SET THEORY AND TURING DEGREES

ASHUTOSH KUMAR

Abstract. We survey some set-theoretic problems about Turing degrees.

1. Introduction

The goal of this survey is to discuss some problems about Turing degrees where set theory has played a role. Our starting point is the following question of Sacks [12]. Does every locally countable partial ordering of size continuum embed into the Turing degrees $(\mathcal{D}, \leq_T)$? Recall that a poset $(\mathbb{P}, \leq_{\mathbb{P}})$ is locally countable iff for every $x \in \mathbb{P}$, $\{y \in \mathbb{P} : y \leq_{\mathbb{P}} x\}$ is countable. Sacks showed that the answer is yes under the continuum hypothesis. But can we prove this in ZFC?

As far as we know, it is not even known if every well-founded locally countable poset of size continuum embeds into the Turing degrees. It turns out that to be able to embed such posets, we need to construct Turing independent sets with additional properties. These matters are discussed in Section 3.

In Section 4, we discuss some Ramsey type problems about Turing independent sets of the following type: Given a large set of reals, does it have a large Turing independent subset? Largeness is defined in terms of cardinality, measure and category. With a couple of exceptions, most of these problems turned out to be undecidable in ZFC. Some questions remain open and have been stated at relevant places.

Section 2 reviews some classical constructions of Spector and Sacks that are used later. We have tried to present the main ideas behind the proofs of as many results as possible. Anyone interested in seeing the details can find them in the relevant citations.

1.1. **Notation.** Φ_e denotes the eth Turing functional. For $x \in 2^\omega$, $k < \omega$ and $\ell < 2$, if the eth Turing functional with oracle x on input k halts/converges and outputs ℓ, then we write $\Phi_e^x(k) \downarrow = \ell$. If $\sigma \subseteq x$ contains the (finite) oracle use of this computation, then we also write $\Phi_e^\sigma(k) \downarrow = \ell$. $\mathsf{dom}(\Phi_e^x) = \{k < \omega : \Phi_e^x(k) \downarrow\}$ and Φ_e^x is total iff $\mathsf{dom}(\Phi_e^x) = \omega$. For $x, y \in 2^\omega$, we say that x is computable from/Turing reducible to y and write $x \leq_T y$ if there exists $e < \omega$ such that $\Phi_e^x = y$. Turing equivalence is defined by $x \equiv_T y$ iff $x \leq_T y$ and $y \leq_T x$. $\mathcal{D}$ is the set of all $\equiv_T$-equivalence classes in 2^ω with the induced partial order also denoted by $\leq_T$. For $p \subseteq {}^{<\omega}2$, define $[p] = \{x \in 2^\omega : (\forall n < \omega)(x \upharpoonright n \in p)\}$. For $\sigma \in {}^{<\omega}2$, we write $[\sigma] = \{x \in 2^\omega : \sigma \subseteq x\}$. For a finite list $\langle x_k : k < n \rangle$ of functions whose domains are subsets of ω, we define the join of F, denoted $\bigoplus_{k<n} x_k$, to be the partial function y on ω satisfying $y(nj+k) = x_k(j)$ for every $k < n$ and $j \in \mathsf{dom}(x_k)$. $(m, n) \mapsto \langle m, n \rangle$ is a fixed computable bijection from $\omega \times \omega$ to ω that is used to extend statements

The author was partially supported by SERB grant no. MTR/2023/000334. We thank Ted Slaman for pointing out an error in an earlier version of Fact 2.7.

about functions with domain $\subseteq \omega$ to corresponding statements about functions with domain $\subseteq \omega \times \omega$. Similar conventions apply to other countable sets like ${}^{<\omega}2$.

2. Some classical constructions

Let $\mathbb{S}$ denote Sacks forcing: $p \in \mathbb{S}$ iff p is a perfect subtree of ${}^{<\omega}2$ and for $p, q \in \mathbb{S}$, $p \leq_{\mathbb{S}} q$ iff $p \subseteq q$. If G is an $\mathbb{S}$-generic filter over V, then the unique real $s_G \in \bigcap\{[p] : p \in G\}$ is called a Sacks real over V. Sacks [13] showed that if s is a Sacks real over V, then for every $x \in 2^\omega \cap V[s]$, either $x \in V$ or there exists $y \in 2^\omega \cap V$ such that $s \leq_T x \oplus y$. In the same paper, he also introduced recursively pointed trees.

Definition 2.1. *We say that $p \in \mathbb{S}$ is recursively pointed iff every $x \in [p]$ computes p. Let $\mathbb{S}_{rp}$ be the forcing whose conditions are recursively pointed trees in $\mathbb{S}$ ordered by inclusion.*

The following facts are useful in constructions involving $\mathbb{S}_{rp}$.

Fact 2.2 (Sacks)**.** *Let $p \in \mathbb{S}_{rp}$.*

(a) *For every $y \in 2^\omega$, $p \leq_T y$ iff $(\exists x \in [p])(x \equiv_T y)$.*
(b) *For every $y \in 2^\omega$, if $p \leq_T y$, then there exists $q \in \mathbb{S}_{rp}$ such that $q \subseteq p$ and $q \equiv_T y$.*
(c) *If $q \in \mathbb{S}$, $q \subseteq p$ and $q \leq_T p$, then $q \in \mathbb{S}_{rp}$ and $q \equiv_T p$.*

How different is $\mathbb{S}_{rp}$ from $\mathbb{S}$? If G is an $\mathbb{S}_{rp}$-generic filter over V, then there is a unique real $x_G \in \bigcap\{[p] : p \in G\}$. Any such real is called an $\mathbb{S}_{rp}$-generic real over V. Using Fact 2.2(b), it is easy to see that every $\mathbb{S}_{rp}$-generic real over V computes every real in V. It follows that in $V^{\mathbb{S}_{rp}}$ the ground model continuum becomes countable. Therefore, unlike Sacks forcing, $\mathbb{S}_{rp}$ is improper.

Definition 2.3. *Let $p \in \mathbb{S}$ and $e < \omega$. We say that p is an e-splitting tree iff for every $x \neq y$ in $[p]$, there exists $k < \omega$ such that $\Phi_e^x(k) \downarrow \neq \Phi_e^y(k) \downarrow$.*

Suppose p is e-splitting, $x \in [p]$ and $\Phi_e^x = y \in 2^\omega$. We claim that $x \leq_T y \oplus p$. To see this, suppose $\sigma = x \upharpoonright n$ has been computed and we want to know whether $x(n)$ is 0 or 1. We can assume that both $\sigma^\frown 0, \sigma^\frown 1$ are in p otherwise this is easy. As p is e-splitting and $\Phi_e^x = y$, we can perform a successful search for some $\ell < 2$, $\langle \tau_i : i < N\rangle$, $\langle k_i : i < N\rangle$ and $\rho \in p$ such that each real in $[p]$ above $\sigma^\frown \ell$ extends some τ_i, $\sigma^\frown(1-\ell) \preceq \rho$ and $\Phi_e^{\tau_i}(k_i) \downarrow \neq \Phi_e^{\rho}(k_i) \downarrow = y(k_i)$. Then $x(n) = 1 - \ell$.

Definition 2.4. *Let $p \in \mathbb{S}$ and $e < \omega$. We say that p is an e-good tree iff either p is e-splitting or for every $\sigma, \tau \in p$ and $k < \omega$, if $\Phi_e^\sigma(k)$ and $\Phi_e^\tau(k)$ both converge, then they are equal.*

Suppose p is e-good and not e-splitting. Then for every $x \in [p]$, if $\Phi_e^x = y \in 2^\omega$, then $y \leq_T p$ since to compute $y(k)$, we perform a (successful) search for some $\tau \in p$ such that $\Phi_e^\tau(k) \downarrow = \ell$ and output ℓ. Relativizing Spector's minimal degree construction to a recursively pointed tree gives the following.

Fact 2.5 (Spector minimal degree)**.** *Let $p \in \mathbb{S}_{rp}$.*

(a) *For every $e < \omega$, there exists $q \in \mathbb{S}_{rp}$ such that $q \subseteq p$, $q \equiv_T p$ and q is e-good. It follows that for every $x \in [q]$, if $\Phi_e^x = y \in 2^\omega$, then either $y \leq_T q$ or $x \leq_T y \oplus q$.*

(b) *There exists $r \in \mathbb{S}$ such that $r \subseteq p$ and for every $x \in [r]$ and $y \in 2^\omega$, if $y \leq_T x$, then either $y \leq p$ or $x \leq_T y \oplus p$.*

Proof. For $p \in \mathbb{S}$, define $\mathsf{Split}_k(p)$ to be the set of all kth level splitting nodes of p. So $\mathsf{Split}_0(p)$ is the singleton containing the stem of p and $\mathsf{Split}_k(p)$ has 2^k nodes.

(a) Call $\sigma \in p$ an ambiguous node if there are there are $k < \omega$ and $\tau_1, \tau_2 \in p$ above σ such that $\Phi_e^{\tau_1}(k) \downarrow \neq \Phi_e^{\tau_2}(k) \downarrow$. We consider two cases.

Case 1. Some $\sigma \in p$ is not ambiguous. Define $q = \{\tau \in p : \tau \preceq \sigma \text{ or } \sigma \preceq \tau\}$. By Fact 2.2(c), $q \in \mathbb{S}_{rp}$ and $q \equiv_T p$. That q is e-good follows from the fact that σ is not ambiguous.

Case 2. All nodes in p are ambiguous. Inductively construct a sequence $\langle p_n : n < \omega \rangle$ of members of $\mathbb{S}_{rp}$ as follows.

(i) $p_0 = p$.
(ii) Given p_n, define p_{n+1} as follows. Let $\{\sigma_j : j < 2^{n+1}\}$ list $\mathsf{Split}_{n+1}(p_n)$. For each $j < 2^n$, search for the least $k < \omega$ and $\tau_{j,0}, \tau_{j,1} \in p$ above σ_j such that $\Phi_e^{\tau_{j,0}}(k) \downarrow \neq \Phi_e^{\tau_{j,1}}(k) \downarrow$ and define

$$p_{n+1} = \{\sigma \in p : (\exists \ell < 2)(\exists j < 2^{n+1})(\sigma \preceq \tau_{j,\ell} \text{ or } \tau_{j,\ell} \preceq \sigma)\}.$$

Put $q = \bigcap_{n<\omega} p_n$ and observe that $q \leq_T p$. So by Fact 2.2(c), $q \in \mathbb{S}_{rp}$ and $q \equiv_T p$. Note that for every $x \neq y$ in $[q]$, there exists $k < \omega$ such that $\Phi_e^x(k) \downarrow \neq \Phi_e^y(k) \downarrow$. So q is e-splitting and therefore e-good.

(b) Using (a), construct a sequence $\langle p_n : n < \omega \rangle$ of members of $\mathbb{S}_{rp}$ as follows. $p_0 = p$, $p_{n+1} \subseteq p_n$, $p_{n+1} \equiv_T p_n$, $\mathsf{Split}_n(p_n) = \mathsf{Split}_n(p_{n+1})$ and for each $\sigma \in \mathsf{Split}_{n+1}(p_{n+1})$, $\{\tau \in p_{n+1} : \tau \preceq \sigma \text{ or } \sigma \preceq \tau\}$ is n-good. Define $r = \bigcap_{n<\omega} p_n$. Then r is as required. ☐

Definition 2.6. *For $A \subseteq 2^\omega$, the Turing ideal generated by A, denoted $\mathcal{I}_A$, is the set of all reals that are computable from the join of a finite subset of A. We say that $y \in 2^\omega$ is a minimal upper bound of $\mathcal{I}_A$ iff*

(a) *for every $x \in \mathcal{I}_A$, $x \leq_T y$ and*
(b) *for every $z <_T y$, there exists some $x \in \mathcal{I}_A$, $x \not\leq_T z$.*

Fact 2.7 (Sacks minimal upper bound)**.** *Let $A \subseteq 2^\omega$ be countable. Then there exists $p \in \mathbb{S}$ such that for every $y \in [p]$, the following hold.*

(i) *y computes every real in $\mathcal{I}_A$.*
(ii) *If $z \leq_T y$, then there exists $x \in \mathcal{I}_A$ such that either $z \leq_T x$ or $y \leq_T x \oplus z$.*
(iii) *If $\mathcal{I}_A$ is not finitely generated, then y is a minimal upper bound of $\mathcal{I}_A$.*

Proof. Fix a $\leq_T$-increasing sequence $\langle a_k : k < \omega \rangle$ such that $a_0 \equiv_T 0$, each $a_k \in \mathcal{I}_A$ and $(\forall y \in \mathcal{I}_A)(\exists k < \omega)(y \leq_T a_k)$. We can further assume that if $\mathcal{I}_A$ is not finitely generated, then $a_n <_T a_{n+1}$ for every n.

Inductively construct a sequence $\langle p_n : n < \omega \rangle$ of members of $\mathbb{S}_{rp}$ satisfying the following.

(1) $p_0 = {}^{<\omega}2$.
(2) $p_n \in \mathbb{S}_{rp}$ and $p_{n+1} \subseteq p_n$.
(3) $\mathsf{Split}_n(p_n) = \mathsf{Split}_n(p_{n+1})$.
(4) For every $\sigma \in \mathsf{Split}_{n+1}(p_{n+1})$, $\{\tau \in p_{n+1} : \sigma \preceq \tau \text{ or } \tau \preceq \sigma\}$ is n-good.
(5) $p_{n+1} \equiv_T a_{n+1}$.

To obtain p_{n+1} from p_n, we use Facts 2.5(a) and 2.2(b). Let $p' \in \mathbb{S}$ be the intersection of $\{p_n : n < \omega\}$. Choose $p \in \mathbb{S}$ such that $p \subseteq p'$ and no member of $[p]$ is computable from any member of $\mathcal{I}_A$.

Let us check that p is as required. Fix $y \in [p]$. Since $\{a_n : n < \omega\}$ is $\leq_T$-cofinal in $\mathcal{I}_A$, it is clear than y computes every real in $\mathcal{I}_A$. Thus (i) holds. Next, assume $z \leq_T y$ and fix $e < \omega$ such that $\Phi_e^y = z$. Fix $n > e + 1$. Then by Clauses (4) and (5), either $z \leq_T a_n$ or $y \leq_T z \oplus a_n$. So (ii) holds. Finally assume $\mathcal{I}_A$ is not finitely generated. Then $a_n <_T a_{n+1}$ for every n. Towards a contradiction, fix some $z <_T y$ such that $(\forall n)(a_n \leq_T z)$. Using (ii), fix $N < \omega$ such that either $z \leq_T a_N$ (impossible since $a_N <_T a_{N+1} \leq_T z$) or $y \leq_T z \oplus a_N$. As $a_N \leq_T z$, the latter implies $y \leq_T z <_T y$ which is a contradiction. □

The next fact is a minor generalization of a result of Spector (see Lemma 2.3 in [8]). Suppose $A \subseteq 2^\omega$ is countable and $p \in \mathbb{S}$, we say that $[p]$ is a perfect set of exact pairs for A iff for every distinct $x, y \in [p]$,

$$\mathcal{I}_A = \{z \in 2^\omega : z \leq_T x \text{ and } z \leq_T y\}.$$

Fact 2.8 (Spector exact pair). *Suppose $A \subseteq 2^\omega$ is countable. Then there exists $p \in \mathbb{S}$ such that $[p]$ is a perfect set of exact pairs for A.*

Definition 2.9. *Let $X \subseteq 2^\omega$ and $1 \leq n < \omega$.*

(1) *X is n-Turing independent iff for every $F \subseteq X$ of size $|F| \leq n$, the Turing join of F does not compute any real in $X \setminus F$. 1-Turing independent sets are also called Turing antichains.*
(2) *X is Turing independent iff it is n-Turing independent for every $1 \leq n < \omega$.*
(3) *X is σ-Turing independent iff for every countable $A \subseteq X$, there exists $y_A \in 2^\omega$ such that y_A computes every real in A and $(X \setminus A) \cup \{y_A\}$ is Turing independent.*

How large can a Turing independent set of reals be? In [12], Sacks constructed a Turing independent set of size continuum. One can also construct $p \in \mathbb{S}$ via finite approximations such that $[p]$ is Turing independent. Relativizing this construction to an arbitrary perfect tree gives the following.

Fact 2.10 (Sacks). *For every $p \in \mathbb{S}$, there exists $q \in \mathbb{S}$ such that $q \subseteq p$ and $[q]$ is Turing independent.*

Proof. Fix $p \in \mathbb{S}$. Let $\mathbb{Q}_p$ be the forcing whose conditions are finite subtrees $T \subseteq p$ ordered by end-extension. So $T \leq S$ iff $S \subseteq T$ and for every $\tau \in T \setminus S$ there exists a terminal node $\sigma \in S$ such that $\sigma \preceq \tau$. One can write down a countable family $\mathcal{D}$ of dense sets in $\mathbb{Q}_p$ such that whenever G is a filter on $\mathbb{Q}_p$ that meets every set in $\mathcal{D}$, $q = \bigcup G$ is as required. The dense sets needed to ensure that q is a perfect subtree of p are easy to write down and left to the reader. To ensure the Turing independence of $[q]$ we require that $\{D_{e,n} : e, n < \omega\} \subseteq \mathcal{D}$ where $D_{e,n}$ consists of those $T \in \mathbb{Q}_p$ which satisfy the following.

(i) Every terminal node in T has length $\geq n$
(ii) For every injective sequence $\bar{\sigma} = \langle \sigma_k : k \leq N+1 \rangle$ of terminal nodes in T, if there exists $\langle x_k : k \leq N \rangle$ such that each $\sigma_k \preceq x_k \in [p]$ and Φ_e^X is total where $X = \bigoplus_{k \leq N} x_k$, then there exists $m \in \mathsf{dom}(\sigma_{N+1})$ such that $\Phi_e^\rho(m) \downarrow \neq \sigma_{N+1}(m)$ where $\rho = \bigoplus_{k \leq N} \sigma_k$.

□

3. Suborders of Turing degrees

Definition 3.1. *Let $\mathcal{C}_{lc}$ be the class of all locally countable posets of size continuum. We say that $\mathbb{P}$ is universal for $\mathcal{C}_{lc}$ if $\mathbb{P} \in \mathcal{C}_{lc}$ and every $\mathbb{Q} \in \mathcal{C}_{lc}$ embeds into $\mathbb{P}$.*

Fact 3.2 ([3])**.** *There is a $\mathbb{P} \in \mathcal{C}_{lc}$ which is universal for $\mathcal{C}_{lc}$.*

Note that the poset $(\mathcal{D}, \leq_T)$ is in $\mathcal{C}_{lc}$.

Question 3.3 (Sacks)**.** *Is $(\mathcal{D}, \leq_T)$ universal for $\mathcal{C}_{lc}$?*

Sacks [12] showed that the answer is yes under CH or just MA. The problem remains open in ZFC. As suggested in [3], it may be useful to first try to see if all well-founded members of $\mathcal{C}_{lc}$ can be embedded into the Turing degrees. In fact, the problem seems non-trivial even for well-founded locally countable posets of finite rank.

Definition 3.4 (Higuchi)**.** *For an infinite set X and $n < \omega$, define the poset $\mathbb{H}^n_X$ as follows. Put $X_0 = X$ and $X_{k+1} = [X_k]^{\aleph_0}$ for every $k \geq 0$. The universe of $\mathbb{H}^n_X$ is defined to be the disjoint union $\bigsqcup\{X_k : k \leq n\}$. We say that p belongs to the kth level of $\mathbb{H}^n_X$ iff $p \in X_k$. For $a, B \in \mathbb{H}^n_X$, define $a \preceq B$ iff either $a = B$ or for some k, $a \in X_k$, $B \in X_{k+1}$ and $a \in B$. Define $\leq_{\mathbb{H}^n_X}$ to be the transitive closure of $\preceq$.*

Note that $\mathbb{H}^n_X$ is a locally countable poset for every X and $\mathbb{H}^n_X \in \mathcal{C}_{lc}$ iff $|X| \leq \mathfrak{c}$. Furthermore, $\mathbb{H}^n_{\mathfrak{c}}$ embeds into $\mathbb{H}^{n+1}_\omega$. The following is a minor generalization of Theorem 2.2 in [8]. Also compare with Lemma 2.5 in [6].

Lemma 3.5. *For every Turing independent $X \subseteq 2^\omega$, there is an embedding of $\mathbb{H}^1_X$ into the Turing degrees that is the identity on X.*

Proof. By Fact 2.7, for each $A \in [X]^{\aleph_0}$, we can fix $p_A \in \mathbb{S}$ such that for every $x_A \in [p_A]$, x_A computes every real in $\mathcal{I}_A$ and for every $z \leq_T x_A$, there exists $y \in \mathcal{I}_A$ such that either $z \leq_T y$ or $x_A \leq_T y \oplus z$.

Fix an injective enumeration $\langle A_i : i < \mathfrak{c}\rangle$ of $[X]^{\aleph_0}$. We will inductively construct $\langle x_i : i < \mathfrak{c}\rangle$ such that each $x_i \in [p_{A_i}]$ and for every $j < i < \mathfrak{c}$, x_j and x_i are Turing incomparable. At stage i, to be able to choose an appropriate $x_i \in [p_{A_i}]$, it is enough to show that for any $j < i$, both $W_1 = \{x \in p_{A_i} : x \leq_T x_j\}$ and $W_2 = \{x \in p_{A_i} : x_j \leq_T x\}$ are countable. That W_1 is countable is obvious. Towards a contradiction, suppose W_2 is uncountable. Choose an uncountable $W \subseteq W_2$ and $y_\star \in \mathcal{I}_{A_i}$ such that for every $x \in W$, $x_j <_T x$ and either $x \leq_T x_j \oplus y_\star$ or $x_j \leq y_\star$. Now $x_j \leq y_\star$ is impossible because A_j is infinite and X is Turing independent. So every real in W must be computable from $x_j \oplus y_\star$. But this is impossible since W is uncountable. Therefore, $\langle x_i : i < \mathfrak{c}\rangle$ can be constructed.

Define $f : \mathbb{H}^1_X \to 2^\omega$ by $f \upharpoonright X$ is identity and $f(A_i) = x_{A_i}$ for each $i < \mathfrak{c}$. To show that f is an embedding, it is enough to check the following.

(a) For every $A \in [X]^{\aleph_0}$ and $y \in X$, $y \leq_T f(A)$ iff $y \in A$.
(b) For every $A \neq B$ in $[X]^{\aleph_0}$, $f(A)$ and $f(B)$ are Turing incomparable.

Clause (b) follows from the fact that x_i's are pairwise Turing incomparable. Let us check (a). If $y \in A$, then clearly $y \leq_T f(A)$. Next, towards a contradiction, assume $y \in X \setminus A$ and $y \leq_T f(A)$. Fix $z \in \mathcal{I}_A$ such that either $y \leq_T z$ or $f(A) \leq_T y \oplus z$. Since $y \notin A$, $z \in \mathcal{I}_A$ and X is Turing independent, we cannot have $y \leq_T z$. As X is Turing independent, only finitely many reals in A are computable from $y \oplus z$. But $f(A) \leq_T y \oplus z$ computes every real in A. A contradiction. $\square$

3.1. **An embedding.** Working a little harder, we can improve Lemma 3.5 to the following.

Lemma 3.6. *For every Turing independent $X \subseteq 2^\omega$ with $|X| = \omega$, there is an embedding of $\mathbb{H}^2_X$ into the Turing degrees that is the identity on X.*

Proof. Fix an injective enumeration $\langle a_k : k < \omega \rangle$ of X. Define a forcing $\mathbb{P}$ as follows. $p \in \mathbb{P}$ iff $p = (K, f) = (K_p, f_p)$ where

(i) $K < \omega$

(ii) f is a function with domain ${}^K2 \times {}^K2$ and for every $\sigma, \tau \in {}^K2$, the following hold.

(a) $f(\sigma, \tau)$ is a partial function from $\omega \times \omega$ to 2.

(b) If $\tau \neq \tau'$, then $\mathsf{dom}(f(\sigma, \tau)) = \mathsf{dom}(f(\sigma, \tau))$ and $f_{\sigma,\tau} \neq f_{\sigma,\tau'}$.

(c) $K \times \omega \subseteq \mathsf{dom}(f(\sigma, \tau))$.

(d) $\mathsf{dom}(f(\sigma, \tau)) \setminus (K \times \omega)$ is finite.

(e) For every $k < K$, if $\sigma(k) = 1$, then $\{n < \omega : f(\sigma, \tau)(k, n) \neq a_k(n)\}$ is finite.

(f) For every $k < K$, if $\sigma(k) = 0$, then $\{n < \omega : f(\sigma, \tau)(k, n) \neq 0\}$ is finite.

For $p, q \in \mathbb{P}$, define $p \leq q$ iff $K_q \leq K_p$ and for every $\sigma, \tau \in {}^{K_p}2$, $f_q(\sigma \upharpoonright K_q, \tau \upharpoonright K_q) \subseteq f_p(\sigma, \tau)$.

Definition 3.7. *Suppose $p \in \mathbb{P}$, $e, N < \omega$, $i < K_p$, $\bar{\sigma} = \langle \sigma_k : k \leq N+1 \rangle$ and $\bar{\tau} = \langle \tau_k : k \leq N+1 \rangle$ are sequences of members of ${}^{K_p}2$ and $\bar{\sigma}$ is injective. We write $\mathsf{Split}_e(p, \bar{\sigma}, \bar{\tau}, i)$ to denote the following statement. If there exist $x_k : \omega \times \omega \to 2$ for $k \leq N$ such that $f_p(\sigma_k, \tau_k) \subseteq x_k$ and Φ^X_e is total where $X = \bigoplus_{k \leq N} x_k$, then there exist $x_k : \omega \times \omega \to 2$ for $k \leq N$ such that $f_p(\sigma_k, \tau_k) \subseteq x_k$ and the following hold.*

(a) *There exists $(j, m) \in \mathsf{dom}(f_p(\sigma_{N+1}, \tau_{N+1}))$ such that*
$$\Phi^X_e(\langle j, m \rangle) \downarrow \neq f_p(\sigma_{N+1}, \tau_{N+1})(j, m)$$
and the use of the computation $\Phi^X_e(\langle j, m \rangle)$ is contained in $\bigoplus_{k \leq N} f_p(\sigma_k, \tau_k)$.

(b) *If $(\forall k \leq N)(\sigma_k(i) = 0)$, then there exists $n < \omega$ such that $\Phi^X_e(n) \downarrow \neq a_i(n)$ and the use of the computation $\Phi^X_e(n)$ is contained in $\bigoplus_{k \leq N} f_p(\sigma_k, \tau_k)$.*

The following is easy to check.

Claim 3.8. *Suppose p, e, i, N, $\bar{\sigma}$ and $\bar{\tau}$ are as in Definition 3.7 and $\mathsf{Split}_e(p, \bar{\sigma}, \bar{\tau}, i)$ holds. Assume $q \in \mathbb{P}$, $q \leq p$ and $K_q = K_p$. Then $\mathsf{Split}_e(q, \bar{\sigma}, \bar{\tau}, i)$ holds.*

Definition 3.9. *$p \in \mathbb{P}$ is a splitting condition iff whenever $e, i < K_p$, $N < 2^{K_p}$ and $\bar{\sigma}$, $\bar{\tau}$ are sequences of members of ${}^{K_p}2$ of length $N+1$ where $\bar{\sigma}$ is injective, $\mathsf{Split}_e(p, \bar{\sigma}, \bar{\tau}, i)$ holds.*

Every condition can be extended to a splitting condition.

Claim 3.10. *For every $p \in \mathbb{P}$, there exists $q \leq p$ such that $K_q = K_p$ and q is a splitting condition.*

Proof of Claim 3.10. In view of Claim 3.8, it suffices to show the following. If $e, i < K_p$, $N < 2^{K_p}$ and $\bar{\sigma}$, $\bar{\tau}$ are sequences of members of ${}^{K_p}2$ of length $N+1$ and $\bar{\sigma}$ is injective, then there exists $q \leq p$ such that $K_q = K_p$ and $\mathsf{Split}_e(q, \bar{\sigma}, \bar{\tau}, i)$ holds. So fix such e, i, $\bar{\sigma} = \langle \sigma_k : k \leq N+1 \rangle$ and $\bar{\tau} = \langle \tau_k : k \leq N+1 \rangle$. Fix an arbitrary $(j, m) \in (\omega \times \omega) \setminus \mathsf{dom}(f_p(\sigma_{N+1}, \tau_{N+1}))$ and ask the following.

Does there exist $\langle x_k : k \leq N\rangle$ such that for all k, $f_p(\sigma_k, \tau_k) \subseteq x_k \in {}^{\omega\times\omega}2$ and $\Phi_e^X(\langle j, m\rangle)$ converges where $X = \bigoplus_{k\leq N} x_k$?

If the answer is no, then define $r = p$. Otherwise, fix such $\langle x_k : k \leq N\rangle$, put $X = \bigoplus_{k\leq N} x_k$ and choose $r \leq p$ such that

(1) $K_p = K_r$,
(2) $f_r(\sigma_{N+1}, \tau_{N+1})(j, m) \neq \Phi_e^X(\langle j, m\rangle)$ and
(3) the use of the computation $\Phi_e^X(\langle j, m\rangle)$ is contained in $\bigoplus_{k\leq N} f_r(\sigma_k, \tau_k)$.

This is possible because $\bar{\sigma}$ is injective and the oracle use in computing $\Phi_e^X(\langle j, m\rangle)$ is finite.

Next, assume $(\forall k \leq N)(\sigma_k(i) = 0)$ and ask the following. Do there exist $n < \omega$ and $\langle x_k : k \leq N\rangle$ such that for all k, $f_r(\sigma_k, \tau_k) \subseteq x_k \in {}^{\omega\times\omega}2$ and $\Phi_e^X(n) \downarrow\neq a_i(n)$ where $X = \bigoplus_{k\leq N} x_k$?

If the answer is no, then define $q = r$. Otherwise, fix such n and $\langle x_k : k \leq N\rangle$, put $X = \bigoplus_{k\leq N} x_k$ and choose $q \leq r$ such that

(4) $K_q = K_r$ and
(5) the use of the computation $\Phi_e^X(n)$ is contained in $\bigoplus_{k\leq N} f_r(\sigma_k, \tau_k)$.

One can now check that $\mathsf{Split}_e(q, \bar{\sigma}, \bar{\tau}, i)$ holds. To see that Definition 3.7 clause (b) holds, note that since $(\forall k \leq N)(\sigma_k(i) = 0)$ and X is Turing independent, a_i is not computable from $\bigoplus_{k\leq N} f_r(\sigma_k, \tau_k)$. This concludes the proof of Claim 3.10. $\square$

Using Claim 3.10, we can recursively construct a sequence $\bar{p} = \langle p_n : n < \omega\rangle$ of splitting conditions in $\mathbb{P}$ such that each $p_{n+1} \leq p_n$ and $K_{p_n} \to \infty$. Fix such a sequence and define the following.

- Define $F : 2^\omega \times 2^\omega \to 2^{\omega\times\omega}$ by $F(x, y) = \bigcup_{n<\omega} f_{p_n}(x \restriction K_{p_n}, y \restriction K_{p_n})$.
- For every $A \subseteq X$, define $P_A = \{F(1_A, y) : y \in 2^\omega\}$ where $1_A : \omega \to 2$ is the characteristic function of A.

Claim 3.11. *For every $A \subseteq X$, $P_A \subseteq 2^{\omega\times\omega}$ is a perfect set of reals. Furthermore, if $A_0, \cdots, A_\ell$ are pairwise distinct subsets of X and $z_k \in A_k$ for $k \leq \ell$, then the following hold.*

(a) *$\{z_k : k \leq \ell\}$ is Turing independent.*
(b) *For every $a \in X$, a is computable from $\bigoplus_{k\leq\ell} z_k$ iff $a \in \bigcup_{k\leq\ell} A_k$.*

Proof of Claim 3.11. That P_A is perfect follows from the fact that the map $y \mapsto F(1_A, y)$ is both continuous and injective (by Clause (ii)(b) in the definition of $\mathbb{P}$ above).

Next, suppose $A_0, \cdots, A_\ell$ are pairwise distinct subsets of X and $z_k \in A_k$ for $k \leq \ell$. Fix $y_k \in 2^\omega$ such that $z_k = F(1_{A_k}, y_k)$.

(a) Towards a contradiction, suppose $\{z_k : k \leq \ell\}$ is not Turing independent. Then $\ell \geq 1$. Put $N = \ell - 1$ and assume that z_{N+1} is computable from $\bigoplus_{k\leq N} z_k$ say via the functional Φ_e. Choose n such that $e < K_{p_n}$ and $\bar{\sigma} = \langle \sigma_k : k \leq N+1\rangle$ has pairwise distinct members where $\sigma_k = 1_{A_k} \restriction K_{p_n}$. Define $\bar{\tau} = \langle \tau_k : k \leq N+1\rangle$ by $\tau_k = y_k \restriction K_{p_n}$. Now use the fact that $\mathsf{Split}_e(p_n, \bar{\sigma}, \bar{\tau}, 0)$ holds to get a contradiction via Clause (a) in Definition 3.7.

(b) Suppose $a \in A$. Fix $i < \omega$ such that $a = a_i$. First assume that $a_i \in \bigcup_{k\leq\ell} A_k$ and fix $k \leq \ell$ such that $a_i \in A_k$. Observe that for every $y \in 2^\omega$, the ith column of $F(1_{A_k}, y)$ differs from a_i on a finite set (by Clause ii(e) in the definition of $\mathbb{P}$ above). Hence $a \leq_T F(1_{A_k}, y_k)$.

Next, suppose $a_i \notin \bigcup_{k \leq \ell} A_k$ and towards a contradiction, assume that a_i is computable from $\bigoplus_{k \leq \ell} z_k$ say via the functional Φ_e. Choose n such that $e, \ell < K_{p_n}$ and $\langle 1_{A_k} \upharpoonright K_{p_n} : k \leq \ell \rangle$ has pairwise distinct members. Define $\sigma_k = 1_{A_k} \upharpoonright K_{p_n}$ and $\tau_k = y_k \upharpoonright K_{p_n}$ for $k \leq \ell$. Fix some $\sigma_{\ell+1} \in {}^{K_{p_n}}2 \setminus \{\sigma_k : k \leq \ell\}$ and $\tau_{\ell+1} \in {}^{K_{p_n}}2$. Define $\bar{\sigma} = \langle \sigma_k : k \leq \ell + 1 \rangle$ and $\bar{\tau} = \langle \tau_k : k \leq \ell + 1 \rangle$. Now use the fact that $\mathsf{Split}_e(p_n, \bar{\sigma}, \bar{\tau}, i)$ holds to get a contradiction via Clause (b) in Definition 3.7. $\square$

We can now construct the desired embedding $h : \mathbb{H}^2_X \to 2^\omega$ as follows. Put $X_1 = [X]^{\aleph_0}$ and $X_2 = [X_1]^{\aleph_0}$. Fix injective enumerations $\langle A_i : i < \mathfrak{c} \rangle$ and $\langle \mathcal{B}_i : i < \mathfrak{c} \rangle$ of X_1 and X_2 respectively. We will inductively construct a sequence of partial functions $\langle h_i : i < \mathfrak{c} \rangle$ from $\mathbb{H}^2_X$ to 2^ω such that the following hold.

(1) $h_i \upharpoonright X$ is the identity and $i < j < \mathfrak{c} \implies h_i \subseteq h_j$.
(2) $\mathsf{dom}(h_i) \cap X_1 = \{A_j : j < i\}$ and $h_i(A_j) \in P_{A_j}$ for all $j < i < \mathfrak{c}$.
(3) $\mathsf{dom}(h_i) \cap X_2 = \{\mathcal{B}_j : j < i \wedge \mathcal{B}_j \subseteq \{A_k : k < i\}\}$.
(4) h_i is a partial embedding of $\mathbb{H}^2_X$ into $(2^\omega, \leq_T)$.

At any limit stage $i < \mathfrak{c}$, define $h_i = \bigcup_{j<i} h_j$. At a successor stage $i + 1$, we define h_{i+1} as follows.

First choose $h_{i+1}(A_i) \in P_{A_i}$ such that for every $\mathcal{B} \in \mathsf{dom}(h_i) \cap X_1$, $h_{i+1}(A_i)$ is not computable from $h_i(\mathcal{B})$. This constraint only rules out fewer than continuum candidates from P_{A_i}.

Next assume $\mathcal{B}_i \subseteq \mathsf{dom}(h_i) \cup \{A_i\}$ and define $h(\mathcal{B}_i)$ as follows. Fix $q \in \mathbb{S}$ such that $[q]$ is a perfect set of exact pairs for $\{h_{i+1}(A) : A \in \mathcal{B}_i\}$. To ensure that h_{i+1} remain a partial embedding, we will choose $h_{i+1}(\mathcal{B}_i) \in [q]$ satisfying the following.

(a) $h_{i+1}(\mathcal{B}_i)$ is Turing incomparable with $h_i(\mathcal{B})$ for every $\mathcal{B} \in \mathsf{dom}(h_i) \cap X_2$.
(b) For each $j \leq i$, $A_j \in \mathcal{B}_i$ iff $h_{i+1}(A_j) \leq_T h_{i+1}(\mathcal{B}_i)$.
(c) For every $y \in X \setminus \bigcup \mathcal{B}_i$, y is not computable from $h_{i+1}(\mathcal{B}_i)$.

Observe that for each $\mathcal{B} \in \mathsf{dom}(h_i) \cap X_2$, there is at most one member of $[q]$ that computes $h_i(\mathcal{B})$. Since if there were two such members, then $h_i(\mathcal{B})$ would be computable from the join of a finite subset of $\{h_{i+1}(A) : A \in \mathcal{B}_i\}$ which contradicts the fact that $\{h_{i+1}(A_i) : i \leq j\}$ is Turing independent (by Claim 3.11(a)). Also, there are at most countable many members of $[q]$ that are computable from $h_i(\mathcal{B})$. So requirement (a) rules out fewer than continuum choices in $h_{i+1}(\mathcal{B}_i) \in [q]$.

A similar reasoning shows that for each $j \leq i$, if $A_j \notin \mathcal{B}_i$, then $h_{i+1}(A_j)$ is computable from at most one member of $[q]$. So requirement (b) also rules out fewer than continuum choices.

Finally, note that X is countable and for each $y \in X \setminus \bigcup \mathcal{B}_i$, by Claim 3.11(b), y is computable from at most one member of $[q]$. So requirement (c) rules out only countably many members of $[q]$.

This completes the construction of $\langle h_i : i < \mathfrak{c} \rangle$. Define $h = \bigcup_{i<\mathfrak{c}} h_i$. Then $h : \mathbb{H}^2_X \to 2^\omega$ is the required embedding. $\square$

3.2. **Embedding $\mathbb{H}^2_{\mathfrak{c}}$.** We do not know if $\mathbb{H}^2_{\mathfrak{c}}$ can be embedded into the Turing degrees. Let us try to explain some difficulties in doing so.

Definition 3.12. *We say that $X \subseteq 2^\omega$ is n-embeddable iff X is infinite and there is an embedding h of $\mathbb{H}^n_X$ into the Turing degrees such that $h \upharpoonright X$ is the identity.*

Using Lemma 3.5, it is easy to see that an infinite set of reals is 1-embeddable iff it is Turing independent.

Lemma 3.13. *Every 2-embeddable set is σ-Turing independent (Definition 2.9(3)).*

Proof. Assume $X \subseteq 2^\omega$ is 2-embeddable. Then X is 1-embeddable and hence Turing independent. First assume that X is countable. For each $A \subseteq X$, choose $y_A \in P_A$ as defined in Claim 3.11. Then it is easy to check that y_A computes every real in A and $X \setminus A \cup \{y_A\}$ is Turing independent. So X is σ-Turing independent.

Now suppose X is uncountable and fix an embedding $h : \mathbb{H}^2_X \to 2^\omega$ such that $h \restriction X$ is the identity. Let $A \subseteq X$ be countable. If A is finite, we can take y_A to be the join of A. So assume A is infinite and define $y_A = h(A)$. Since h is an embedding, y_A computes every real in A. Towards a contradiction, suppose there exist a finite $F \subseteq X \setminus A$ and $x \in X \setminus (A \cup F)$ such that x is computable from the join of $F \cup \{y_A\}$. Choose $\{B_k : k < \omega\}$ such that each $B_k \in [X \setminus (A \cup \{x\})]^{\aleph_0}$ and $F \subseteq B_0$. Define $\mathcal{B} = \{A, B_k : k < \omega\}$. Then $h(\mathcal{B})$ computes $h(A) = y_A$ and every real in $B_0 \supseteq F$. Hence $h(\mathcal{B})$ computes x. But this is impossible as $x \not\leq_{\mathbb{H}^2_X} \mathcal{B}$. $\square$

The following puts some limitations on the possibility of generalizing Claim 3.11 to uncountable sets.

Lemma 3.14. *In the Groszek-Slaman model (see Theorem 4.1), there is a Turing independent set of reals of size ω_1 that is not σ-independent.*

Proof. Fix V, κ, $\bar{s}$ and $\bar{t}$ as in Theorem 4.1. Let $W \in V \cap [\omega_1]^{\omega_1}$ be arbitrary. Put $X = \{s_i : i \in W\}$. Then X is Turing independent. We claim that X is not σ-Turing independent in $V[\bar{s}][\bar{t}]$. Suppose not and for each countable $A \subseteq W$, fix $y_A \in 2^\omega$ such that y_A computes every real in $\{s_i : i \in A\}$ and $\{s_i : i \in W \setminus A\} \cup \{y_A\}$ is Turing independent. Note that the function $A \mapsto y_A$ is injective with inverse $y \mapsto \{i \in W : s_i \leq_T y\}$. Let $\alpha = \min(W)$. Since $V[\bar{s}][\bar{t}] \models [W]^{\aleph_0} = \mathfrak{c} > \omega_1$, we can find $A \in [W]^{\aleph_0} \cap V[\bar{s}][\bar{t}]$ such that $\alpha \notin A$ and $y_A \notin V[\bar{s}]$. Using Theorem 4.1(d), fix $x \in V \cap 2^\omega$ such that $s_\alpha \leq_T x \oplus y_A$. Since $W \in V$, an easy density argument shows that there are uncountably many $\beta \in W$ such that $x \leq_T s_\beta$. Fix such a $\beta > \alpha$ in $W \setminus A$. Then $s_\alpha \leq_T s_\beta \oplus y_A$. It follows that $\{s_i : i \in W \setminus A\} \cup \{y_A\}$ is not Turing independent. A contradiction. $\square$

3.3. **Definability.** Lemma 3.5 implies that $\mathbb{H}^1_{2^\omega}$ can be embedded into the Turing degrees. Using Lusin-Novikov uniformization theorem, Higuchi and Lutz have shown (Lemma 2.7 in [6]) that one can arrange such an embedding to be Borel. Remarkably though, there is no such embedding for $\mathbb{H}^2_{2^\omega}$.

Theorem 3.15 ([6])**.** *There is no Borel embedding of $\mathbb{H}^2_{2^\omega}$ into the Turing degrees.*

The proof uses the following result of Lutz and Siskind.

Theorem 3.16 ([11])**.** *Suppose $p \in \mathbb{S}$ and D is a countable dense subset of $[p]$. Assume $x \in 2^\omega$ computes every member of D. Then for every $z \in 2^\omega$, there are $\{y_1, y_2, y_3, y_4\} \subseteq [p]$ such that $z \leq_T x \oplus y_1 \oplus y_2 \oplus y_3 \oplus y_4$.*

Note that Theorem 3.16 implies that a σ-Turing independent set cannot contain a perfect set (see Theorem 1.2 in [6]). So it is natural ask the following.

Question 3.17. *Must there exist a σ-Turing independent set of size $\mathfrak{c}$?*

3.4. **Order dimension.** For a poset $(\mathbb{P}, \leq_{\mathbb{P}})$, the order dimension of $\mathbb{P}$, denoted $\mathsf{odim}(\mathbb{P})$, is the smallest cardinality of a family $\mathcal{O}$ of linear orderings on $\mathbb{P}$ whose intersection is $\leq_{\mathbb{P}}$. It is easy to see that if $\mathbb{P}$ embeds into $\mathbb{Q}$, then $\mathsf{odim}(\mathbb{P}) \leq \mathsf{odim}(\mathbb{Q})$. A possible way of obtaining the consistency of a negative answer to Question 3.3 would be to construct a model of ZFC in which $\mathsf{odim}(\mathcal{D}) \neq \max\{\mathsf{odim}(\mathbb{P}) : \mathbb{P} \in \mathcal{C}_{lc}\}$. This optimism is ill-founded.

Fact 3.18 ([8])**.** *$\mathsf{odim}(\mathcal{D})$ is the smallest cardinality of a family $\mathcal{L}$ of linear orders on 2^ω such that for every $A \in [2^\omega]^{\aleph_0}$ and $x \in 2^\omega \setminus A$, there exists $\preceq$ in $\mathcal{L}$ such that $(\forall a \in A)(a \prec x)$. Furthermore, for every $\mathbb{P} \in \mathcal{C}_{lc}$, $\mathsf{odim}(\mathbb{P}) \leq \mathsf{odim}(\mathcal{D}) = \mathsf{odim}(\mathbb{H}^1_{\mathfrak{c}})$.*

$\mathsf{odim}(\mathcal{D})$ appears to be quite different from other classical cardinal invariants. For example, in [5], it was shown that if $\mathfrak{c} = \kappa^+$ where κ is regular uncountable, then $\mathsf{odim}(\mathcal{D}) \leq \kappa$. Also, Martin's axiom has no effect on $\mathsf{odim}(\mathcal{D})$. For a proof of these and more, see [5, 8].

4. Turing independent sets

4.1. **Maximality.** $X \subseteq 2^\omega$ is a maximal Turing independent set if it is Turing independent and for every $y \in 2^\omega \setminus X$, $X \cup \{y\}$ is not Turing independent. Maximal n-independent and σ-independent sets are analogously defined. Every maximal Turing independent set is uncountable and under Martin's axiom, it has size $\mathfrak{c}$. Groszek and Slaman showed that they can consistently have size $< \mathfrak{c}$.

Theorem 4.1 ([4])**.** *Assume $V \models \kappa \geq \mathsf{cf}(\kappa) \geq \omega_1 = \mathfrak{c}$. Let $\mathbb{P}$ be the countable support product of ω_1 copies of $\mathbb{S}$ and let $\bar{s} = \langle s_i : i < \omega_1 \rangle \in V^{\mathbb{P}}$ be the generic sequence of Sacks reals. In $V^{\mathbb{P}}$, let $\mathbb{Q}$ be the countable support product of κ copies of Sacks forcing. Let $\bar{t} = \langle t_i : i < \kappa \rangle \in V^{\mathbb{P} \star \mathbb{Q}}$ be the generic sequence of Sacks reals added by $\mathbb{Q}$. Then, the following hold.*

(a) *Forcing with $\mathbb{P} \star \mathbb{Q}$ preserves all cofinalities and $V[\bar{s}][\bar{t}] \models \mathfrak{c} = \kappa$.*
(b) *For every $x \in V \cap 2^\omega$, $|\{i < \omega_1 : x \leq_T s_i\}| = \omega_1$.*
(c) *$S = \{s_i : i < \omega_1\}$ is a maximal Turing independent set of reals in $V[\bar{s}]$.*
(d) *For every $i < \omega$ and $x \in 2^\omega \cap (V[\bar{s}][\bar{t}] \setminus V[\bar{s}])$, there exists $y \in V \cap 2^\omega$ such that $s_i \leq_T x \oplus y$.*
(e) *S remains maximal in $V[\bar{s}][\bar{t}]$.*

Clause (a) is standard and (b) follows from an easy density argument. The proofs of clauses (c) and (d) use a fusion argument. To see how (e) follows, assume $x \in 2^\omega \setminus S$. If $x \in V[\bar{s}]$, then clause (c) implies that $S \cup \{x\}$ is not Turing independent. So assume $x \in V[\bar{s}][\bar{t}] \setminus V[\bar{s}]$. Using clause (d), we can find $y \in V \cap 2^\omega$ such that $s_0 \leq x \oplus y$. Using clause (c), choose $0 < i < \omega_1$ such that $y \leq_T s_i$. Then $s_0 \leq x \oplus s_i$. Hence $S \cup \{x\}$ is not even 2-Turing independent.

Note that while consistently, a maximal 2-Turing independent set can have size $< \mathfrak{c}$, every maximal 1-Turing independent set must have size $\mathfrak{c}$. To see this, apply Fact 2.5(b) to $p = 2^{<\omega}$, to get $r \in \mathbb{S}$ such that for every $x \in [r]$ and $y \leq_T x$, either $x \leq_T y$ or y is computable. Now observe that if $X \in [2^\omega]^{<\mathfrak{c}}$ is 1-Turing independent, then for every $y \in [r] \setminus \{z : (\exists x \in X)(z \leq_T x)\}$, $X \cup \{y\}$ is also 1-Turing independent. So X cannot be maximal.

Let $\mathcal{T}_m$ be the set of cardinalities of a maximal Turing independent set. Then $\mathfrak{c} \in \mathcal{T}_m$ and under Martin's axiom, $\mathcal{T}_m = \{\mathfrak{c}\}$. Theorem 4.1 shows that consistently, $\mathfrak{c} > \omega_1 \in \mathcal{T}_m$.

Question 4.2. *Is it consistent that $\mathfrak{c} > \omega_2 \in \mathcal{T}_m$? More generally, what are the possible values of $\mathcal{T}_m$?*

4.2. **Cardinality.** In view of Fact 2.10, one might ask if we can carry out a similar construction within every set of reals of size continuum? More precisely, given any $X \in [2^\omega]^{\mathfrak{c}}$, can be find $Y \in [X]^{\mathfrak{c}}$ such that Y is Turing independent? We can recursively build $X = \{x_i : i < \omega_1\} \subseteq 2^\omega$ such that x_i's are $<_T$-increasing. So under CH, $|X| = \mathfrak{c}$ and X doesn't even have a Turing antichain of size 2. But what happens when $\mathfrak{c} > \omega_1$? The following theorem says that the answer is yes in the Cohen/random real model. For a more general version, see Theorem 3.1 in [9].

Theorem 4.3 ([9])**.** *Let $V \models CH$. Suppose $\mathbb{P}$ is the forcing for adding ω_2 Cohen/random reals. Then the following hold in $V^{\mathbb{P}}$.*

(a) $\mathfrak{c} = \kappa$.
(b) *For every $X \subseteq [2^\omega]^{\mathfrak{c}}$, there exists $Y \subseteq X$ such that $|Y| = \mathfrak{c}$ and Y is Turing independent.*

Proof. We will sketch the argument for random forcing. The proof in the case of Cohen forcing is identical except that one has to use Kuratowski-Ulam theorem instead of Fubini's theorem. Let $\mathbb{P}$ be the forcing for adding ω_2 random reals $\bar{r} = \langle r_i : i < \omega_2 \rangle$ where each $r_i \in 2^\omega$. Fix $X = \{x_i : i < \omega_2\} \subseteq 2^\omega$. Using Borel reading of names (Lemma 3.1.7 in [1]) and the fact that $V \models CH$, choose $W \in [\omega_2]^{\omega_2}$, $\langle B_i : i \in W \rangle$, $\gamma < \omega_1$ and f such that the following hold.

(i) $\langle A \sqcup B_i : i \in W \rangle$ is a Δ-system of countable subsets of ω_2 with root $A \in [\omega_2]^{\aleph_0}$ and $\mathsf{otp}(A \cup B_i) = \gamma$ does not depend on $i \in W$.
(ii) $f : 2^\gamma \to 2^\omega$ is a Borel function coded in V.
(iii) For each $i \in W$, $f(\bar{r} \upharpoonright (A \cup B_i)) = x_i$.

By replacing V with $V[\bar{r} \upharpoonright A]$ and modifying f, we can assume that $A = \emptyset$. WLOG, we can also assume that the trivial condition forces (i)-(iii) above. This implies that $f \upharpoonright K$ is not constant for any positive measure $K \subseteq 2^\gamma$.

Claim 4.4. *For every $n \geq 1$, the set*

$$E_n = \{(a_1, \cdots, a_n) \in (2^\gamma)^n : \{f(a_1), \cdots, f(a_n)\} \text{ is not Turing independent}\}$$

has measure zero.

Proof of Claim 4.4: Suppose not and fix the least $n \geq 2$ for which E_n has positive measure. Choose $K \subseteq E_n$ and $1 \leq j \leq n$ such that K has positive measure and for every $(a_1, \cdots, a_n) \in K$, $f(a_j)$ is computable from the join of $\{f(a_i) : i \neq j\}$. Using Fubini's theorem, choose $\bar{b} = (b_1, \cdots, b_n) \in K$ such that

$$K_{\bar{b},j} = \{a \in 2^\gamma : (b_1, \cdots, b_{j-1}, a, b_{j+1}, \cdots, b_n) \in K\}$$

has positive measure. Since the join of $\{b_i : i \neq j\}$ can only compute countably many reals, f must be constant on a positive measure subset of $K_{\bar{b},j}$ which is impossible. So the claim holds. $\square$

Note that for any $i(1) < \cdots < i(n)$ in W, $\langle \bar{r} \upharpoonright A_{i(k)} : 1 \leq k \leq n \rangle$ avoids any null subset of $(2^\gamma)^n$ coded in V. It follows that $Y = \{x_i : i \in W\}$ is Turing independent. $\square$

What is the best that we can do in ZFC? First observe that Hajnal's set mapping theorem (Theorem 44.3 in [2]) implies the following.

Fact 4.5. *Suppose $X \subseteq 2^\omega$ and $|X| \geq \omega_2$. Then there exists $Y \subseteq X$ such that $|Y| = |X|$ and Y is 1-Turing independent.*

For $n \geq 2$, this can consistently fail. In fact, we have the following.

Theorem 4.6 ([9])**.** *Assume Martin's axiom and $\mathfrak{c} \geq \omega_n$ where $2 \leq n < \omega$. Then there exists $X \in [2^\omega]^{\omega_n}$ such that every Turing independent subset of X has size $\leq n$.*

The proof of this theorem involves two steps. The first is a purely combinatorial ZFC construction. Recall that a poset $(\mathbb{P}, \preceq)$ is an upper semi-lattice iff every finite $F \subseteq \mathbb{P}$ has a $\preceq$-least upper bound in $\mathbb{P}$ (called the join of F).

Lemma 4.7 ([9])**.** *For each $2 \leq n < \omega$, there exists a locally countable upper semi-lattice $(\mathbb{P}, \preceq)$ such that $|\mathbb{P}| = \omega_n$ and for every $F \in [\mathbb{P}]^{n+1}$, there exists $x \in F$ such that x is $\preceq$-below the join of $F \setminus \{x\}$.*

The upper semi-lattices witnessing Lemma 4.7 are constructed using a theorem of Kuratowski on set mappings (see Lemma 3.6 in [9]). The next step is to find an isomorphic copy of this lattice inside the Turing degrees. This is consistently possible by the following theorem of W. Wei.

Theorem 4.8 ([15])**.** *Assume Martin's axiom. Let $(\mathbb{P}, \preceq)$ be a locally countable upper semi-lattice where $|\mathbb{P}| \leq \mathfrak{c}$. Then there exists a join preserving embedding of $\mathbb{P}$ into the Turing degrees.*

Theorem 4.6 immediately follows. We conclude this section with the following result of Groszek and Slaman.

Theorem 4.9 ([4])**.** *In the Cohen/random real model, there is a locally countable upper semi-lattice $(\mathbb{P}, \preceq)$ of size ω_2 such that there is no join preserving embedding of $(\mathbb{P}, \preceq)$ into the Turing degrees.*

Proof. Let $(\mathbb{P}, \preceq)$ be as in Lemma 4.7 for $n = 2$. Towards a contradiction, fix a join preserving embedding $f : \mathbb{P} \to 2^\omega$ into the Turing degrees. Put $X = \mathsf{range}(f)$. Then $|X| = \omega_2$ and Lemma 4.7 implies that X does not have any Turing independent subset of size 3. But this contradicts Theorem 4.3. $\square$

4.3. **Category and measure.** Suppose $X \subseteq 2^\omega$ is Turing independent. How large can X be in the sense of Baire category and Lebesgue measure? If X is Lebesgue measurable and $\mu(X) > 0$, then for some $\sigma \in {}^{<\omega}2$, $\mu(X \cap [\sigma]) > 0.5\mu([\sigma])$. So there must exist $x, y \in X$ above σ such that the XOR sum of x and y is $0^{|\sigma|}{}^\frown 1^\omega$. Thus X cannot even be 1-Turing independent. A similar argument shows that if X is 1-Turing independent and has the Baire property, then X is meager. So the question becomes the following. Does there exist a Turing independent $X \subseteq 2^\omega$ such that X is non-meager (resp. non-null)?

In [9], it was shown that the answer is yes in the case of category. A rough outline of the construction is as follows – For more details, see [9]. First construct $F : {}^{<\omega}2 \to {}^{<\omega}2$ and $K : \omega \to \omega$ both computable such that for every $e < n < \omega$ and pairwise distinct $\rho_0, \rho_1, \cdots, \rho_N$ in ${}^n2$, letting $F(\rho_i) = \sigma_i$ we have $\rho_i \preceq \sigma_i \in {}^{K(n)}2$ and the e-th Turing functional cannot compute any real extending σ_0 using any oracle extending the join of $\langle \sigma_k : 1 \leq k \leq n \rangle$.

Let $\mathcal{C}$ consist of all pairs $(\mathbf{m}, x)$ where $x \in 2^\omega$ and $\mathbf{m} = \langle m_k : k < \omega \rangle$ is a strictly increasing sequence in ω. Observe that $\mathsf{range}(H)$ is non-meager for every $H : \mathcal{C} \to 2^\omega$ satisfying

$$H(\mathbf{m}, x) = y \implies (\exists^\infty k)\,(y \upharpoonright [m_k, m_{k+1}) = x \upharpoonright [m_k, m_{k+1}))\,.$$

Now the key step is the following. Given a non-principal ultrafilter $\mathcal{U}$ on ω, one can construct an $H : \mathcal{C} \to 2^\omega$, definable in $\mathcal{U}$, as above such that for every $y \in \mathsf{range}(H)$, $\{n < \omega : F(y \upharpoonright K(n)) \subseteq y\} \in \mathcal{U}$. Then $\mathsf{range}(H)$ is both non-meager and Turing independent.

The measure version is open.

Question 4.10. *Does there exist a non-null Turing independent set of reals?*

In analogy with the discussion in subsection 4.2, we can ask the following. Suppose $X \subseteq 2^\omega$ is non-meager (resp. non-null). Must there exist a Turing independent $Y \subseteq X$ such that Y is non-meager (resp. non-null)?

4.3.1. *Martin's axiom.* Under Martin's axiom, the answer is yes. The following (see Lemma 4.1 in [9]) is a minor generalization of the fact that non-trivial Turing cones are both meager and null.

Fact 4.11 (Sacks)**.** *Suppose $x, y \in 2^\omega$ and x is not computable from y. Then $\{z \in 2^\omega : x \leq_T y \oplus z\}$ is both meager and null.*

Now assume $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$ (a consequence of Martin's axiom) and fix any non-null $X \subseteq 2^\omega$. Let $\langle K_i : i < \mathfrak{c} \rangle$ list all compact $K \subseteq 2^\omega$ such that $K \cap X$ is non-null. Now recursively construct a Turing independent subset $\{y_i : i < \mathfrak{c}\} \subseteq X$ such that for every $i < \mathfrak{c}$, $y_i \in K_i \cap X$. To see that y_i's can be chosen, use Fact 4.11 and $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$. A similar argument shows that, assuming $\mathsf{add}(\mathcal{M}) = \mathfrak{c}$, every non-meager set has a non-meager Turing independent subset.

4.3.2. *Antichains.* Fix any non-meager (resp. non-null) $X \subseteq 2^\omega$. As in the case of cardinality, just in ZFC, we can always get a non-meager (resp. non-null) 1-Turing independent $Y \subseteq X$. But the proof is more involved and relies on some facts about effectively random/Cohen reals. The following consequence of a result of Yu (Lemma 3.11 in [16]) suffices for this application. For more details see Lemma 4.5 in [9].

Fact 4.12. *Let M be a countable transitive model of ZFC. Suppose $E \subseteq 2^\omega$ is a meager (resp. null) set of Cohen (resp. random) reals over M. Then the set of all reals that compute some member of E is meager (resp. null).*

Question 4.13. *Find a "set-theoretic proof" of Fact 4.12.*

Let $X \subseteq 2^\omega$ be non-meager (resp. non-null). Fix a countable transitive $M \models$ ZFC. By throwing away a meager subset of X, we can assume that each real in X is Cohen (resp. random) over M. Towards a contradiction, assume that every 1-Turing independent subset of X is meager (resp. null). Call $S \subseteq X$ good iff no two distinct reals in S compute the same real in X. Let Y be a maximal good subset of X. Let $W = \{x \in X : (\exists y \in Y)(x \leq_T y)\}$. Since Y is good, W is 1-Turing independent and hence meager (resp. null) by assumption. Let T be the set of all reals that compute some member of W. By Fact 4.12, T is also meager (resp. null). We claim that $X \subseteq T$ and therefore we get a contradiction. To see this, suppose $x \in X \setminus T$. Since $Y \subseteq W \subseteq T$, we must have $x \notin Y$. Since Y is a maximal good subset of X, there exist $y \in Y$ and $w \in X$ such that both x and y compute w. As $x \geq_T w \in W$, we get $x \in T$. A contradiction.

4.3.3. *Measurable cardinal.* Assuming the consistency of a measurable cardinal, it is consistent that there is a non-meager (resp. non-null) $X \subseteq 2^\omega$ all of whose 2-Turing independent subsets are meager (resp. null). This follows from the following.

Theorem 4.14 ([9])**.** *Suppose there is a measurable cardinal. Then there is a ccc forcing $\mathbb{P}$ such that the following hold in $V^{\mathbb{P}}$. There are $X \subseteq 2^\omega$ and a total Turing functional Φ such that X is non-meager (resp. non-null) and for every non-meager (resp. non-null) $Y \subseteq X$, there are distinct $x, y, z \in Y$ such that $\Phi^{x \oplus y} = z$.*

Proof. Since the pairing function $(x, y) \mapsto x \oplus y$ is a computable measure preserving homeomorphism from $2^\omega \times 2^\omega$ to 2^ω, we can work in $2^\omega \times 2^\omega$ instead of 2^ω.

By a result of Komjath [7] (resp. Shelah [14]), starting with a measurable cardinal, one can construct a ccc forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is a non-meager (resp. non-null) $A \subseteq 2^\omega$ such that for every $f : A \to A$, (the graph of) f is meager (resp. null) in the product space $2^\omega \times 2^\omega$.

Put $X = A \times A$. Then it follows that for every $Y \subseteq X$ that is non-meager (resp. non-null) in the product space $2^\omega \times 2^\omega$, there are $x, y, z \in X$ that form the vertices of a right triangle at z with $\overline{xz}$ parallel to the vertical axis and $\overline{yz}$ parallel to the horizontal axis. Let $\Phi : 2^\omega \times 2^\omega$ be defined by $\Phi(x, y) = (x_0, y_1)$ where $x = (x_0, x_1)$ and $y = (y_0, y_1)$. Then X, Φ are as required. □

4.3.4. *Avoiding large cardinals.* In the case of category, one can avoid the use of large cardinals.

Theorem 4.15 ([10])**.** *Relative to ZFC, it is consistent that there is a non-meager $X \subseteq 2^\omega$ such that for every noon-meager $Y \subseteq X$, there are distinct $x, y, z \in Y$ such that $z \leq_T x \oplus y$.*

The measure analogue remains open.

Question 4.16. *Can we prove the consistency of the following statement without assuming the consistency of any large cardinals? There is a non-null set of reals all of whose non-null subsets are 2-Turing independent.*

References

[1] T. Bartoszynski and H. Judah, *Set theory: On the structure of the real line*, A K Peters Wellesley MA, 1995.

[2] P. Erdős, A. Hajnal, A. Mate and R. Rado, *Combinatorial set theory: Partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, Vol. 106, North-Holland Publishing Co., Amsterdam, 1984.

[3] M. J. Groszek, *Independence results from ZFC in computability theory: Some open problems*, Contemp. Math. 257, American Math. Soc., Providence RI, 2000, 171-179.

[4] M. J. Groszek and T. A. Slaman, *Independence results on the global structure of the Turing degrees*, Trans. of the AMS 277(2), 1983, 579-588.

[5] K. Higuchi, S. Lempp, D. Raghavan and F. Stephan, *The order dimension of locally countable partial orders*, Proc. Amer. Math. Soc. 148 no. 7, 2020, 2823-2833.

[6] K. Higuchi and P. Lutz, *A note on a conjecture of Sacks: It Is harder to embed height theree partial orders than height two partial orders*, Preprint arXiv: 2309.01876.

[7] P. Komjáth, *A second category set with only first category functions*, Proc. Amer. Math. Soc., 112 (1991), 1129–1136.
[8] A. Kumar and D. Raghavan, *Separating families and order dimension of Turing degrees*, Ann. Pure Appl. Logic, Vol. 172 Issue 5 (2021), 102911.
[9] A. Kumar and S. Shelah, *Large Turing independent sets*, Proc. Amer. Math. Soc. , 151 (2023), 355-367.
[10] A. Kumar and S. Shelah, *Turing independence and Baire category*, Journal of Mathematical Logic, To appear.
[11] P. Lutz and B. Siskind, *Part 1 of Martin's conjecture for order preserving and measure preserving functions*, Preprint aXiv: 2305.19646.
[12] G. E. Sacks, *Degrees of unsolvability*, Ann. of Math. Studies no. 55, Princeton Univ. Press, Princeton N.J., 1963.
[13] G. E. Sacks, *Forcing with perfect closed sets*, Proc. Symp. Pure Math. 17, 1971, 331-355.
[14] S. Shelah, *The null ideal restricted to some non-null set may be $\aleph_1$-saturated*, Fund. Math., 179 (2) 2003, 97–129.
[15] W. Wei, *Martin's axiom and embeddings of upper semi-lattices into the Turing degrees*, Ann. of pure and applied logic, 161 (10) 2010, 1291-1298.
[16] L. Yu, *Measure theory aspects of locally countable orderings*, Journal of Symb. logic 71 (3), 2006, 958-968.

(Kumar) Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, UP, India.

Email address: `krashu@iitk.ac.in`

URL: `https://home.iitk.ac.in/~krashu/`