) = \{ \gamma_p : p \in A \} \cup \{ \gamma_\ast \} \) and for every \( \xi \in \text{dom}(p_*) \), \( p(\xi) = 1 \) iff \( \xi = \gamma_\ast \). Note that \( \text{otp}(\text{dom}(p_*)) \leq \omega + 1 < \omega^2 \) and hence \( p_* \in Q \). Choose \( n < \omega \) such that \( p_* \in W_n \). But now \( A_n \cup \{ p_* \} \subseteq W_n \) is an antichain which contradicts the maximality of \( A_n \).

\( \square \)

Theorem 4.3. Suppose \( \omega_1 \leq \kappa \leq \lambda \), \( I \) is an \( \omega_1 \)-saturated uniform ideal on \( \lambda \) and \( \text{add}(I) = \kappa \). Let \( \mathbb{P}_\kappa \) be as in Definition 4.2. Let \( J \) be the ideal generated by \( I \) in \( V^{\mathbb{P}_\kappa} \). Then there exists \( A \subseteq J^+ \) such that \(|A| = \omega_1 \) and there is no countable set that meets every member of \( A \). Hence \( V^{\mathbb{P}_\kappa} \models J \) is an \( \omega_1 \)-saturated \( \kappa \)-complete uniform ideal on \( \lambda \) which is not supersaturated.

Proof. As \( \mathbb{P}_\kappa \) is ccc, it is easy to see that in \( V^{\mathbb{P}_\kappa} \), \( J \) is an \( \omega_1 \)-saturated \( \kappa \)-complete uniform ideal on \( \lambda \). So it suffices to show that in \( V^{\mathbb{P}_\kappa} \), there exists \( A \subseteq J^+ \) such that \(|A| = \omega_1 \) and there is no countable set that meets every member of \( A \).

Since \( \text{add}(I) = \kappa \), we can fix \( Y \in I^+ \) and a partition \( Y = \bigcup_{\alpha < \kappa} W_\alpha \) such that for each \( \Gamma \in [\kappa]^{<\kappa} \), \( \bigcup_{\alpha \in \Gamma} W_\alpha \in I \). Let \( G \) be \( \mathbb{P}_\kappa \)-generic over \( V \). Let \( G_\delta = \{ p(\delta) : p \in G \} \). So \( G_\delta \) is \( Q_{\delta} \)-generic over \( V \). Define \( A_\delta \in V^{\mathbb{P}_\kappa} \cap P(\lambda) \) by

\[
\gamma \in A_\delta \iff (\exists p \in G)(p(\delta)(\alpha) = 1 \land \gamma \in W_\alpha)
\]

Suppose \( Y \in I \) and \( p \in \mathbb{P}_\kappa \) with \( \delta \in \text{dom}(p) \). Choose \( \alpha < \kappa \) such that \( W_\alpha \setminus Y \neq \emptyset \) and \( \alpha \notin \text{dom}(p(\delta)) \). Let \( q \leq p \) be such that \( q(\delta)(\alpha) = 1 \). Then \( q \Vdash_{\mathbb{P}_\kappa} A_\delta \setminus Y \neq \emptyset \). Hence \( \Vdash_{\mathbb{P}_\kappa} A_\delta \in J^+ \).

Towards a contradiction suppose that in \( V^{\mathbb{P}_\kappa} \), there is a countable \( X \subseteq \lambda \) that meets each \( A_\delta \). Since \( \mathbb{P} \) satisfies ccc, we can assume that \( X \in V \). Fix \( p \in \mathbb{P}_\kappa \) such that \( p \Vdash (\forall \delta)(X \cap A_\delta \neq \emptyset) \). Put \( W = \{ \alpha < \kappa : W_\alpha \cap X \neq \emptyset \} \). So \( W \subseteq \kappa \) is countable. Choose \( \delta \in \omega_1 \setminus \text{dom}(p) \) indecomposable such that \( \delta > \text{otp}(W) \). Define
Suppose fact that $J$ is nowhere prime iff every $J$-positive set can be partitioned into two $J$-positive subsets.

**Fact 4.6.** Suppose $I_1, I_2$ are $\omega_1$-saturated $\sigma$-ideals on $X$ and $I_1 \subseteq I_2$. Then there is a partition $X = A \sqcup B$ such that $A \in I_2$ and $I_2 \upharpoonright B = I_1 \upharpoonright B$.

**Proof.** Take $A$ to be the union of a maximal family of pairwise disjoint sets in $I_2 \setminus I_1$. □

The following lemma will be used in the proofs of Theorems 4.5 and 4.8(d).

**Lemma 4.7.** Suppose $\mathcal{J}$ is a nowhere prime supersaturated ideal on $X$ and $\mu = \text{add}(\mathcal{J})$. Then $\mu \leq \kappa$ and there exists a $\mu$-additive supersaturated ideal on $\mu$.

**Proof.** Towards a contradiction, suppose $\mu > \kappa$. Construct a tree $\langle A_\sigma : \sigma \in 2^{<\omega_1} \rangle$ of subsets of $X$ as follows.

(i) $A_\emptyset = X$.

(ii) If $A_\sigma \in \mathcal{J}^+$, then $\{A_{\sigma_0}, A_{\sigma_1}\}$ is a partition of $A_\sigma$ into two $\mathcal{J}$-positive sets.

This is possible since $\mathcal{J}$ is nowhere prime.

(iii) If $A_\sigma \in \mathcal{J}$, then $A_{\sigma_0} = A_{\sigma_1} = A_\sigma$.

(iv) If $\alpha < \omega_1$ is limit and $\sigma \in 2^\alpha$, then $A_\sigma = \bigcap\{A_{\sigma|\beta} : \beta < \alpha\}$.

Put $\mathcal{F} = \{A_\sigma : \sigma \in 2^{<\omega_1} \text{ and } A_\sigma \in \mathcal{J}\}$. We claim that $X = \bigcup \mathcal{F}$. Suppose not and fix $y \in X \setminus \bigcup \mathcal{F}$. Now observe that $\{A_{\sigma_k} : \sigma \in 2^{<\omega_1} \land k < 2 \land y \in (A_\sigma \setminus A_{\sigma_k})\}$ is an uncountable family of pairwise disjoint $\mathcal{J}$-positive sets which contradicts the fact that $\mathcal{J}$ is $\omega_1$-saturated. So $X = \bigcup \mathcal{F}$. But since $|\mathcal{F}| \leq |2^{<\omega_1}| = \kappa$, this contradicts the fact that $\text{add}(\mathcal{J}) = \mu > \kappa$. Hence $\mu \leq \kappa$.

Since $\text{add}(\mathcal{J}) = \mu$, there are $Y \in \mathcal{J}^+$ and a partition $Y = \bigcup_{\alpha < \mu} W_\alpha$ such that for every $\Gamma \in |\mu|^{<\mu}$, $\bigcup_{\alpha \in \Gamma} W_\alpha \in \mathcal{J}$. Define

$$\mathcal{K} = \{\Gamma \subseteq \mu : \bigcup_{\alpha \in \Gamma} W_\alpha \in \mathcal{J}\}$$

Then $\mathcal{K}$ is a $\mu$-additive $\omega_1$-saturated ideal on $\mu$. So $\mu$ is weakly inaccessible.

We claim that $\mathcal{K}$ must also be supersaturated. To see this, suppose $A \subseteq \mathcal{K}^+$ and $|A| < \mu$. For each $A \in \mathcal{A}$, define $Y_A = \bigcup_{\alpha \in A} W_\alpha$. Then $\{Y_A : A \in \mathcal{A}\} \subseteq \mathcal{J}^+$. 

\[ q \in \mathbb{P}_\kappa \text{ by } \text{dom}(q) = \text{dom}(p) \cup \{\delta\}, q \upharpoonright \text{dom}(p) = p \text{ and } q(\delta) \in \mathbb{Q}_\delta \text{ is constantly zero on } W. \]
Since \( J \) is supersaturated, we can choose a countable \( T \subseteq Y \) that meets \( Y_A \) for every \( A \in \mathcal{A} \). Let \( B = \{ \alpha < \mu : T \cap W_\alpha \neq \emptyset \} \). Then \( B \subseteq \mu \) is countable (as \( W_\alpha \)'s are pairwise disjoint) and it meets every \( A \in \mathcal{A} \). Hence \( \mathcal{K} \) is a \( \mu \)-additive supersaturated ideal on \( \mu \). \( \square \)

**Proof of Theorem 4.5** Clause (a) is easy to check. Let us prove Clause (b). Suppose \( J \) is a supersaturated ideal on \( X \). Put \( \mu = \text{add}(J) \). We claim that it suffices to show that \( V^\mathcal{S} \models \mu > \mathfrak{c} \). First note that, by Lemma 4.7, this would imply that for every \( Y \in J^+ \), there exists \( J \)-positive \( Z \subseteq Y \) such that \( J \upharpoonright Z \) is a prime ideal. Hence by \( \omega_1 \)-saturation of \( J \), we can find a countable partition of \( X \) into \( J \)-positive sets such that the restriction of \( J \) to each one of them is a prime ideal.

So towards a contradiction, assume \( V^\mathcal{S} \models \mu \leq \mathfrak{c} \). Fix \( Y \in J^+ \) such that for every \( J \)-positive \( Z \subseteq Y \), \( \text{add}(J \upharpoonright Z) = \mu \). Since \( \mu \leq \mathfrak{c} \), it follows that \( J \upharpoonright Y \) is a nowhere prime supersaturated ideal. Using Lemma 4.7 again, we can get a \( \mu \)-additive supersaturated ideal \( \mathcal{K} \) on \( \mu \). Let us assume that the trivial condition in \( \mathcal{S} \) forces all of this about \( \mathcal{K} \).

Since \( V^\mathcal{S} \models \text{“} \mu \leq \mathfrak{c} = \kappa^+ \text{”} \) and \( \mu \) is weakly inaccessible”, we must have \( \mu \leq \kappa \). We consider two cases.

Case \( \mu < \kappa \): In \( V \), define \( \mathcal{I}' = \{ X \subseteq \mu : 1_\mathcal{S} \vDash X \in \mathcal{K} \} \). Since \( \mathcal{S} \) is ccc, \( V \models \mathcal{I}' \) is a \( \mu \)-additive \( \omega_1 \)-saturated ideal on \( \mu \). As \( V \models \mu > \omega_1 = \mathfrak{c} \), \( \mu \) is measurable in \( V \).

Since \( \kappa \) is the least measurable cardinal in \( V \), \( \mu \geq \kappa \): Contradiction.

Case \( \mu = \kappa \): In \( V \), define \( \mathcal{I}' = \{ X \subseteq \kappa : 1_\mathcal{S} \vDash X \in \mathcal{K} \} \). Since \( V \models \kappa > \mathfrak{c} = \omega_1 \), we must have \( V \models \mathcal{I}' \) is a \( \kappa \)-additive prime ideal on \( \kappa \). Let \( \mathcal{K}' \) be the ideal generated by \( \mathcal{I}' \) in \( V^\mathcal{S} \). Then \( V^\mathcal{S} \models \mathcal{K}' \subseteq \mathcal{K} \) are \( \omega_1 \)-saturated \( \kappa \)-additive ideals on \( \kappa \). Using Fact 4.6, fix \( B \in \mathcal{K}' \) such that \( \mathcal{K}' \upharpoonright B = \mathcal{K} \upharpoonright B \).

Choose \( \gamma < \kappa^+ \) such that \( B \in V^{\mathcal{S}^\gamma} \). Let \( \mathcal{K}'' \) be the ideal generated by \( \mathcal{I}' \) in \( V^{\mathcal{S}^\gamma} \). By Theorem 4.3, it follows that in \( V^{\mathcal{S}^\gamma+1} \), the ideal generated by \( \mathcal{K}'' \upharpoonright B \) is not supersaturated. Now observe that \( \mathcal{K} \upharpoonright B = \mathcal{K}' \upharpoonright B \) is the ideal generated by \( \mathcal{K}'' \upharpoonright B \) in \( V^\mathcal{S} \). It follows that \( \mathcal{K} \) is not a supersaturated ideal: Contradiction. \( \square \)

Using some results about separating families and supersaturated ideals from [2, 4], we can also get the following.

**Theorem 4.8.** Suppose \( \kappa \) is a measurable cardinal with a witnessing normal prime ideal \( \mathcal{I} \). Let \( \mathbb{P}_\kappa \) be the forcing in Definition 4.4. Then the following hold in \( V^{\mathbb{P}_\kappa} \).

(a) \( \mathfrak{c} = \kappa \) and the ideal generated by \( \mathcal{I} \) is a normal \( \omega_1 \)-saturated ideal on \( \kappa \).

(b) There is a family \( \mathcal{F} \subseteq \mathcal{P}(\kappa) \) such that \( |\mathcal{F}| = \omega_1 \) and for every countable \( X \subseteq \kappa \) and \( \alpha \in \kappa \setminus X \), there exists \( S \in \mathcal{F} \) such that \( \alpha \in S \) and \( S \cap X = \emptyset \).

(c) The order dimension of Turing degrees is \( \omega_1 \).

(d) There are no nowhere prime supersaturated ideals.

**Proof.** (a) Since \( \mathbb{Q}_\kappa^\kappa \) adds \( \kappa \) Cohen reals, \( \mathfrak{c} \geq \kappa \). The other inequality follows by a name counting argument using the facts that \( \mathbb{P}_\kappa \) is a ccc forcing, \( |\mathbb{P}_\kappa| = \kappa \) and \( \kappa^\omega = \kappa \). That the ideal generated by \( \mathcal{I} \) is a normal \( \omega_1 \)-saturated ideal on \( \kappa \) follows from the fact that \( \mathbb{P}_\kappa \) is ccc.
(b) For each indecomposable \( \delta < \omega_1 \), define
\[
S_\delta = \{ \alpha < \kappa : (\exists p \in G_{P_\kappa})(\delta \in \text{dom}(p) \land p(\delta)(\alpha) = 1) \}
\]
Let \( \mathcal{F} = \{ S_\delta : \delta < \omega_1 \text{ is indecomposable} \} \). Suppose \( X \subseteq \kappa \) is countable and \( \alpha \in \kappa \setminus X \). We’ll find an \( S_\delta \in \mathcal{F} \) such that \( \alpha \in S_\delta \) and \( X \cap S_\delta = \emptyset \). Since \( P_\kappa \) is ccc, we can find a countable \( Y \in V \) such that \( X \subseteq Y \subseteq \kappa \setminus \{ \alpha \} \). Now an easy density argument shows that the set
\[
D_{\alpha,Y} = \{ p \in P_\kappa : (\exists \delta \in \text{dom}(p))(p(\delta)(\alpha) = 1 \land (\forall \beta \in Y)(p(\delta)(\beta) = 0)) \}
\]
is dense in \( P_\kappa \). So we can choose \( p \in D_{\alpha,Y} \cap G_{P_\kappa} \). Let \( \delta \) witness that \( p \in D_{\alpha,Y} \). Then it is clear that \( \alpha \in S_\delta \) and \( X \cap S_\delta \subseteq Y \cap S_\delta = \emptyset \).

(c) This follows from Theorem 3.9 in [2] and part (b) above.

(d) Suppose not. Then by Lemma 4.7, we can find some \( \mu \leq \mathfrak{c} = \kappa \) and a \( \mu \)-additive supersaturated ideal on \( \mu \). Define \( \mathcal{E} = \{ S \cap \mu : S \in \mathcal{F} \} \). Then \( |\mathcal{E}| = \omega_1 \) and for every countable \( X \subseteq \mu \) and \( \alpha \in \mu \setminus X \), there exists \( S \in \mathcal{E} \) such that \( \alpha \in S \) and \( S \cap X = \emptyset \). Now applying Lemma 4.2 in [4] gives us a contradiction. \( \square \)

We conclude with the following questions.

1. Suppose \( I, J \) are normal ideals on \( \kappa \), \( I \) is supersaturated and \( P(\kappa)/I \) is isomorphic to \( P(\kappa)/J \). Must \( J \) be supersaturated?
2. Suppose \( \kappa \) is regular uncountable, \( I \) is a \( \kappa \)-complete normal ideal on \( \kappa \) and \( P(\kappa)/I \) is a Cohen algebra. Must \( I \) be supersaturated?
3. Do \( \sigma \)-finite/bounded-cc forcings preserve supersaturation? What about Boolean algebras that admit a strictly positive finitely additive measure?

References