SUPERSATURATED IDEALS

ASHUTOSH KUMAR AND DILIP RAGHAVAN

Dedicated to the memory of Ken Kunen

Abstract. An ideal $\mathcal{I}$ on a set X is supersaturated iff $\mathsf{add}(\mathcal{I}) \geq \omega_2$ and for every family $\mathcal{F}$ of $\mathcal{I}$-positive sets with $|\mathcal{F}| < \mathsf{add}(\mathcal{I})$, there exists a countable set that meets every set in $\mathcal{F}$. We show that many well-known ccc forcings preserve supersaturation. We also show that the existence of supersaturated ideals is independent of ZFC plus "There exists an ω_1-saturated σ-ideal".

1. Introduction

Saturation properties of ideals are ubiquitous in modern set theory and there is a considerable body of work (for example, see [3, 5, 6, 7]) on the study of a large number of such properties. Throughout this paper, by an ideal $\mathcal{I}$ on X, we mean an ideal $\mathcal{I}$ on X that contains every finite subset of X. Supersaturation is a strengthening of ω_1-saturation defined as follows.

Definition 1.1. *Suppose $\mathcal{I}$ is an ideal on X and λ is a cardinal. We say that $\mathcal{I}$ is λ-supersaturated iff $\mathsf{add}(\mathcal{I}) \geq \lambda^+$ and for every $\mathcal{A} \subseteq \mathcal{I}^+$, if $|\mathcal{A}| < \mathsf{add}(\mathcal{I})$, then there exists $W \in [X]^{<\lambda}$ such that for every $A \in \mathcal{A}$, $A \cap W \neq \emptyset$. $\mathcal{I}$ is supersaturated iff it is ω_1-supersaturated.*

Suppose $\mathcal{I}$ is a supersaturated ideal on X. Since $\mathsf{add}(\mathcal{I}) \geq \omega_2$, it follows that $\mathcal{I}^+$ cannot have an uncountable subfamily of pairwise disjoint sets because no countable set can meet all of them. So $\mathcal{I}$ is ω_1-saturated. Let $\mu = \mathsf{add}(\mathcal{I})$. Ulam showed that either μ is a measurable cardinal or μ is a weakly inaccessible cardinal $\leq \mathfrak{c}$. Solovay showed that μ admits a normal ω_1-saturated ideal $\mathcal{J}$ and μ is a measurable cardinal in the inner model $L[\mathcal{J}]$. For proofs of these facts, see [7].

Though closely related to some of the works of Fremlin, supersaturated ideals were formally introduced in [4] where it was shown that if $\kappa \leq \mathfrak{c}$ admits a normal supersaturated ideal then the order dimension of the Turing degrees is at least κ. An earlier motivation for investigating these ideals comes from the following question of Fremlin – See Problem EG(h) in [1].

Question 1.2 (Fremlin)**.** *Suppose κ is real valued measurable and $m : \mathcal{P}(\kappa) \to [0, 1]$ is a witnessing normal measure. Let $\mathcal{F}$ be a family of subsets of κ such that $|\mathcal{F}| < \kappa$ and for every $A \in \mathcal{F}$, $m(A) > 0$. Must there exist a countable $N \subseteq \kappa$ such that for every $A \in \mathcal{F}$, $N \cap A \neq \emptyset$?*

So Question 1.2 is asking if the null ideal of every normal witnessing measure on a real valued measurable cardinal must be supersaturated. One of the standard ways of obtaining ω_1-saturated ideals on cardinals below the continuum is to start

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with a measurable cardinal κ and a witnessing normal prime ideal $\mathcal{I}$ on κ, and force with a ccc forcing $\mathbb{P}$ that adds $\geq \kappa$ reals. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Then $\mathcal{J}$ is always an ω_1-saturated normal ideal on $\kappa \leq \mathfrak{c}$. But whether or not $\mathcal{J}$ is supersaturated will depend on the choice of $\mathbb{P}$. This motivates the notion of supersaturation preserving forcings (Definition 2.1). In Section 2, we show that a large class of ccc forcings for adding new reals are supersaturation preserving. In particular, the following holds.

Theorem 1.3. *Let Random_λ denote the forcing for adding λ random reals.*
1. *Every σ-linked forcing is supersaturation preserving.*
2. *Random_λ is supersaturation preserving for every λ.*

The question of whether every ω_1-saturated ideal must be supersaturated was raised in [4]. Our main result shows that this is independent.

Theorem 1.4. *Each of the following is consistent.*
1. *There is an ω_1-saturated ideal on a cardinal below the continuum and there are no supersaturated ideals.*
2. *There is an ω_1-saturated ideal on a cardinal below the continuum and every ω_1-saturated ideal is supersaturated.*

Notation: Let $\mathcal{I}$ be an ideal on X. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. $\mathsf{add}(\mathcal{I})$ denotes the least cardinality of a subfamily of $\mathcal{I}$ whose union is in $\mathcal{I}^+$. For $A \subseteq X$, define $\mathcal{I} \restriction A = \{Y \subseteq X : Y \cap A \in \mathcal{I}\}$. Suppose $V \subseteq W$ are transitive models of set theory, $X, \mathcal{I} \in V$ and $V \models$ "$\mathcal{I}$ is an ideal on X". Recall that the ideal generated by $\mathcal{I}$ in W is $\mathcal{J} = \{A \in W : (\exists B \in \mathcal{I})(A \subseteq B)\}$.

For a set of ordinals X, $\mathsf{otp}(X)$ denotes the order type of X. An ordinal δ is indecomposable iff for every $X \subseteq \delta$, either $\mathsf{otp}(X) = \delta$ or $\mathsf{otp}(\delta \setminus X) = \delta$. If $\mathbb{P}, \mathbb{Q}$ are forcing notions, we write $\mathbb{P} \lessdot \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain in $\mathbb{P}$ is also a maximal antichain in $\mathbb{Q}$. Cohen_λ denotes the forcing for adding λ Cohen reals. Random_λ is the measure algebra on 2^λ equipped with the usual product measure denoted by μ_λ. If λ is clear from the context, then we drop it and just write μ.

2. CCC forcings and supersaturation

Definition 2.1. *A forcing $\mathbb{P}$ is κ-ssp (ssp = supersaturation preserving) iff for every normal supersaturated ideal $\mathcal{I}$ on κ, $V^{\mathbb{P}} \models$ "the ideal generated by $\mathcal{I}$ is supersaturated". $\mathbb{P}$ is ssp iff it is κ-ssp for every κ.*

In [4], the following forcings were shown to be κ-ssp for every κ.

(a) Cohen_λ for any λ.
(b) Any finite support iteration of ccc forcings of size $< \kappa$.

It was also shown that Random_λ is κ-ssp for any measurable κ. The next theorem improves this to all κ.

Theorem 2.2. *Random_λ is κ-ssp for every κ and λ.*

Proof. Fix a normal supersaturated ideal $\mathcal{I}$ on κ. Put $\mathbb{B} = \mathsf{Random}_\lambda$ and let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{B}}$. Suppose $\theta < \kappa$ and $\Vdash_{\mathbb{B}} \langle \mathring{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets. It suffices to find $B \in [\kappa]^{\aleph_0}$ such that $\Vdash_{\mathbb{B}} (\forall i < \theta)(\mathring{A}_i \cap B \neq \emptyset)$.

For $i < \theta$ and $\alpha < \kappa$, put $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{B}}$. So each $p_{i,\alpha}$ is a Baire subset of 2^λ. Put $T_i = \{\alpha < \kappa : p_{i,\alpha} \neq 0_{\mathbb{B}}\}$.

Claim 2.3. *For each $p \in \mathbb{B} \setminus \{0_{\mathbb{B}}\}$, $\{\alpha \in T_i : p_{i,\alpha} \cap p \neq 0_{\mathbb{B}}\} \in \mathcal{I}^+$.*

Proof. Put $X_p = \{\alpha \in T_i : p_{i,\alpha} \cap p \neq 0_{\mathbb{B}}\}$ and suppose $X_p \in \mathcal{I}$. Since the empty condition forces that $\mathring{A}_i \in \mathcal{J}^+$, it follows that for every $X \in \mathcal{I}$, $\{p_{i,\alpha} : \alpha \in T_i \setminus X\}$ is predense in $\mathbb{B}$. But every condition in $\{p_{i,\alpha} : \alpha \in T_i \setminus X_p\}$ is incompatible with p which is impossible. □

For a finite partial function f from λ to 2, define $[f] = \{x \in 2^\lambda : x \restriction \mathsf{dom}(f) = f\}$. For a clopen $K \subseteq 2^\lambda$, define $\mathsf{supp}(K)$ to be the smallest finite set $S \subseteq \lambda$ such that $(\forall x, y \in 2^\lambda)(x \restriction S = y \restriction S \implies (x \in K \iff y \in K))$. If $\mathsf{supp}(K) = S$, then there there is finite list $\{f_{K,n} : n < n_\star\}$ where $f_{K,n}$'s are pairwise distinct functions from S to 2 and $K = \bigsqcup_{n<n_\star} [f_{K,n}]$.

Definition 2.4. *Suppose $\mathcal{C}$ is a family of clopen sets in 2^λ. We say that $\mathcal{C}$ is a strong Δ-system of width $(n_\star, N_\star)$ iff $n_\star, N_\star < \omega$ and the following hold.*

(a) *$\langle \mathsf{supp}(K) : K \in \mathcal{C}\rangle$ is a Δ-system with root R.*
(b) *For every $K \in \mathcal{C}$, $|\mathsf{supp}(K) \setminus R| = n_\star$.*
(c) *For every $K \in \mathcal{C}$, $K = \bigsqcup_{n<N_\star} [f_{K,n}]$ where each $f_{K,n} : \mathsf{supp}(K) \to 2$ and $f_{K,n}$'s are pairwise distinct.*
(d) *For every $K_1, K_2 \in \mathcal{C}$ and $n < N_\star$,*
    (i) $f_{K_1,n} \restriction R = f_{K_2,n} \restriction R$ and
    (ii) if for $m \in \{1, 2\}$, $\{\xi^m_j : j < |R| + n_\star\}$ lists $\mathsf{supp}(K_m)$ in increasing order, then $f_{K_1,j}(\xi^1_j) = f_{K_2,j}(\xi^2_j)$ for every $j < |R| + n_\star$.

Lemma 2.5. *Suppose $p \subseteq 2^\lambda$ is Baire and $\mathcal{C}$ is an infinite strong Δ-system of clopen sets in 2^λ of width $(n_\star, N_\star)$. Let $\varepsilon > 0$ and assume that for infinitely many $K \in \mathcal{C}$, $\mu(p \cap K) \geq \varepsilon$. Then for all but finitely many $K \in \mathcal{C}$, $\mu(p \cap K) \geq \varepsilon/2$.*

Proof. Let R be the root of $\langle \mathsf{supp}(K) : K \in \mathcal{C}\rangle$. For each $K \in \mathcal{C}$, fix $\langle f_{K,n} : n < N_\star\rangle$ such that $K = \bigsqcup_{n<N_\star} [f_{K,n}]$. First suppose that p is clopen. Let $\mathcal{C}_p = \{K \in \mathcal{C} : (\mathsf{supp}(K) \setminus R) \cap \mathsf{supp}(p) = \emptyset\}$. Then $\mathcal{C} \setminus \mathcal{C}_p$ is finite and for each $K \in \mathcal{C}_p$,

$$\mu(p \cap K) = \sum_{n<N_\star} \mu(p \cap [f_{K,n}]) = 2^{-n_\star} \sum_{n<N_\star} \mu(p \cap [f_{K,n} \restriction R])$$

which does not depend on $K \in \mathcal{C}_p$. It follows that the result holds if p is clopen. The general case follows by applying the previous case to a clopen $q \subseteq 2^\lambda$ satisfying $\mu(p \Delta q) < \varepsilon/2$. □

For each $\alpha \in T_i$, fix $S_{i,\alpha} \in [\lambda]^{\aleph_0}$ such that $p_{i,\alpha}$ is supported in $S_{i,\alpha}$. For every $i < \theta$, $\alpha \in T_i$ and $\varepsilon > 0$ rational, choose a clopen set $K_{i,\alpha,\varepsilon} \subseteq 2^\lambda$ with $\mathsf{supp}(K_{i,\alpha,\varepsilon}) \subseteq S_{i,\alpha}$ such that

$$\frac{\mu(p_{i,\alpha} \Delta K_{i,\alpha,\varepsilon})}{\mu(K_{i,\alpha,\varepsilon})} < \varepsilon$$

Claim 2.6. *For each $i < \theta$ and $\varepsilon > 0$ rational, we can find $\mathcal{F}_{i,\varepsilon} \subseteq \mathcal{I}^+$ and $\langle (n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y}) : Y \in \mathcal{F}_{i,\varepsilon}\rangle$ such that the following hold.*

(1) *$\mathcal{F}_{i,\varepsilon}$ is a countable family of pairwise disjoint sets and $T_i \setminus \bigcup \mathcal{F}_{i,\varepsilon} \in \mathcal{I}$.*
(2) *For each $Y \in \mathcal{F}_{i,\varepsilon}$, $\{K_{i,\alpha,\varepsilon} : \alpha \in Y\}$ is a strong Δ-system of width $(n_{i,\varepsilon,Y}, N_{i,\varepsilon,Y})$.*

Proof. Fix $i < \theta$ and $\varepsilon > 0$ rational. To simplify notation, we write K_α instead of $K_{i,\alpha,\varepsilon}$. It suffices to show that for every $\mathcal{I}$-positive $X \subseteq T_i$, there exists $Y \subseteq X$ such that $Y \in \mathcal{I}^+$ and there exist (n_Y, N_Y) such that $\{K_\alpha : \alpha \in Y\}$ is a strong Δ-system of width (n_Y, N_Y). Since then we can take $\mathcal{F}_{i,\varepsilon}$ to be a maximal disjoint family of such Y's. That each $\mathcal{F}_{i,\varepsilon}$ is countable follows from the fact that $\mathcal{I}$ is ω_1-saturated.

Fix a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $\alpha \in T_i \cap \gamma$, $\max(\mathsf{supp}(K_\alpha)) < \gamma$. Suppose $X \subseteq T_i \cap E$ and $X \in \mathcal{I}^+$. Since $\mathcal{I}$ is normal and the map $\alpha \mapsto \max(\mathsf{supp}(K_\alpha \cap \alpha))$ is regressive on X, we can find $R \subseteq \kappa$ finite and $Y_1 \subseteq X$ such that $Y_1 \in \mathcal{I}^+$, $(\forall \alpha \in Y_1)(\mathsf{supp}(K_\alpha) \cap \alpha = R)$ and $|\mathsf{supp}(K_\alpha) \setminus R| = n_\star$ does not depend on $\alpha \in Y_1$. It also follows that $\langle \mathsf{supp}(K_\alpha) : \alpha \in Y_1 \rangle$ forms a Δ-system with root R. For each $\alpha \in Y_1$, let $K_\alpha = \bigsqcup_{n<N_\alpha} [f_{\alpha,n}]$ where each $f_{\alpha,n} : \mathsf{supp}(K_\alpha) \to 2$. Choose $Y_2 \subseteq Y_1$ such that $Y_2 \in \mathcal{I}^+$ and $N_\alpha = N_\star$ does not depend on $\alpha \in Y_2$. Finally, choose $Y \subseteq Y_2$ such that $Y \in \mathcal{I}^+$ and $\{K_\alpha : \alpha \in Y\}$ is a strong Δ-system of width $(n_\star, N_\star)$. □

Since $\mathcal{I}$ is supersaturated, we can choose $B \in [\kappa]^{\aleph_0}$ such that for every $i < \theta$, $\varepsilon > 0$ rational and $Y \in \mathcal{F}_{i,\varepsilon}$, we have $|B \cap Y| = \aleph_0$. It suffices to show that for each $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in $\mathbb{B}$.

Suppose not. Fix $i < \theta$ and $p \subseteq 2^\lambda$ Baire such that $\mu(p) > 0$ and for every $\alpha \in B$, $\mu(p_{i,\alpha} \cap p) = 0$. Let $X = \{\alpha \in T_i : \mu(p_{i,\alpha} \cap p) > 0\}$. By Claim 2.3, $X \in \mathcal{I}^+$. Using the argument in the proof of Claim 2.6, we can choose $\varepsilon > 0$ rational, $X_\star \subseteq X$ and $n_\star, N_\star < \omega$ such that

(a) $X_\star \in \mathcal{I}^+$ and for each $\alpha \in X_\star$, $\mu(p_{i,\alpha} \cap p) \geq 4\varepsilon$.
(b) $\{K_{i,\alpha,\varepsilon} : \alpha \in X_\star\}$ is a strong Δ-system of width $(n_\star, N_\star)$.

Choose $Y \in \mathcal{F}_{i,\varepsilon}$ such that $Y \cap X_\star \in \mathcal{I}^+$. Since $|Y \cap X_\star| \geq \aleph_0$ and $|Y \cap B| = \aleph_0$, by Lemma 2.5, we can choose $\alpha \in Y \cap B$ such that $\mu(p \cap K_{i,\alpha,\varepsilon}) \geq 2\varepsilon$. But since $\mu(p_{i,\alpha} \Delta K_{i,\alpha,\varepsilon}) \leq \varepsilon\mu(K_{i,\alpha,\varepsilon}) \leq \varepsilon$, it follows that $\mu(p \cap p_{i,\alpha}) \geq \varepsilon > 0$: Contradiction. This completes the proof of Theorem 2.2. □

Theorem 2.7. *Every σ-linked forcing is κ-ssp for every κ.*

Proof. Let $\mathcal{I}$ be a normal supersaturated ideal on κ. Suppose $\mathbb{P}$ is a σ-linked forcing and $\mathcal{J}$ is the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Fix $\theta < \kappa$ and WLOG, assume that the trivial condition forces that $\langle \mathring{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets. It suffices to construct $X \in [\kappa]^{\aleph_0}$ such that $\Vdash_{\mathbb{P}} (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$.

Since $\mathbb{P}$ is σ-linked, we can write $\mathbb{P} = \bigcup\{L_n : n < \omega\}$ where each $L_n \subseteq \mathbb{P}$ has pairwise compatible members. For each $i < \theta$ and $n < \omega$, define

$$B_{i,n} = \{\alpha < \kappa : (\exists p \in L_n)(p \Vdash \alpha \in \mathring{A}_i)\}$$

Claim 2.8. *$W_i = \bigcup\{L_n : n < \omega, B_{i,n} \in \mathcal{I}^+\}$ is dense in $\mathbb{P}$.*

Proof. Suppose not and fix $p \in \mathbb{P}$ such that no extension of p lies in W_i. Put $C = \{\alpha < \kappa : (\exists q \leq p)(q \Vdash \alpha \in \mathring{A}_i)\}$. Since no extension of p lies in W_i, it follows that $C \subseteq \bigcup\{B_{i,n} : n < \omega, B_{i,n} \in \mathcal{I}\}$ and hence $C \in \mathcal{I}$. It now follows that $p \Vdash \mathring{A}_i \in \mathcal{J}$ which is impossible. □

Since $\mathcal{I}$ is supersaturated, we can find a countable $X \subseteq \kappa$ such that for every $i < \theta$ and $n < \omega$, if $B_{i,n} \in \mathcal{I}^+$, then $X \cap B_{i,n} \neq \emptyset$. We claim that $\Vdash (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$.

Suppose not and fix $p \in \mathbb{P}$ and $i < \theta$ such that $p \Vdash X \cap \mathring{A}_i = \emptyset$. Using Claim 2.8, choose $n < \omega$ and $p' \leq p$ such that $p' \in L_n$ and $B_{i,n} \in \mathcal{I}^+$. Choose $\alpha \in B_{i,n} \cap X$ and $q \in L_n$ such that $q \Vdash \alpha \in \mathring{A}_i$. Since L_n is linked, we can find a common extension $r \in \mathbb{P}$ of p', q. But $r \Vdash \alpha \in X \cap \mathring{A}_i$: Contradiction. □

Corollary 2.9. *Each of the following forcings is ssp: Cohen, random, Amoeba, Hechler, Eventually different real forcing.*

We do not know if we can improve Theorem 2.7 to the class of σ-finite-cc forcings. For example, one can ask the following.

Question 2.10. *Suppose $\mathbb{B}$ is a boolean algebra and $m : \mathbb{B} \to [0,1]$ is a strictly positive finitely additive measure on $\mathbb{B}$. Must $\mathbb{B}$ be supersaturation preserving?*

The next two facts are well known.

Fact 2.11. *Suppose $\mathbb{P}$ is a separative σ-linked forcing. Then $|\mathbb{P}| \leq \mathfrak{c}$.*

Fact 2.12. *Let $\langle (\mathbb{P}_\xi, \mathring{\mathbb{Q}}_\xi) : \xi < \lambda \rangle$ be a finite support iteration with limit $\mathbb{P}_\lambda$ where for every $\xi < \lambda$, $V^{\mathbb{P}_\xi} \models \mathring{\mathbb{Q}}_\xi$ is σ-linked. Assume $\lambda < \mathfrak{c}^+$. Then $\mathbb{P}_\lambda$ is also σ-linked.*

Theorem 2.13. *Let $\mathcal{I}$ be a normal supersaturated ideal on κ and let $\lambda \leq \kappa^+$. Suppose $\langle (\mathbb{P}_\xi, \mathring{\mathbb{Q}}_\xi) : \xi < \lambda \rangle$ is a finite support iteration with limit $\mathbb{P}_\lambda$ where for every $\xi < \lambda$, $V^{\mathbb{P}_\xi} \models \mathring{\mathbb{Q}}_\xi$ is σ-linked. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}_\lambda}$. Then $\mathcal{J}$ is supersaturated.*

Proof. By induction on λ. First suppose $\kappa \leq \mathfrak{c}$. If $\lambda < \kappa^+$, then by Fact 2.12, $\mathbb{P}_\lambda$ is σ-linked and the claim holds by Theorem 2.7. So assume $\lambda = \kappa^+$ and fix any $\mathbb{P}_\lambda$-generic filter G_λ over V. Let $\langle A_i : i < \theta \rangle$ be a sequence of $\mathcal{J}$-positive sets in $V[G_\lambda]$ where $\theta < \kappa$. Since $\mathbb{P}_\lambda$ is a finite support iteration of ccc forcings, there exists $\eta < \lambda = \kappa^+$ such that $\langle A_i : i < \theta \rangle \in V[G_\eta]$ where $G_\eta = \mathbb{P}_\eta \cap G_\lambda$. Note that each A_i is $\mathcal{J}_\eta$-positive where $\mathcal{J}_\eta$ is the ideal generated by $\mathcal{I}$ in $V[G_\eta]$. By inductive hypothesis, there is a countable set that meets A_i for every $i < \theta$. Hence $\mathcal{J}$ is supersaturated.

Next assume $\kappa > \mathfrak{c}$. Then κ is measurable and $\mathcal{I}$ is a normal prime ideal on κ. First suppose $\lambda \leq \kappa$. By Fact 2.11, $|\mathbb{P}_\xi| \leq |\xi \cdot \mathfrak{c}| < \kappa$ for every $\xi < \kappa$. Hence by Theorem 4.9 in [4], it follows that $\mathcal{J}$ is supersaturated. Next suppose $\kappa < \lambda \leq \kappa^+$. Note that $V^{\mathbb{P}_\kappa} \models \mathfrak{c} \geq \kappa$ since Cohen reals are added at each stage of cofinality ω. So we can work in $V^{\mathbb{P}_\kappa}$ and repeat the argument for the case $\kappa \leq \mathfrak{c}$. □

It is now natural to ask the following.

Question 2.14 ([4])**.** *Suppose κ is measurable. Is every ccc forcing κ-ssp?*

In Section 4, we'll show that the answer is negative. We end this section with the following weaker positive result.

Theorem 2.15. *Suppose κ is measurable and $\mathcal{I}$ is a normal prime ideal on κ. Let $\mathbb{B}$ be a ccc complete boolean algebra. Then $V^{\mathbb{B}} \models$ "the ideal generated by $\mathcal{I}$ is ω_2-supersaturated."*

Proof. It suffices to show that the following holds in $V^{\mathbb{B}}$: For every $\mathcal{A} \subseteq \mathcal{J}^+$, if $|\mathcal{A}| < \kappa$, then there exists $X \in [\kappa]^{\aleph_1}$ such that X meets every member of $\mathcal{A}$.

Suppose $\theta < \kappa$ and $\Vdash_{\mathbb{B}} \{\mathring{A}_i : i < \theta\} \subseteq \mathcal{J}^+$. Choose $Y \subseteq \kappa$ of $\mathcal{I}$-measure one such that for every $i < \theta$ and $\alpha \in Y$, $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]] > 0_{\mathbb{B}}$. Using the inaccessibility of κ, the following claim is easy to check.

Claim 2.16. *There exists $\langle \mathbb{B}_\alpha : \alpha < \kappa \rangle$ such that the following hold.*

(i) $\mathbb{B}_\alpha \lessdot \mathbb{B}$ *and* $|\mathbb{B}_\alpha| < \kappa$.
(ii) $\mathbb{B}_\alpha$*'s are increasing and continuous at* α *when* $\mathsf{cf}(\alpha) > \aleph_0$.
(iii) $\{p_{i,\beta} : \beta < \alpha, i < \theta\} \subseteq \mathbb{B}_\alpha$.

Let $\pi_\alpha : \mathbb{B} \to \mathbb{B}_\alpha$ be a projection map witnessing $\mathbb{B}_\alpha \lessdot \mathbb{B}$. Choose $f : \kappa \to \kappa$ such that for every $i < \theta$ and $\alpha < \kappa$, we have $\alpha < f(\alpha)$ and $p_{i,\alpha} \in \mathbb{B}_{f(\alpha)}$. Choose $Y_1 \subseteq Y$ of measure one and $\alpha_\star < \kappa$ such that for every $i < \theta$, $\pi_\alpha(p_{i,\alpha}) = p_{i,\star} \in \mathbb{B}_{\alpha_\star}$ does not depend on $\alpha \in Y_1$ and $\mathsf{range}(f \restriction \alpha) \subseteq \alpha$ for every $\alpha \in Y_1$. Note that $p_{i,\star} = 1_{\mathbb{B}}$ since $\Vdash_{\mathbb{B}} \mathring{A}_i \in \mathcal{J}^+$. Let $X \subseteq Y \setminus \alpha_\star$ be such that $\mathsf{otp}(X) = \omega_1$ and for every $\alpha < \beta$ in X, $f(\alpha) < f(\beta)$.

Claim 2.17. *For every $i < \theta$, $\{p_{i,\alpha} : \alpha \in X\}$ is predense in $\mathbb{B}$.*

Proof. Let $\sup(X) = \gamma_\star$. Then $\mathsf{cf}(\gamma_\star) = \aleph_1$ and hence $\mathbb{B}_{\gamma_\star} = \bigcup\{\mathbb{B}_\gamma : \gamma \in X\}$. Fix $i < \theta$. Given $p \in \mathbb{B}$, choose $\gamma \in X$ such that $\pi_{\gamma_\star}(p) \in \mathbb{B}_\gamma$. Now since

$$\mathbb{B} = \mathbb{B}_\gamma \star \mathbb{B}_{\gamma_\star}/\mathbb{B}_\gamma \star \mathbb{B}/\mathbb{B}_{\gamma_\star}$$

we can decompose $p = (\pi_{\gamma_\star}(p), 1, x)$ and $p_{i,\gamma} = (1, y, 1)$. Hence $p, p_{i,\gamma}$ are compatible. □

It follows that $\mathcal{J}$ is ω_2-supersaturated. □

3. Consistently, there are ω_1-saturated ideals on $\mathfrak{c}$ and all of them are supersaturated

The aim of this section is to show that it is consistent that every ω_1-saturated σ-ideal is supersaturated.

Theorem 3.1. *It is consistent that there is a normal supersaturated ideal on $\mathfrak{c}$ and every ω_1-saturated σ-ideal is supersaturated.*

Lemma 3.2. *Suppose that every σ-ideal $\mathcal{I}$ satisfying (i)-(iv) below is supersaturated.*

(i) $\mathcal{I}$ *is a uniform ideal on* λ,
(ii) $\mu \leq \lambda$,
(iii) *for every* $X \in \mathcal{I}^+$, $\mathsf{add}(\mathcal{I} \restriction X) = \mu$ *and*
(iv) $\mathcal{I}$ *is* ω_1*-saturated.*

Then every ω_1-saturated σ-ideal is supersaturated.

Proof. Suppose $\mathcal{J}$ is an ω_1-saturated σ-ideal on X. Note that for every $A \in \mathcal{J}^+$, there exists $B \subseteq A$ such that $(\star)_B$ holds where

$(\star)_B$ says the following: $B \in \mathcal{J}^+$, $[B]^{<|B|} \subseteq \mathcal{J}$ and for every $C \subseteq B$, if $C \in \mathcal{J}^+$, then $\mathsf{add}(\mathcal{J} \restriction C) = \mathsf{add}(\mathcal{J} \restriction B)$.

Since $\mathcal{J}$ is ω_1-saturated, we can find a countable partition $\mathcal{F}$ of X such that for each $B \in \mathcal{F}$, $(\star)_B$ holds. Now by assumption, each $\mathcal{J} \restriction B$ is supersaturated. Hence $\mathcal{J}$ is also supersaturated. □

Lemma 3.3. *Suppose $\mathbb{P}$ is a ccc forcing, $\kappa > \mathfrak{c}$ and $V^{\mathbb{P}} \models \mathcal{J}$ is a κ-complete ω_1-saturated uniform ideal on λ. Let $\mathcal{I} = \{X \subseteq \kappa : 1_{\mathbb{P}} \Vdash X \in \mathcal{J}\}$. Then there is a countable partition $\mathcal{F}$ of λ such that for every $A \in \mathcal{F}$, $\mathcal{I} \restriction A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a κ-complete prime ideal on λ.*

Proof. It is clear that $\mathcal{I}$ is a κ-complete uniform ideal on λ. Suppose $\mathcal{F} \subseteq \mathcal{I}^+$ is an uncountable family of pairwise disjoint sets. For each $A \in \mathcal{F}$, choose $p_A \in \mathbb{P}$ such that $p_A \Vdash A \notin \mathcal{J}$. Since $\mathbb{P}$ is ccc, some $p \in \mathbb{P}$ forces uncountably many p_A's into the $\mathbb{P}$-generic filter. But this contradicts the fact that $\mathcal{J}$ is ω_1-saturated in $V^{\mathbb{P}}$. So $\mathcal{I}$ is ω_1-saturated. Since $\mathcal{I}$ is κ-complete and $\kappa > \mathfrak{c}$, $\mathcal{I}$ is nowhere atomless. Hence there is a countable partition $\mathcal{F}$ of λ such that for every $A \in \mathcal{F}$, $\mathcal{I} \restriction A = \{Y \subseteq \lambda : Y \cap A \in \mathcal{I}\}$ is a κ-complete prime ideal on λ. □

Lemma 3.4. *Suppose κ is an inaccessible cardinal and $\mathcal{U}$ is a κ-complete uniform ultrafilter on λ. Let $\mathbb{P} = \mathsf{Cohen}_\kappa$. Let $\mathcal{J}$ be the ideal generated by the dual ideal of $\mathcal{U}$ in $V^{\mathbb{P}}$. Then for each $\mathcal{A} \subseteq \mathcal{J}^+$, if $|\mathcal{A}| < \kappa$, then there exists a countable set that meets every member of $\mathcal{A}$.*

Proof. We identify conditions $p \in \mathbb{P}$ as members of the Baire algebra on 2^κ which is the σ-algebra generated by clopen subsets of 2^κ. Note that for every Baire $p \subseteq 2^\kappa$ there is a countable $S \subseteq \kappa$ such that for every $x, y \in 2^\kappa$ satisfying $x \restriction S = y \restriction S$, we have $x \in p$ iff $y \in p$. We call such an S, a support of p. The ordering on Cohen_κ is defined by $p \leq q$ iff $p \setminus q$ is meager in 2^κ. Recall that if $p \subseteq 2^\kappa$ is Baire and $S \in [\kappa]^{\aleph_0}$ is a support of p then there is a countable family $\mathcal{P}$ of clopen subsets of 2^κ each supported in S such that the symmetric difference of p and $\bigcup \mathcal{P}$ is meager. So p is completely determined by the family $\mathcal{P}$.

It is clear that $\mathcal{J}$ is a κ-complete uniform ideal on λ. Suppose $\theta < \kappa$ and $\langle \mathring{A}_i : i < \theta \rangle$ is a sequence of $\mathcal{J}$-positive sets in $V^{\mathbb{P}}$. WLOG, assume that the trivial condition forces this. For $i < \theta$ and $\alpha < \lambda$, let $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{P}}$. Note that for each $i < \theta$, and $Z \in \mathcal{U}$, $\{p_{i,\alpha} : \alpha \in Z\}$ is predense in $\mathbb{P}$ since otherwise some condition will force $\mathring{A}_i \in \mathcal{J}$. Since $\mathcal{U}$ is κ-complete, we can choose $X \in \mathcal{U}$ such that for every $i < \theta$ and $\alpha \in X$, $p_{i,\alpha} > 0_{\mathbb{P}}$. Let $S_{i,\alpha} \in [\kappa]^{\aleph_0}$ be a support of $p_{i,\alpha}$. Since κ is inaccessible, we can choose $Y \subseteq X$ such that $Y \in \mathcal{U}$ and for each $i < \theta$, the following hold.

(a) For every $\alpha, \beta \in Y$, $(S_{i,\alpha}, 2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (S_{i,\beta}, 2^{S_{i,\beta}}, p_{i,\beta})$. Put $\mathsf{otp}(S_{i,\alpha}) = \gamma_i$. Let $h_{i,\alpha} : \gamma_i \to S_{i,\alpha}$ be the order isomorphism and define $H_{i,\alpha} : 2^{\gamma_i} \to 2^{S_{i,\alpha}}$ by $H_{i,\alpha}(x) = x \circ h_{i,\alpha}^{-1}$. Choose $p_i \subseteq 2^{\gamma_i}$ such that $H_{i,\alpha}[p_i] = p_{i,\alpha}$.

(b) For each $\gamma < \gamma_i$, either $|\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1$ or for every $Z \in \mathcal{U}$, $|\{h_{i,\alpha}(\gamma) : \alpha \in Z \cap Y\}| \geq \kappa$. Put $\Gamma_i = \{\gamma < \gamma_i : |\{h_{i,\alpha}(\gamma) : \alpha \in Y\}| = 1\}$ and $h_{i,\alpha}[\Gamma_i] = R_i$.

Define

$$B_{i,\alpha} = \{x \in 2^{R_i} : \{y \restriction (S_{i,\alpha} \setminus R_i) : y \in p_{i,\alpha} \wedge y \restriction R_i = x\} \text{ is meager}\}.$$

Then $B_{i,\alpha} = B_i$ does not depend on $\alpha \in Y$ and B_i is meager in 2^{R_i} since otherwise $\{p_{i,\alpha} : \alpha \in Y\}$ will not be predense in $\mathbb{P}$.

Using (b), choose $B \in [Y]^{\aleph_0}$ such that for every $i < \theta$ and $\alpha \neq \beta$ in B, $S_{i,\alpha} \cap S_{i,\beta} = R_i$. It follows now that for every $i < \theta$, $\{p_{i,\alpha} : \alpha \in B\}$ is predense in $\mathbb{P}$. Hence $\Vdash (\forall i < \theta)(B \cap \mathring{A}_i \neq \emptyset)$. □

Proof of Theorem 3.1: Let $V \models$ "$\mathfrak{c} = \omega_1$ and κ is the least measurable cardinal". Let $\mathbb{P} = \mathsf{Cohen}_\kappa$. We already know that there is a normal supersaturated ideal on $\kappa = \mathfrak{c}$ in $V^{\mathbb{P}}$. Let us check that, $V^{\mathbb{P}} \models$ "Every ω_1-saturated σ-ideal is supersaturated". By Lemma 3.2, it suffices to consider ideals $\mathcal{J}$ that satisfy the following for some $\omega_1 \leq \mu \leq \lambda$.

(i) $\mathcal{J}$ is a uniform ideal on λ,
(ii) for every $X \in \mathcal{J}^+$, $\mathsf{add}(\mathcal{J} \restriction X) = \mu$ and
(ii) $\mathcal{J}$ is ω_1-saturated.

Since $V^{\mathbb{P}} \models \mathfrak{c} = \kappa$, we can assume that $\mu \leq \kappa$. Otherwise there is a countable partition $\mathcal{E}$ of λ into $\mathcal{J}$-positive sets such that for each $X \in \mathcal{E}$, $\mathcal{J} \restriction X$ is a μ-complete prime ideal and it easily follows that $\mathcal{J}$ is supersaturated.

Towards a contradiction, suppose $\mu < \kappa$. Working in $V^{\mathbb{P}}$, define an ideal $\mathcal{K}$ on μ as follows. Since $\mathsf{add}(\mathcal{J}) = \mu$, we can choose a family $\{A_i : i < \mu\} \subseteq \mathcal{J}$ of pairwise disjoint sets such that $\bigcup_{i<\mu} A_i \in \mathcal{J}^+$. Define

$$\mathcal{K} = \{\Gamma \subseteq \mu : \bigcup\{A_i : i \in \Gamma\} \in \mathcal{J}\}$$

It is easy to see that $\mathcal{K}$ is a μ-additive ω_1-saturated ideal on μ. For simplicity, assume that $1_{\mathbb{P}} \Vdash \mathring{\mathcal{K}}$ is a μ-additive ω_1-saturated ideal on μ. Coming back to V, define $\mathcal{K}' = \{X \subseteq \mu : 1_{\mathbb{P}} \Vdash X \in \mathring{\mathcal{K}}\}$. It is clear that $V \models \mathcal{K}'$ is a μ-additive ideal on μ. We claim that $V \models \mathcal{K}'$ is ω_1-saturated. Suppose not and fix $\langle (A_\xi, p_\xi) : \xi < \omega_1 \rangle$ such that A_ξ's are pairwise disjoint subsets of μ and for every $\xi < \omega_1$, $p_\xi \Vdash A_\xi \notin \mathring{\mathcal{K}}$. Since $\mathbb{P}$ is ccc, we can find some $p_\star \in \mathbb{P}$ that forces uncountable many p_ξ's into the generic $G_{\mathbb{P}}$. But this means that $p_\star \Vdash \mathring{K}$ is not ω_1-saturated which is impossible. So $V \models \mathcal{K}'$ is ω_1-saturated. So μ is weakly inaccessible in V. Since $V \models \mu > \omega_1 = \mathfrak{c}$, it follows that μ must be measurable in V. But κ is the least measurable cardinal in V. Hence $\mu \geq \kappa$: Contradiction.

So we must have $\mu = \kappa$. Let $\mathcal{I} = \{Y \subseteq \lambda : 1_{\mathbb{P}} \Vdash X \in \mathcal{J}\}$. By Lemma 3.3, there is a countable partition $\mathcal{F}$ of λ such that for each $X \in \mathcal{F}$, $\mathcal{I} \restriction X$ is a κ-complete prime ideal on λ. For each $X \in \mathcal{F}$, let $\mathcal{I}_X$ be the ideal generated by $\mathcal{I} \restriction X$ in $V^{\mathbb{P}}$. By Lemma 3.4, for every $\mathcal{A} \subseteq \mathcal{I}_X^+$, if $|\mathcal{A}| < \kappa$, then there is a countable set that meets every member of $\mathcal{A}$. Since $\mathcal{I}_A \subseteq \mathcal{J} \restriction A$ and $\mathsf{add}(\mathcal{J} \restriction A) = \kappa$, it follows that $\mathcal{J} \restriction A$ is supersaturated for each $A \in \mathcal{F}$. Since $\mathcal{F}$ is a countable partition of λ, it follows that $\mathcal{J}$ is also supersaturated. □

4. Killing supersaturated ideals

Definition 4.1. *Suppose $\delta < \omega_1$ is indecomposable and κ is an infinite cardinal. Let $\mathbb{Q}^\kappa_\delta$ consist of all countable partial maps from κ to 2 such that*

(1) *$\mathsf{otp}(\mathsf{dom}(p)) < \delta$ and*
(2) *$\{\xi \in \mathsf{dom}(p) : p(\xi) = 1\}$ is finite.*

For $p, q \in \mathbb{Q}^\kappa_\delta$ define $p \leq q$ iff $q \subseteq p$. Let $\mathbb{P}_\kappa$ be the finite support product of $\{Q^\kappa_\delta : \delta < \omega_1, \delta$ indecomposable$\}$.

Lemma 4.2. *Let $\mathbb{P}_\kappa$ be as in Definition 4.1.*

(1) *$\mathbb{P}_\kappa$ is ccc.*
(2) *If $\kappa \geq \omega_1$, then $\mathbb{P}_\kappa$ is not σ-finite-cc.*

Proof. (1) Towards a contradiction, suppose $A = \{p_i : i < \omega_1\}$ is an uncountable antichain in $\mathbb{P}_\kappa$. Put $D_i = \mathsf{dom}(p_i)$. By passing to an uncountable subset of A, we can assume that D_i's form a Δ-system with root D. For each $\delta \in D$ and $i < \omega_1$, put $s_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 1\}$ and $X_{i,\delta} = \{\gamma : p_i(\delta)(\gamma) = 0\}$. Note that $\mathsf{otp}(X_{i,\delta}) < \delta$. Choose $B \in [A]^{\omega_1}$ such that for each $\delta \in D$, $\langle s_{i,\delta} : i \in B\rangle$ is a Δ-system with root s_δ and for every $i < j$ in B, $s_{j,\delta} \cap X_{i,\delta} = \emptyset$.

Choose $j \in B$ and $\delta \in D$ such that letting $C = \{i \in B \cap j : p_i(\delta) \perp_{\mathbb{Q}_\delta} p_j(\delta)\}$, every transversal of $\{s_{i,\delta} \setminus s_\delta : i \in C\}$ has order type $\geq \delta$. Now observe that $X_{j,\delta}$ has to meet $s_{i,\delta} \setminus s_\delta$ for every $i \in C$. Hence $\mathsf{otp}(X_{j,\delta}) \geq \delta$: Contradiction.

(2) It is enough to show that $\mathbb{Q} = \mathbb{Q}^{\omega_1}_{\omega^2}$ is not σ-finite-cc. Towards a contradiction, suppose $\mathbb{Q} = \bigsqcup_{n<\omega} W_n$ where no W_n has an infinite antichain. Choose $\langle A_n : n < \omega\rangle$ as follows.

(a) $A_0 \subseteq W_0$ is a maximal antichain of conditions p such that $\max(\mathsf{dom}(p)) = \gamma_p$ exists and $p(\gamma_p) = 1$. Define $\gamma_0 = \max(\{\gamma_p : p \in A_0\})$.
(b) $A_{n+1} \subseteq W_{n+1}$ is a maximal antichain of conditions $p \in W_{n+1}$ such that $\max(\mathsf{dom}(p)) = \gamma_p$ exists, $\gamma_p > \gamma_n$ and $p(\gamma_p) = 1$. If $A_{n+1} \neq \emptyset$, define $\gamma_{n+1} = \max(\{\gamma_p : p \in A_{n+1}\})$. Otherwise, $\gamma_{n+1} = \gamma_n$.

Put $A = \bigcup_{n<\omega} A_n$ and $\gamma = \sup(\{\gamma_n : n < \omega\})$. Fix $\gamma_\star \in (\gamma, \omega_1)$. Let $p_\star$ be defined by $\mathsf{dom}(p_\star) = \{\gamma_p : p \in A\} \cup \{\gamma_\star\}$ and for every $\xi \in \mathsf{dom}(p_\star)$, $p(\xi) = 1$ iff $\xi = \gamma_\star$. Note that $\mathsf{otp}(\mathsf{dom}(p)) \leq \omega + 1 < \omega^2$ and hence $p_\star \in \mathbb{Q}$. Choose $n < \omega$ such that $p_\star \in W_n$. But now $A_n \cup \{p_\star\} \subseteq W_n$ is an antichain which contradicts the maximality of A_n. □

Theorem 4.3. *Suppose $\omega_1 \leq \kappa \leq \lambda$, $\mathcal{I}$ is an ω_1-saturated uniform ideal on λ and $\mathsf{add}(\mathcal{I}) = \kappa$. Let $\mathbb{P}_\kappa$ be as in Definition 4.1. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}_\kappa}$. Then there exists $\mathcal{A} \subseteq \mathcal{J}^+$ such that $|\mathcal{A}| = \omega_1$ and there is no countable set that meets every member of $\mathcal{A}$. Hence $V^{\mathbb{P}_\kappa} \models \mathcal{J}$ is an ω_1-saturated κ-complete uniform ideal on λ which is not supersaturated.*

Proof. As $\mathbb{P}_\kappa$ is ccc, it is easy to see that in $V^{\mathbb{P}_\kappa}$, $\mathcal{J}$ is an ω_1-saturated κ-complete uniform ideal on λ. So it suffices to show that in $V^{\mathbb{P}_\kappa}$, there exists $\mathcal{A} \subseteq \mathcal{J}^+$ such that $|\mathcal{A}| = \omega_1$ and there is no countable set that meets every member of $\mathcal{A}$.

Since $\mathsf{add}(\mathcal{I}) = \kappa$, we can fix $Y \in \mathcal{I}^+$ and a partition $Y = \bigsqcup_{\alpha<\kappa} W_\alpha$ such that for each $\Gamma \in [\kappa]^{<\kappa}$, $\bigcup_{\alpha\in\Gamma} W_\alpha \in \mathcal{I}$. Let G be $\mathbb{P}_\kappa$-generic over V. Let $G_\delta = \{p(\delta) : p \in G\}$. So G_δ is $\mathbb{Q}_\delta$-generic over V. Define $\mathring{A}_\delta \in V^{\mathbb{P}_\kappa} \cap \mathcal{P}(\lambda)$ by

$$\gamma \in \mathring{A}_\delta \iff (\exists p \in G)(p(\delta)(\alpha) = 1 \wedge \gamma \in W_\alpha)$$

Suppose $Y \in \mathcal{I}$ and $p \in \mathbb{P}_\kappa$ with $\delta \in \mathsf{dom}(p)$. Choose $\alpha < \kappa$ such that $W_\alpha \setminus Y \neq \emptyset$ and $\alpha \notin \mathsf{dom}(p(\delta))$. Let $q \leq p$ be such that $q(\delta)(\alpha) = 1$. Then $q \Vdash_{\mathbb{P}_\kappa} \mathring{A}_\delta \setminus Y \neq \emptyset$. Hence $\Vdash_{\mathbb{P}_\kappa} \mathring{A}_\delta \in \mathcal{J}^+$.

Towards a contradiction suppose that in $V^{\mathbb{P}_\kappa}$, there is a countable $X \subseteq \lambda$ that meets each $\mathring{A}_\delta$. Since $\mathbb{P}$ satisfies ccc, we can assume that $X \in V$. Fix $p \in \mathbb{P}_\kappa$ such that $p \Vdash_{\mathbb{P}} (\forall\delta)(X \cap \mathring{A}_\delta \neq \emptyset)$. Put $W = \{\alpha < \kappa : W_\alpha \cap X \neq \emptyset\}$. So $W \subseteq \kappa$ is countable. Choose $\delta \in \omega_1 \setminus \mathsf{dom}(p)$ indecomposable such that $\delta > \mathsf{otp}(W)$. Define

$q \in \mathbb{P}_\kappa$ by $\mathsf{dom}(q) = \mathsf{dom}(p) \cup \{\delta\}$, $q \restriction \mathsf{dom}(p) = p$ and $q(\delta) \in \mathbb{Q}_\delta$ is constantly zero on W. Then $q \leq p$ and $q \Vdash_{\mathbb{P}_\kappa} X \cap \mathring{A}_\delta = \emptyset$: Contradiction. It follows that $\mathcal{A} = \{A_\delta : \delta < \omega_1, \delta \text{ indecomposable}\}$ is as required. □

Definition 4.4. *Let $\langle (\mathbb{S}_i, \mathbb{R}_j) : i \leq \kappa^+, j < \kappa \rangle$ be the finite support iteration defined by*

(a) *$\mathbb{S}_0$ is the trivial forcing.*
(b) *For each $i < \kappa^+$, $V^{\mathbb{S}_i} \models \mathbb{R}_i = \mathbb{P}_\kappa$.*

The next theorem shows how to kill all atomless supersaturated ideals.

Theorem 4.5. *Suppose $V \models$ "$\mathfrak{c} = \omega_1$ and κ is the least measurable cardinal with a witnessing normal prime ideal $\mathcal{I}$". Put $\mathbb{S} = \mathbb{S}_{\kappa^+}$. Then the following hold in $V^{\mathbb{S}}$.*

(a) *$\mathfrak{c} = \kappa^+$ and the ideal generated by $\mathcal{I}$ is a normal ω_1-saturated ideal on κ.*
(b) *Whenever $\mathcal{J}$ is a supersaturated ideal on a set X, there is a countable partition $\mathcal{F}$ of X such that for each $A \in \mathcal{F}$, $\mathcal{J} \restriction A$ is a prime ideal. In particular, there is no supersaturated ideal on any cardinal $\leq \mathfrak{c}$.*

Fact 4.6. *Suppose $\mathcal{I}_1, \mathcal{I}_2$ are ω_1-saturated σ-ideals on X and $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Then there is a partition $X = A \sqcup B$ such that $A \in \mathcal{I}_2$ and $\mathcal{I}_2 \restriction B = \mathcal{I}_1 \restriction B$.*

Proof. Take A to be the union of a maximal family of pairwise disjoint sets in $\mathcal{I}_2 \setminus \mathcal{I}_1$. □

The following lemma will be used in the proofs of Theorems 4.5 and 4.8(d). Recall that an ideal $\mathcal{J}$ is nowhere prime iff every $\mathcal{J}$-positive set can be partitioned into two $\mathcal{J}$-positive subsets.

Lemma 4.7. *Suppose $\mathcal{J}$ is a nowhere prime supersaturated ideal on X and $\mu = \mathsf{add}(\mathcal{J})$. Then $\mu \leq \mathfrak{c}$ and there exists a μ-additive supersaturated ideal on μ.*

Proof. Towards a contradiction, suppose $\mu > \mathfrak{c}$. Construct a tree $\langle A_\sigma : \sigma \in 2^{<\omega_1} \rangle$ of subsets of X as follows.

(i) $A_\emptyset = X$.
(ii) If $A_\sigma \in \mathcal{J}^+$, then $\{A_{\sigma 0}, A_{\sigma 1}\}$ is a partition of A_σ into two $\mathcal{J}$-positive sets. This is possible since $\mathcal{J}$ is nowhere prime.
(iii) If $A_\sigma \in \mathcal{J}$, then $A_{\sigma 0} = A_{\sigma 1} = A_\sigma$.
(iv) If $\alpha < \omega_1$ is limit and $\sigma \in 2^\alpha$, then $A_\sigma = \bigcap \{A_{\sigma \restriction \beta} : \beta < \alpha\}$.

Put $\mathcal{F} = \{A_\sigma : \sigma \in 2^{<\omega_1} \text{ and } A_\sigma \in \mathcal{J}\}$. We claim that $X = \bigcup \mathcal{F}$. Suppose not and fix $y \in X \setminus \bigcup \mathcal{F}$. Now observe that $\{A_{\sigma k} : \sigma \in 2^{<\omega_1} \wedge k < 2 \wedge y \in (A_\sigma \setminus A_{\sigma k})\}$ is an uncountable family of pairwise disjoint $\mathcal{J}$-positive sets which contradicts the fact that $\mathcal{J}$ is ω_1-saturated. So $X = \bigcup \mathcal{F}$. But since $|\mathcal{F}| \leq |2^{<\omega_1}| = \mathfrak{c}$, this contradicts the fact that $\mathsf{add}(\mathcal{J}) = \mu > \mathfrak{c}$. Hence $\mu \leq \mathfrak{c}$.

Since $\mathsf{add}(\mathcal{J}) = \mu$, there are $Y \in \mathcal{J}^+$ and a partition $Y = \bigsqcup_{\alpha < \mu} W_\alpha$ such that for every $\Gamma \in [\mu]^{<\mu}$, $\bigcup_{\alpha \in \Gamma} W_\alpha \in \mathcal{J}$. Define

$$\mathcal{K} = \{\Gamma \subseteq \mu : \bigcup_{\alpha \in \Gamma} W_\alpha \in \mathcal{J}\}$$

Then $\mathcal{K}$ is a μ-additive ω_1-saturated ideal on μ. So μ is weakly inaccessible. We claim that $\mathcal{K}$ must also be supersaturated. To see this, suppose $\mathcal{A} \subseteq \mathcal{K}^+$ and $|\mathcal{A}| < \mu$. For each $A \in \mathcal{A}$, define $Y_A = \bigsqcup_{\alpha \in A} W_\alpha$. Then $\{Y_A : A \in \mathcal{A}\} \subseteq \mathcal{J}^+$.

Since $\mathcal{J}$ is supersaturated, we can choose a countable $T \subseteq Y$ that meets Y_A for every $A \in \mathcal{A}$. Let $B = \{\alpha < \mu : T \cap W_\alpha \neq \emptyset\}$. Then $B \subseteq \mu$ is countable (as W_α's are pairwise disjoint) and it meets every $A \in \mathcal{A}$. Hence $\mathcal{K}$ is a μ-additive supersaturated ideal on μ. □

Proof of Theorem 4.5: Clause (a) is easy to check. Let us prove Clause (b). Suppose $\mathcal{J}$ is a supersaturated ideal on X. Put $\mu = \mathsf{add}(\mathcal{J})$. We claim that it suffices to show that $V^{\mathbb{S}} \models \mu > \mathfrak{c}$. First note that, by Lemma 4.7, this would imply that for every $Y \in \mathcal{J}^+$, there exists $\mathcal{J}$-positive $Z \subseteq Y$ such that $\mathcal{J} \restriction Z$ is a prime ideal. Hence by ω_1-saturation of $\mathcal{J}$, we can find a countable partition of X into $\mathcal{J}$-positive sets such that the restriction of $\mathcal{J}$ to each one of them is a prime ideal.

So towards a contradiction, assume $V^{\mathbb{S}} \models \mu \leq \mathfrak{c}$. Fix $Y \in \mathcal{J}^+$ such that for every $\mathcal{J}$-positive $Z \subseteq Y$, $\mathsf{add}(\mathcal{J} \restriction Z) = \mu$. Since $\mu \leq \mathfrak{c}$, it follows that $\mathcal{J} \restriction Y$ is a nowhere prime supersaturated ideal. Using Lemma 4.7 again, we can get a μ-additive supersaturated ideal $\mathcal{K}$ on μ. Let us assume that the trivial condition in $\mathbb{S}$ forces all of this about $\mathcal{K}$.

Since $V^{\mathbb{S}} \models$ "$\mu \leq \mathfrak{c} = \kappa^+$ and μ is weakly inaccessible", we must have $\mu \leq \kappa$. We consider two cases.

Case $\mu < \kappa$: In V, define $\mathcal{I}' = \{X \subseteq \mu : 1_{\mathbb{S}} \Vdash X \in \mathcal{K}\}$. Since $\mathbb{S}$ is ccc, $V \models \mathcal{I}'$ is a μ-additive ω_1-saturated ideal on μ. As $V \models \mu > \omega_1 = \mathfrak{c}$, μ is measurable in V. Since κ is the least measurable cardinal in V, $\mu \geq \kappa$: Contradiction.

Case $\mu = \kappa$: In V, define $\mathcal{I}' = \{X \subseteq \kappa : 1_{\mathbb{S}} \Vdash X \in \mathcal{K}\}$. Since $V \models \kappa > \mathfrak{c} = \omega_1$, we must have $V \models \mathcal{I}'$ is a κ-additive prime ideal on κ. Let $\mathcal{K}'$ be the ideal generated by $\mathcal{I}'$ in $V^{\mathbb{S}}$. Then $V^{\mathbb{S}} \models \mathcal{K}' \subseteq \mathcal{K}$ are ω_1-saturated κ-additive ideals on κ. Using Fact 4.6, fix $B \in \mathcal{K}^+$ such that $\mathcal{K}' \restriction B = \mathcal{K} \restriction B$.

Choose $\gamma < \kappa^+$ such that $\mathring{B} \in V^{\mathbb{S}_\gamma}$. Let $\mathcal{K}''$ be the ideal generated by $\mathcal{I}'$ in $V^{\mathbb{S}_\gamma}$. By Theorem 4.3, it follows that in $V^{\mathbb{S}_{\gamma+1}}$, the ideal generated by $\mathcal{K}'' \restriction B$ is not supersaturated. Now observe that $\mathcal{K} \restriction B = \mathcal{K}' \restriction B$ is the ideal generated by $\mathcal{K}'' \restriction B$ in $V^{\mathbb{S}}$. It follows that $\mathcal{K}$ is not a supersaturated ideal: Contradiction. □

Using some results about separating families and supersaturated ideals from [2, 4], we can also get the following.

Theorem 4.8. *Suppose κ is a measurable cardinal with a witnessing normal prime ideal $\mathcal{I}$. Let $\mathbb{P}_\kappa$ be the forcing in Definition 4.1. Then the following hold in $V^{\mathbb{P}_\kappa}$.*

(a) *$\mathfrak{c} = \kappa$ and the ideal generated by $\mathcal{I}$ is a normal ω_1-saturated ideal on κ.*
(b) *There is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| = \omega_1$ and for every countable $X \subseteq \kappa$ and $\alpha \in \kappa \setminus X$, there exists $S \in \mathcal{F}$ such that $\alpha \in S$ and $S \cap X = \emptyset$.*
(c) *The order dimension of Turing degrees is ω_1.*
(d) *There are no nowhere prime supersaturated ideals.*

Proof. (a) Since $\mathbb{Q}_\omega^\kappa$ adds κ Cohen reals, $\mathfrak{c} \geq \kappa$. The other inequality follows by a name counting argument using the facts that $\mathbb{P}_\kappa$ is a ccc forcing, $|\mathbb{P}_\kappa| = \kappa$ and $\kappa^\omega = \kappa$. That the ideal generated by $\mathcal{I}$ is a normal ω_1-saturated ideal on κ follows from the fact that $\mathbb{P}_\kappa$ is ccc.

(b) For each indecomposable $\delta < \omega_1$, define

$$S_\delta = \{\alpha < \kappa : (\exists p \in G_{\mathbb{P}_\kappa})(\delta \in \mathsf{dom}(p) \wedge p(\delta)(\alpha) = 1)\}$$

Let $\mathcal{F} = \{S_\delta : \delta < \omega_1 \text{ is indecomposable}\}$. Suppose $X \subseteq \kappa$ is countable and $\alpha \in \kappa \setminus X$. We'll find an $S_\delta \in \mathcal{F}$ such that $\alpha \in S_\delta$ and $X \cap S_\delta = \emptyset$. Since $\mathbb{P}_\kappa$ is ccc, we can find a countable $Y \in V$ such that $X \subseteq Y \subseteq \kappa \setminus \{\alpha\}$. Now an easy density argument shows that the set

$$D_{\alpha,Y} = \{p \in \mathbb{P}_\kappa : (\exists \delta \in \mathsf{dom}(p))[p(\delta)(\alpha) = 1 \wedge (\forall \beta \in Y)(p(\delta)(\beta) = 0)]\}$$

is dense in $\mathbb{P}_\kappa$. So we can choose $p \in D_{\alpha,Y} \cap G_{\mathbb{P}_\kappa}$. Let δ witness that $p \in D_{\alpha,Y}$. Then it is clear that $\alpha \in S_\delta$ and $X \cap S_\delta \subseteq Y \cap S_\delta = \emptyset$.

(c) This follows from Theorem 3.9 in [2] and part (b) above.

(d) Suppose not. Then by Lemma 4.7, we can find some $\mu \leq \mathfrak{c} = \kappa$ and a μ-additive supersaturated ideal on μ. Let $\mathcal{F}$ be as in part (b) above. Define $\mathcal{E} = \{S \cap \mu : S \in \mathcal{F}\}$. Then $|\mathcal{E}| = \omega_1$ and for every countable $X \subseteq \mu$ and $\alpha \in \mu \setminus X$, there exists $S \in \mathcal{E}$ such that $\alpha \in S$ and $S \cap X = \emptyset$. Now applying Lemma 4.2 in [4] gives us a contradiction. □

We conclude with the following questions.

(1) Suppose $\mathcal{I}, \mathcal{J}$ are normal ideals on κ, $\mathcal{I}$ is supersaturated and $\mathcal{P}(\kappa)/\mathcal{I}$ is isomorphic to $\mathcal{P}(\kappa)/\mathcal{J}$. Must $\mathcal{J}$ be supersaturated?
(2) Suppose κ is regular uncountable, $\mathcal{I}$ is a κ-complete normal ideal on κ and $\mathcal{P}(\kappa)/\mathcal{I}$ is a Cohen algebra. Must $\mathcal{I}$ be supersaturated?
(3) Do σ-finite/bounded-cc forcings preserve supersaturation? What about Boolean algebras that admit a strictly positive finitely additive measure?

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(Kumar) Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, UP, India.
Email address: krashu@iitk.ac.in
URL: https://home.iitk.ac.in/~krashu/

(Raghavan) Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076.
Email address: dilip.raghavan@protonmail.com
URL: https://dilip-raghavan.github.io/