On some problems in set-theoretic real analysis

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Abstract

This thesis contains a few applications of set-theoretic methods to certain problems in real analysis.

In the first two sections of Chapter 1, we discuss some results related to a question of Fremlin about partitions of a set of reals into null sets. In Section 3, we answer a question of Komjáth in dimension one. Our proof uses some results of Gitik and Shelah in an essential way. There seems to be more open problems than answers here.

In Chapter 2, we answer a couple of questions about finitely additive total extensions of Lebesgue measure. These problems arose from a question of Juhász in set-theoretic topology.

In Chapter 3, we give some "natural" examples of additive subgroups of reals of arbitrarily high finite Borel rank. The existence of such groups is an old and well known result.

In Chapter 4, we construct a non principal ultrafilter from any free maximal ideal in the ring of bounded continuous functions on reals.

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Contents

Chapter 1

On partitions into small sets

1.1 Introduction

Suppose $Y \subseteq X \subseteq \mathbb{R}^n$. We say that Y has full outer measure in X if for every compact $K \subseteq \mathbb{R}^n$ of positive measure, if $X \cap K$ is non null then $Y \cap K$ is non null. If X has finite outer measure then $Y \subseteq X$ has full outer measure in X iff $\mu^\star(Y) = \mu^\star(X)$ where $\mu^\star$ denotes Lebesgue outer measure. The category analogue of this is defined as follows: For $Y \subseteq X \subseteq \mathbb{R}^n$, we say that Y is everywhere non meager in X if for every open set $U \subseteq \mathbb{R}^n$, if $X \cap U$ is non meager then $Y \cap U$ is non meager. For $X \subseteq \mathbb{R}^n$, $\text{env}(X)$ (envelope of X) denotes a G_δ set containing X such that the inner measure of $\text{env}(X) \setminus X$ is zero. Observe that the outer measure of a set is equal to the measure of its envelope.

One of the early questions on the possibility of dividing a large set into two everywhere large subsets was asked by Kuratowski [20]: Suppose $X \subseteq \mathbb{R}^n$. Can we always partition X into two subsets which are everywhere non meager in X? He observed that if X is Borel or if one assumes the continuum hypothesis (CH) then this can be done. The measure analogue of Kuratowski's question would be: Suppose $X \subseteq \mathbb{R}^n$. Can we always partition X into two subsets which have full outer measure in X? Lusin [22] answered both questions positively without any extra set-theoretic assumption.

Theorem 1 (Lusin)**.** *Suppose $X \subseteq \mathbb{R}^n$. Then X can be partitioned into two subsets* *which have full outer measure in X. It can also be partitioned into two everywhere non meager sets in X.*

Proof: We first prove the measure case and then supply the necessary modifications for the category case. WLOG, we can assume that X is a non null subset of $[0,1]$. It is enough to show that for every non null $X \subseteq [0,1]$ there is a $Z \subseteq X$ such that $\mu^\star(Z) > 0$ and $\mu^\star(X \backslash Z) = \mu^\star(X)$. Since then, we can take Y to be the union of a maximal family of subsets of X with this property such that their envelopes are pairwise disjoint. So assume that for every $Z \subseteq X$, if $\mu^\star(Z) > 0$ then $\mu^\star(X \backslash Z) < \mu^\star(X)$. We claim that this implies that $\mu^\star \restriction \mathcal{P}(X)$ is countably additive. Since $\mu^\star$ is countably subadditive, it is enough to show that $\mu^\star \restriction \mathcal{P}(X)$ is finitely additive. Suppose this fails and let A, B be disjoint subsets of X with $\mu^\star(A) + \mu^\star(B) > \mu^\star(A \cup B)$. Then envelopes of A and B must intersect on a non null Borel set E in which both $A \cap E$ and $B \cap E$ have full outer measure. But then we could have removed $A \cap E$ from X without reducing its outer measure. It follows that for disjoints subsets $A, B \subseteq X$, $\mathrm{env}(A) \cap \mathrm{env}(B)$ is null.

Let κ be the additivity of the sigma ideal I of null subsets of X as witnessed by $\langle N_\alpha : \alpha < \kappa \rangle$ where N_α's form an increasing sequence of null subsets of X. By replacing X with $\bigcup\{N_\alpha : \alpha < \kappa\}$, we can assume that the additivity of the null ideal restricted to every non null subset of X is κ. Notice that $\mathbb{B} = \mathcal{P}(X)/I$ is isomorphic to the measure algebra of Borel subsets of $\mathrm{env}(X)$ modulo null via the map that takes $A \subseteq X$ to $\mathrm{env}(A)$. So forcing with I is isomorphic to random forcing. Let G be $\mathbb{B}$-generic over V. In $V[G]$, the G-ultrapower of V is well founded because I is ω_1-saturated. Let M be its transitive collapse and $j : V \to M$, the diagonal elementary embedding with critical point κ. In V, $X = \bigcup\{N_\alpha : \alpha < \kappa\}$ is an increasing union of a κ-sequence of null sets. Hence in M, $j(X)$ is an increasing union of a $j(\kappa)$-sequence $\langle N'_\alpha : \alpha < j(\kappa)\rangle$ of null sets. Moreover, since j fixes every real in V, $N_\alpha \subseteq N'_\alpha = j(N_\alpha)$ for each $\alpha < \kappa$. Hence X is covered by a null Borel set coded in M. But then X is null in $V[G]$ which contradicts the fact that random forcing preserves non null sets of reals in the ground model.

In the case of category, assuming the negation, we first obtain some non meager $X \subseteq [0,1]$ such that for every non meager $Y \subseteq X$, there is some open set U such that $X \cap U$ is non meager but $(X \setminus Y) \cap U$ is meager. Then forcing with the meager ideal over X is isomorphic to Cohen forcing which preserves non meager sets in the ground model so that we get a similar contradiction. ☕

One might ask if it is also possible to partition any given set into uncountably many everywhere non meager/full outer measure subsets. The question then becomes independent of ZFC. In the case of category, starting with a measurable cardinal, Komjath [16] constructed a model of set theory in which there is a non meager set of reals which cannot be partitioned into uncountably many non meager sets. In the case of measure, Shelah [25] obtained a similar model using more sophisticated arguments.

Lusin's original proof of the above theorem does not use forcing. The argument we gave is inspired from Bukovsky [1] where he proved the following: For every partition $\{A_i : i \in S\}$ of $\mathbb{R}^n$ into null sets, there is a subset T of S such that $\bigcup\{A_i : i \in T\}$ is not Lebesgue measurable. A similar result holds in the case of category. One can interpret this result as saying that both $\bigcup\{A_i : i \in T\}$ and $\bigcup\{A_i : i \in (T \setminus S)\}$ have full outer measure in some positive measure Borel subset of $\mathbb{R}^n$. It is therefore natural to ask if, like Lusin's result, we can get an "everywhere big" partition of S. In the case of category, Cichon et al. [4] showed that this is indeed the case in a more general setup: For every partition $\{A_i : i \in S\}$ of a non meager set $X \subseteq \mathbb{R}^n$ into meager sets, there is a subset T of S such that both $\bigcup\{A_i : i \in T\}$ and $\bigcup\{A_i : i \in (T \setminus S)\}$ are everywhere non meager in X. A key ingredient in their argument was a result of Gitik and Shelah [9] which says that forcing with a sigma ideal cannot be isomorphic to Cohen forcing. The measure case, however, is still open.

1.2 A question of Fremlin

The following is a strengthened version of a question of Fremlin [6]. In [4], the authors prove the category version of this and ask this problem for $X = [0, 1]$.

Question 1. *Suppose $X \subseteq [0, 1]$ and $\{A_i : i \in S\}$ is a partition of X into Lebesgue null sets. Is there a subset T of S such that the sets $\bigcup\{A_i : i \in T\}$, $\bigcup\{A_i : i \in (S\backslash T)\}$ and X all have the same Lebesgue outer measure?*

Fremlin and Todorcevic have shown [7] that when $X = [0, 1]$, for every $\epsilon > 0$, one can get a $T \subseteq S$ such that the outer measure of both $\bigcup\{A_i : i \in T\}$ and $\bigcup\{A_i : i \in (S\backslash T)\}$ is more than $1 - \epsilon$. They also remarked that if there is no quasi measurable cardinal below the continuum, then the answer to the above question is positive. An uncountable cardinal κ is quasi measurable if there is a κ-additive ω_1-saturated ideal I over κ which contains all singletons in κ. We'll show, on the other hand, that if there is a real valued measurable cardinal below the continuum then the answer to the above question is yes for $X = [0, 1]$. In fact, we have the following:

Theorem 2. *Suppose every countably generated sigma algebra extending the Borel algebra on $[0, 1]$ admits a measure extending the Lebesgue measure. Then the answer to above question is yes when $X = [0, 1]$.*

Proof: Suppose this is false. Then one can get a partition $\langle N_i : i \in S\rangle$ of $[0, 1]$ into null sets such that the projected sigma ideal $I = \{A \subseteq S : \bigcup\{N_i : i \in A\} \text{ is null}\}$ is ω_1-saturated (See [7] for this). Hence, it is enough to show:

Theorem 3. *Suppose every countably generated sigma algebra extending the Borel algebra on* $[0,1]$ *admits a measure extending the Lebesgue measure. Then* $[0,1]$ *cannot be partitioned into null sets such that the corresponding projected sigma ideal is* ω_1*-saturated.*

Proof: Suppose not and let $\langle N_i : i \in S\rangle$ be a partition of $[0,1]$ into null sets such that $I = \{A \subseteq S : \bigcup\{N_i : i \in A\} \text{ is null}\}$ is ω_1-saturated. Using the ω_1-saturation of I, get $\langle S_n : n < \omega\rangle$, $\langle \kappa_n : n < \omega\rangle$, $\langle i(n,\alpha) : \alpha < \kappa_n\rangle$ such that

- $\langle S_n : n < \omega\rangle$ is a partition of S,
- $|S_n| = \kappa_n$ and $S_n \notin I$,
- for each $n < \omega$, $\langle i(n,\alpha) : \alpha < \kappa_n\rangle$ is a one-one enumeration of S_n and for every $\alpha < \kappa_n$, $\{i(n,\beta) : \beta < \alpha\} \in I$.

For $n < \omega$, $\alpha \leq \kappa_n$, let $A(n,\alpha) = \bigcup\{N_i : i \in \{i(n,\beta) : \beta < \alpha\}\}$ and $A(n) = A(n,\kappa_n)$. Let m be a sufficiently large extension of Lebesgue measure on $[0,1]$ which allows the Fubini type argument below to go through. We will show that $m(A(n)) = 0$ for each $n < \omega$ which gives us the desired contradiction. Fix $n < \omega$, and consider the set $W \subseteq [0,1]^2$ whose vertical section at x, $W_x = \{y : (x,y) \in W\}$ is empty if x is not in $A(n)$ and is $A(n,\alpha)$ otherwise, where $\alpha < \kappa_n$ is least such that $x \in A(n,\alpha)$. Note that W_x is Lebesgue null for each $x \in [0,1]$. For each x, let G_x be a null G_δ set covering W_x. For each $m \geq 1$, let $U_{x,m}$ be an open set of length less than $1/m$ such that $\bigcap\{U_{x,m} : m \geq 1\} = G_x$. Let $U(m) = \bigcup\{\{x\} \times U_{x,m} : x \in [0,1]\}$. Then every vertical section of $U(m)$ is open hence there are countably many subsets $\langle X_k : k < \omega \rangle$ of $[0,1]$ such that each $U(m)$ and hence G is in the product algebra $\Sigma \otimes \mathcal{B}$ where $\mathcal{B}$ is the Borel algebra and Σ is the sigma algebra generated by X_k's together with the Borel sets. So by Fubini's theorem, W is null in the product measure $m \otimes \mu$ (μ is Lebesgue measure) so that some horizontal section $W^y = \{x : (x,y) \in W\}$ for $y \in A(n)$ and hence $A(n)$ is m-null. ☕

We do not know if it is consistent to have a partition of $[0,1]$ into null sets/meager sets such that the projected ideal is ω_1-saturated. Carlson [3] has constructed a model of ZFC in which every countably generated sigma algebra containing the Borel algebra admits an extension of Lebesgue measure. This is also true in the presence of a real valued measurable cardinal below the continuum. In Carlson's model and in the presence of a real valued measurable cardinal below the continuum, there is a Sierpinski set. In the presence of a Sierpinski set, it is not difficult to see that the answer to Question 1 is yes for $X = [0,1]$. We do not know if the assumption in Theorem 2 guarantees the existence of a Sierpinski set.

1.3 A question of Komjáth

We tried looking at Question 1 in the case when each each member of the partition is countable. In this case the problem is equivalent to the following.

Question 2. *Suppose $X \subseteq [0,1]$ and $\{A_i : i \in S\}$ is a partition of X into countable sets. Is there a full outer measure subset Y of X which meets each A_i at one point?*

Using Theorem 1, it can be shown that this is true if one replaces countable by finite. Komjáth informed us about a problem [17] of a similar flavor, which coincidentally, is a special case of this question in dimension one.

Question 3. *Let $X \subseteq \mathbb{R}^n$. Is there always a full outer measure subset Y of X such that the distance between any two distinct points of Y is irrational?*

In [18] he showed that $\mathbb{R}^n$ can be colored by countably many colors such that the distance between any two points of the same color is irrational. It follows that one can always find a subset of positive outer measure that avoids rational distances. Under the assumption that there is no weakly inaccessible cardinal below the continuum, he also showed in [17] that in dimension one we can always find a subset Y of full outer measure in X, avoiding rational distances. Gitik and Shelah showed the following in [10], [11]: For any sequence $\langle A_n : n \in \omega \rangle$ of sets of reals, there is a disjoint refinement of full outer measure; i.e., there is a sequence $\langle B_n : n \in \omega \rangle$ of pairwise disjoint sets such that $B_n \subseteq A_n$ and they have the same outer measure. It follows that one can omit integer distances in dimension one while preserving outer measure. Their argument relies on one of their results about forcing with sigma ideals which says that forcing with any sigma ideal cannot be isomorphic to a product of Cohen and random forcing. We use this to answer Komjáth's question in dimension one.

Let $\mathcal{T}$ be a subtree of $\omega^{<\omega}$ such that every node in $\mathcal{T}$ has at least two children; i.e., for every $\sigma \in \mathcal{T}$, $|\{n \in \omega : \sigma n \in \mathcal{T}\}| \geq 2$.

Definition 1. *Call a family $\langle A_\sigma : \sigma \in \mathcal{T} \rangle$ a full tree on $A \subseteq \mathbb{R}^n$ if:*

- *$A = A_\phi$, and for every $\sigma \in \mathcal{T}$,*
- *A_σ is a disjoint union of $A_{\sigma n}$'s where $\sigma n \in \mathcal{T}$ and*
- *A_σ has full outer measure in A.*

The following application of Theorem 5 is implicit in [10]:

Theorem 4. *Let $A \subseteq \mathbb{R}^n$ and let $\langle A_\sigma : \sigma \in \mathcal{T}\rangle$ be a full tree on A. Then there is a $B \subseteq A$ of full outer measure in A such that for every $\sigma \in \mathcal{T}$, $A_\sigma \backslash B$ has full outer measure in A_σ.*

This theorem is a consequence of the following theorem in [10], [11]:

Theorem 5. *Suppose I is a sigma ideal over a set X. Then forcing with I cannot be isomorphic to Cohen $\times$ Random.*

Let us explain how Theorem 4 follows from Theorem 5. It is clearly enough to show that there is a non null $X \subseteq A$ such that $A_\sigma \backslash X$ has full outer measure in A_σ for every $\sigma \in \mathcal{T}$, for then the union B of a maximal family $\{X_n : n \in \omega\}$ of such sets with pairwise disjoint envelopes will be as required. Suppose that this fails so that for every non null $X \subseteq A$, there is some $\sigma \in \mathcal{T}$ such that $\mathrm{env}(A_\sigma)$ is strictly larger than $\mathrm{env}(A_\sigma \backslash X)$. Consider the map that sends every positive outer measure subset $X \subseteq A$ to the supremum, in the complete Boolean algebra Cohen $\times$ Random, of all pairs (σ, E) where $\sigma \in \mathcal{T}$ and E is a positive measure Borel subset of $\mathrm{env}(A)$ such that E is disjoint with $\mathrm{env}(A_\sigma \backslash X)$. This gives a dense embedding from $\mathcal{P}(A)/\mathrm{Null}$ to Cohen $\times$ Random contradicting the fact that they cannot be forcing isomorphic.

Theorem 6. *Let $X \subseteq \mathbb{R}$. Then there is a $Y \subseteq X$ such that Y has full outer measure in X and the distance between any two points in Y is irrational.*

Proof: Let X_0 be a set of representatives from the partition on X induced by the equivalence relation $x \sim y$ iff $x - y$ is rational. Let $\langle r_n : n \geq 0\rangle$ be a list of all rationals with $r_0 = 0$. For each $n \geq 0$, let $f_n : X_0 \to \mathbb{R}$ be defined by $f_n(x) = x + r_n$, if $x + r_n \in X$, otherwise $f_n(x) = x$, also put $X_n = \text{range}(f_n)$. For $m, n \geq 0$, let $F_n^m = f_n \circ f_m^{-1} : X_m \to X_n$. Note that $f_n = F_n^0$. Also note that for every $m, n \geq 0$, $x \in X_m$, $F_n^m(x) = x + r$, for some $r \in \{0, r_n, -r_m, r_n - r_m\}$. This will allow us to use Lemma 1 below with $k \leq 4$.

We will inductively define a sequence $\langle K_n : n \geq 0 \rangle$ of pairwise disjoint subsets of X_0 such that for each $n \geq 0$, $f_n[K_n]$ has full outer measure in X_n. Theorem 6 will immediately follow by setting $Y = \bigcup\{f_n[K_n] : n \in \omega\}$. We need a definition for our next lemma.

Definition 2. *Let $Y \subseteq \mathbb{R}$ and $F : Y \to \mathbb{R}$. We say that F is fullness preserving if whenever W is a full outer measure subset of Y, $F[W]$ is a full outer measure subset of $F[Y]$.*

Observe that if $F : Y \to \mathbb{R}$ is fullness preserving, then for any $W \subseteq Y$ of full outer measure in Y, $F \restriction W$ is also fullness preserving.

Lemma 1. *Suppose $F : Y \to \mathbb{R}$ acts by translating k many pieces of Y; i.e., there are a partition $\{T_1, T_2, \ldots, T_k\}$ of Y and reals $s_1, s_2, \ldots, s_k$ such that for every $x \in T_i$, $F(x) = x + s_i$. Then, there is another partition of size k, $\{Y_i : 1 \leq i \leq k\}$ of Y, such that for every $i \leq k$,*

- *Y_i has full outer measure in Y and*
- *$F \restriction Y_i$ is fullness preserving.*

Proof of Lemma 1: Use induction on k. If $k = 1$, $Y_1 = Y$ works. So assume $k = l + 1$. Let $Z = \bigcup\{T_i : 1 \leq i \leq l\}$. Let $\{Z_i : 1 \leq i \leq l\}$ be a partition of Z such that each Z_i has full outer measure in Z and $F \restriction Z_i$ is fullness preserving. Let $E_1 = \text{env}(Z)$,

$E_2 = \mathrm{env}(T_k)$ and $D = E_1 \bigcap E_2$. Let W_1, W_2 be a partition of $Z_1 \bigcap (E_1 \setminus D)$ into two full outer measure subsets. Let $\{V_j : 1 \leq j \leq k\}$ be a partition of $T_k \bigcap (E_2 \setminus D)$ into k full outer measure subsets. Set $Y_1 = W_1 \bigcup (Z_1 \bigcap D) \bigcup V_1$. For $2 \leq i \leq l$, put $Y_i = Z_i \bigcup V_i$ and let $Y_k = W_2 \bigcup (D \bigcap T_k) \bigcup V_k$. Then $\{Y_i : 1 \leq i \leq k\}$ is a partition of Y with the required properties.

Claim 1.1. *There exists $K_0 \subseteq X_0$, such that K_0 has full outer measure in X_0 and for every $n \geq 1$, $X_n \setminus f_n[K_0]$ has full outer measure in X_n.*

Proof of Claim 1.1: Using Lemma 1, construct a full tree $\langle Y_\sigma : \sigma \in 2^{<\omega} \rangle$ on $Y = X_0$ such that for each $n \geq 1$, $\sigma \in 2^n$, $f_n \restriction Y_\sigma$ is fullness preserving.

Now Theorem 4 will imply that there is some $K_0 \subseteq X_0$ of full outer measure in X_0, such that for every $\sigma \in 2^{<\omega}$, $Y_\sigma \setminus X_0$ has full outer measure in Y_σ. Fix any $n \geq 1$. Since for each $\sigma \in 2^n$, $f_n \restriction Y_\sigma$ is fullness preserving, we get that $f_n[Y_\sigma \setminus X_0]$ has full outer measure in $f_n[Y_\sigma]$. It follows that $X_n \setminus f_n[K_0] = \bigcup \{f_n[Y_\sigma \setminus X_0] : \sigma \in 2^n\}$ has full outer measure in $\bigcup \{f_n[Y_\sigma] : \sigma \in 2^n\} = X_n$.

Next suppose that for some $N \geq 1$, we have a pairwise disjoint family $\{K_i : 0 \leq i < N\}$ of subsets of X_0 such that

- for each $0 \leq i < N$, $f_i[K_i]$ has full outer measure in X_i and
- for each $j \geq N$, $f_j[X_0 \setminus \bigcup \{K_i : 1 \leq i < N\}]$ has full outer measure in X_j.

Following the arguments in the proof of Claim 1.1, we first construct, using Lemma 1, a full tree $\langle Y_\sigma : \sigma \in 4^{<\omega} \rangle$ on $Y = f_N[X_0 \setminus \bigcup \{K_i : 1 \leq i < N\}]$ such that for each $n \geq 1$, $\sigma \in 2^n$, $F^N_{N+n} \restriction Y_\sigma$ is fullness preserving. Using Theorem 4, we get some $K \subseteq Y$ of full outer measure in Y, such that for every $\sigma \in 4^{<\omega}$, $Y_\sigma \setminus K$ has full outer measure in Y_σ. Putting $K_N = f_N^{-1}[K]$ it follows that

- for each $0 \leq i < N$, $K_i \cap K_N = \phi$,
- $f_N[K_N]$ has full outer measure in X_N and
- for each $j \geq N+1$, $f_j[X_0 \setminus \bigcup\{K_i : 1 \leq i \leq N\}]$ has full outer measure in X_j.

This concludes the proof of Theorem 6. One can easily see that the above arguments can be applied to avoid any countable set of distances by replacing the rationals with the additive subgroup generated by this countable set. One can also obtain a category analogue in the following sense: Let $X \subseteq \mathbb{R}$. Then there is a subset $Y \subseteq X$ such that Y is everywhere non meager in X and the distance between any two distinct points of Y is irrational. Here we call a subset $Y \subseteq X$ everywhere non meager in X if for every open set U, if $X \cap U$ is non meager then $Y \cap U$ is also non meager. The proof follows essentially the same lines except that one has to use a category analogue of Theorem 4 which depends on the following result of Gitik and Shelah [9]: Suppose I is a sigma ideal over a set X. Then forcing with I cannot be isomorphic to Cohen forcing. We do not know the answer to Komjáth's question (and its category analogue) in higher dimensions.

1.4 Avoiding null distances

Andrews asked if we can avoid null sets of distances.

Question 4. *Suppose $N \subseteq \mathbb{R}^+$ is null. Given $X \subseteq \mathbb{R}$, must there exist $Y \subseteq X$ such that the distance between any two points in Y is not in N and X and Y have same Lebesgue outer measure?*

It is clear that this holds under CH. Using the following result, independently due to Friedman and Shelah, we'll show that this fails in the strongest possible way in the Cohen model.

Theorem 7. *Let $V \models 2^\omega = \omega_1$. Let $P = Fn(\omega_2, 2)$ add ω_2 Cohen reals. Then in V^P, for every F_σ set $F \subseteq \mathbb{R}^2$, if F contains a rectangle of positive outer measure, then it contains a compact rectangle of positive measure.*

For a proof of Theorem 7, see [29] or [2].

Theorem 8. *Let P be as above. The following holds in V^P: Let N be a null dense G_δ subset of $\mathbb{R}^+$. Then for every non null $X \subseteq \mathbb{R}$ there are $x, y \in X$ such that $|x - y| \in N$.*

Proof: The set $F = \{(x, y) : |x-y| \notin N\}$ is an F_σ subset of plane whose complement has zero area. Note that by Steinhaus theorem F cannot contain a compact rectangle of positive measure so, in V^P, it cannot contain a non null rectangle either. Suppose $X \subseteq \mathbb{R}$ is non null and it avoids distances in N. Let A, B be two disjoint non null subsets of X. Then $A \times B$ is a non null rectangle contained in F which is impossible. ☕

We do not know if the category analogue is also independent:

Question 5. *Suppose G is a dense G_δ subset of $\mathbb{R}^2$. Must G contain a non meager rectangle?*

Chapter 2

Around a question of Juhász

2.1 Introduction

We investigate some questions around a problem of Juhász. In particular, we show that the following is consistent: There is no real valued measurable cardinal below the continuum and there is a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is a sigma ideal. We also show that there is a countable partition of $[0,1]$ into interior free sets under the m-density topology of any such extension. Here, the m-density topology is defined by declaring those subsets of $[0,1]$ open, each of whose points is an m-density one point.

The next section contains well known results. For background on elementary embeddings and forcing we refer the reader to Kanamori's book [15].

2.2 Induced ideals in Cohen and random extensions

We start with the following question: Suppose κ is a measurable cardinal and I is a witnessing normal prime ideal over κ. Let P be a forcing notion and G a P-generic filter over V. Let $\hat{I}$ be the ideal generated by I over κ in $V[G]$. Describe forcing with $\hat{I}$; i.e., $\mathcal{P}(\kappa)/\hat{I}$. For the purpose of this chapter, the specific forcings that we consider are all ccc. In this case, the induced ideal is ω_1-saturated and normal.

Proposition 9 (Prikry [24]). *Let I be a κ-additive ω_1-saturated ideal over an uncountable cardinal κ. Let P be a ccc forcing notion and G a P-generic filter over V. Let $\hat{I}$ be the ideal generated by I over κ in $V[G]$. Then $\hat{I}$ is a κ-additive ω_1-saturated ideal over κ. Furthermore, if I is normal so is $\hat{I}$.*

Proof: Let $\langle B_i : i < \theta \rangle \subset \hat{I}$ where $\theta < \kappa$. Let $\langle A_i : i < \theta \rangle \subset I$ be such that $B_i \subseteq A_i$ for each $i < \theta$. Let $p \in P$ force this. In V, for each $i < \theta$, get a maximal antichain $\langle p_{i,n} : n < \omega \rangle$ below p deciding $\mathring{A}_i$. Say $\langle C_{i,n} : n < \omega \rangle \subset I$ is such that $p_{i,n} \Vdash \mathring{A}_i = C_{i,n}$. Let $C_i = \bigcup\{C_{i,n} : n < \omega\}$. Then $p \Vdash \mathring{B}_i \subseteq C_i$ for each $i < \theta$ and hence $\bigcup\{B_i : i < \theta\} \subseteq \bigcup\{C_i : i < \theta\} \in I$. It follows that $\hat{I}$ is κ-additive. Next suppose $\hat{I}$ is not ω_1-saturated and let p force that $\langle X_i : i < \omega_1 \rangle$ is a collection of pairwise disjoint $\hat{I}$-positive sets. Work in V. Let $Y_i = \{\alpha < \kappa : \exists q \leq p(q \Vdash \alpha \in \mathring{X}_i)\}$. Then $Y_i \in I^+$ for each $i < \omega_1$. Now observe that there must exist some $A \in I^+$ such that every I-positive subset of A has I-positive intersections with uncountably many Y_i's. Otherwise we can extract an I-positive disjoint refinement of an uncountable subsequence of $\langle Y_i : i < \omega_1 \rangle$ which is impossible as I is ω_1-saturated. By thinning down we can also assume that $\bigcup\{Y_i : i < \omega_1\} = A$. It follows that for each $j < \omega_1$, $\bigcup\{Y_i : j < i < \omega_1\}$ contains A modulo I. Hence $\limsup\langle Y_i : i < \omega_1 \rangle = A$ modulo I. In particular, some $\alpha < \kappa$ belongs to uncountably many Y_i's and the witnessing conditions q_i's must form an antichain contradicting the ccc-ness of P. Now suppose that I is normal. We'll show that $\hat{I}$ is closed under diagonal unions in $V[G]$. So let $\langle A_\alpha : \alpha < \kappa \rangle \subset I$ and $A_\alpha \subset (\alpha, \kappa) = \{i : \alpha < i < \kappa\}$. Let $p \in P$ force this. Working in V, for each $\alpha < \kappa$, get a maximal antichain $\langle p_{\alpha,n} : n < \omega \rangle$ below p deciding $\mathring{A}_\alpha$.

Say $\langle B_{\alpha,n} : n < \omega \rangle \subset I$ is such that $p_{\alpha,n} \Vdash \mathring{A}_\alpha = B_{\alpha,n}$. Let $B_\alpha = \bigcup\{B_{\alpha,n} : n < \omega\}$. Then $B_\alpha \in I$ and $p \Vdash A_\alpha \subseteq B_\alpha \subset (\alpha, \kappa)$ for each $\alpha < \kappa$. By normality of I, we get $\bigcup\{B_\alpha : \alpha < \kappa\} \in I$. Hence $\bigcup\{A_\alpha : \alpha < \kappa\} \in \hat{I}$. ☕

2.2.1 Prikry's model

Again, let κ be a measurable cardinal with a witnessing normal ideal I. Let $j : V \to M$ be the corresponding ultrapower embedding with critical point κ. We'll repeatedly use $M^\kappa \subset M$. Denote by C_λ, the Cohen algebra on 2^λ for adding λ many Cohen reals. Let G be C_λ-generic over V. We attempt to describe, in $V[G]$, the algebra $\mathcal{P}(\kappa)/\hat{I}$ where $\hat{I}$ is the ideal generated by I. If $\lambda < \kappa$, this algebra is trivial so assume $\lambda \geq \kappa$. By elementarity plus the fact that M is countably closed, $j(C_\lambda) = C_{j(\lambda)}$ is the Cohen algebra for adding $j(\lambda)$ many Cohen reals. Suppose, H is $C_{j(\lambda)}$-generic over V. Consider, $G = \{p \in C_\lambda : j(p) \in H\}$.

Lemma 2. *G is C_λ-generic over V.*

Proof: Clearly, G is a filter over C_λ. Suppose, $\langle p_n : n \in \omega \rangle \subseteq C_\lambda$ is a maximal antichain. Then, $j(\langle p_n : n \in \omega \rangle) = \langle j(p_n) : n \in \omega \rangle$ is a maximal antichain in $C_{j(\lambda)}$. So some $j(p_n) \in H$. Hence $p_n \in G$.

The embedding $j : V \to M$ extends to $j^\star : V[G] \to M[H]$ satisfying $j^\star(G) = H$ by defining $j^\star(X) = val_H(j(\mathring{X}))$. The inclusion $j[G] \subseteq H$ ensures that $j^\star(X)$ does not depend on the choice of the name for X.

Working in $V[G]$, consider the function $\phi : \mathcal{P}(\kappa)/\hat{I} \to C_{j(\lambda)}/j[G]$ defined by:

$$\phi([X]) = [[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}}/j[G]$$

Lemma 3. *ϕ is a Boolean isomoprhism.*

Proof: First note that ϕ is well defined since if $p \Vdash \mathring{X} \Delta \mathring{Y} \subseteq A$ for some $A \in I$, then $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} \wedge j(p) = [[\kappa \in j(\mathring{Y})]]_{C_{j(\lambda)}} \wedge j(p)$ so that $\phi(X) = \phi(Y)$. It is clear that ϕ preserves boolean operations. To see that it is injective, note that if $\phi(X) = 0$, then for some $p \in G$, $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} \wedge j(p) = 0$. Then, $p \Vdash \mathring{X} \in \hat{I}$. Finally, if $q \in j(C_\lambda)/j[G]$, then for some $p_\alpha \in C_\lambda$, for $\alpha < \kappa$, $q = [\langle p_\alpha : \alpha < \kappa \rangle]$. Let $\mathring{X}$ be such that $[[\alpha \in \mathring{X}]]_{C_\lambda} = p_\alpha$, for $\alpha < \kappa$. Then, $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} = j(\mathring{X})([id]) = [\langle p_\alpha : \alpha < \kappa \rangle] = q$. Hence ϕ is surjective.

Corollary 2.1 (Prikry [24])**.** *In $V[G]$, forcing with $\mathcal{P}(\kappa)/\hat{I}$ is same as adding $|j(\lambda) \backslash j[\lambda]|$ Cohen reals. In particular, when $\kappa \leq \lambda \leq 2^\kappa$, $\mathcal{P}(\kappa)/\hat{I}$ adds 2^κ Cohen reals. If $\lambda = 2^\kappa$, $\mathcal{P}(\kappa)/\hat{I}$ is σ-centered.*

Proof: The first statement is clear. The second follows from the fact that whenever $\kappa \leq \lambda \leq (2^\kappa)^V$, we have $2^\kappa < j(\lambda) < (2^\kappa)^+$. For the third statement, use the fact that $2^{(2^\omega)}$ is a separable space. ☕

2.2.2 Solovay's model

Once again, let κ be a measurable cardinal with a witnessing normal ideal I and $j : V \to M$ is the corresponding ultrapower embedding with critical point κ. Let R_λ be the measure algebra on 2^λ for adding λ many random reals with $\lambda \geq \kappa$. So $j(R_\lambda) = R_{j(\lambda)}$ as above. Let H be $R_{j(\lambda)}$-generic over V. Set $G = \{p \in R_\lambda : j(p) \in H\}$. As in Lemma 2, we get

Lemma 4. *G is R_λ-generic over V.*

The embedding $j : V \to M$ extends to $j^\star : V[G] \to M[H]$ satisfying $j^\star(G) = H$ by defining $j^\star(X) = val_H(j(\mathring{X}))$. In $V[G]$, consider the function $\phi : \mathcal{P}(\kappa)/\hat{I} \to R_{j(\lambda)}/j[G]$ defined by:

$$\phi([X]) = [[\kappa \in j(\mathring{X})]]_{R_{j(\lambda)}}/j[G]$$

Lemma 5. *ϕ is a Boolean isomoprhism.*

Corollary 2.2 (Solovay [28])**.** *In $V[G]$, forcing with $\mathcal{P}(\kappa)/\hat{I}$ is same as adding $|j(\lambda)\backslash j[\lambda]|$ random reals. In particular, in $V[G]$, the continuum is real valued measurable.*

2.3 Around a question of Juhász

A topological space is called resolvable it has a dense codense subset. An old result of Sierpinski [27] says that every metric space is maximally resolvable in the following sense.

Theorem 10. *Suppose X is a metric space and κ is an infinite cardinal such that every open ball in X has at least κ many points. Then X can be partitioned into κ many dense sets.*

Proof: We first show that this is the case if each open ball has size exactly κ.

Claim 2.3. *Let κ be an infinite cardinal and X a metric space in which every open ball has size $\kappa \geq \omega$. Let us call such a space κ-homogeneous. Then X can be divided into κ many dense subsets.*

In this case, $|X| = \kappa$ so that X has a basis of size κ (use rational radii balls), say $\{B_\alpha : \alpha < \kappa\}$. Inductively construct $\{D_\alpha : \alpha < \kappa\}$ by putting, at stage α, one point from each B_β, $\beta < \alpha$ into every D_β, $\beta < \alpha$.

Claim 2.4. *Let X be a metric space in which every open ball is infinite. Then there is a family of pairwise disjoint open balls U in X, such that the union of this family is dense in X and each U is a κ-homogeneous metric space for some uncountable cardinal κ.*

Every open ball in X contains a homogeneous ball as there is no infinite decreasing sequence of cardinals. So take a maximal family of pairwise disjoint homogeneous open balls in X. Theorem 10 easily follows from the above claims. ☕

Resolvable spaces were introduced by Hewitt [13] where the following example of a Hausdorff irresolvable space appears: Let T be the usual topology on $\mathbb{R}$. Let $\mathcal{F}$ be the family of all topologies on $\mathbb{R}$ that extend T and are without isolated points. The union of every chain (under inclusion) in $\mathcal{F}$ is also in $\mathcal{F}$ hence $\mathcal{F}$ has a maximal topology say τ. Then τ is irresolvable.

Juhász recently asked the following question (communicated by Miller):

Question 6. *Is there a ccc Hausdorff space X without isolated points such that for every partition $X = \bigcup\{Y_n : n \in \omega\}$ there is some Y_n with non empty interior?*

If X is such a space then by passing to some open subset of X we can assume every open subset of X is such a space. So the family of interior free subsets of X forms a σ-ideal which is also ω_1-saturated as X is ccc. So we need the consistency of a measurable cardinal. Starting with a real valued measurable cardinal below continuum Kunen, Szymanski and Tall have constructed such a space - See Corollary 3.6 in [19]. Kunen described another construction to us which even makes the space T_4.

Theorem 11 (Kunen)**.** *Suppose κ is measurable in V. Let C_κ be Cohen forcing for adding κ Cohen reals. Let G be C_κ-generic over V. Then, in $V[G]$, there is a ccc T_4*

space X wihthout isolated points such that whenever $X = \bigcup\{Y_n : n \in \omega\}$, some Y_n has non empty interior.

Proof: Let I be a witnessing normal ideal over κ and $\hat{I}$, the ideal generated by I in $V[G]$. By Corollary 2.1, $\mathcal{P}(\kappa)/\hat{I}$ is isomorphic to C_{2^κ}. For $\alpha < 2^\kappa$, let $E_\alpha = \{f : 2^{2^\kappa} \to 2 : f(\alpha) = 1\}$. Then $\{[E_\alpha] : \alpha < \kappa\}$ is an independent family that completely generates C_{2^κ}. Let $\{[A_\alpha] : \alpha < 2^\kappa\}$ be the corresponding family in $\mathcal{P}(\kappa)/\hat{I}$. We'll define a topology $\mathcal{T}$ on κ by choosing a member A_α from each equivalence class $[A_\alpha]$ and declaring it to be clopen. We do it in such a way so that for any two disjoint sets $X, Y \in \hat{I}$, there is some A_α, $\alpha < 2^\kappa$ separating them - i.e., $X \subset A_\alpha$ and $Y \subset \kappa \backslash A_\alpha$. Since there are only 2^κ many such pairs, this can clearly be done. Thus $\mathcal{T}$ is Hausdorff. Also, every set in $\hat{I}$ is $\mathcal{T}$-closed since for any $X \in \hat{I}$, the union of A_α's disjoint with X is $\kappa \backslash X$. We claim that for any $B \subseteq \kappa$, $B \in \hat{I}$ iff the $\mathcal{T}$-interior of B is empty. Notice that the family $\mathcal{F}$ of finite boolean combinations of A_α's is a basis for $\mathcal{T}$. Since each member of this basis is $\hat{I}$-positive, every member of $\hat{I}$ has empty $\mathcal{T}$-interior. Conversely, if B is $\hat{I}$-positive, then for some $X \in \hat{I}$ and $A \in \mathcal{F}$, $A \backslash X \subseteq B$. This is because $\{[A] : A \in \mathcal{F}\}$ is dense in $\mathcal{P}(\kappa)/\hat{I}$. As X is closed, $\mathcal{T}$-interior of B is empty. $\mathcal{T}$ is ccc because $\mathcal{P}(\kappa)/\hat{I}$ is ccc. Since $\hat{I}$ is κ-additive, every partition of κ into fewer than κ many sets contains one set with non empty $\mathcal{T}$-interior. It remains to show that $\mathcal{T}$ is normal. Fix $C, D \subseteq \kappa$ disjoint and $\mathcal{T}$-closed. Let C', D' be $\mathcal{T}$-interiors of C and D. Let $C_1 = C \backslash C'$, $D_1 = D \backslash D'$. Then $C_1, D_1 \in \hat{I}$ since their $\mathcal{T}$-interiors are empty. Let A_α separate C_1, D_1. Then $(A_\alpha \bigcup C') \backslash D$ and $((\kappa \backslash A_\alpha) \bigcup D') \backslash C$ are $\mathcal{T}$-open sets separating C and D. ☕

Juhász wondered if the density topology of some countably additive total extension m of Lebesgue measure could give an example of such a space. Kunen pointed out, that using a result of Maharam [23], together with the fact that the measure algebra of m is everywhere inseparable (by a theorem of Gitik and Shelah [9]), one can deduce the existence of some $X \subset [0,1]$ with $m(X) = 1/2$ which divides every Borel set into two pieces of equal measure so this is impossible. He also remarked that it may still be possible to have a finitely additive total extension of Lebesgue measure whose density topology may give an example of such a space. This led us to consider the following question:

Question 7. *Does there exist a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is countably additive but m is nowhere countably additive?*

Kunen suggested the following model: Start with a measurable cardinal κ. Let G be a generic filter for a finite support iteration of random forcing of length κ. Then in $V[G]$, there is no real valued measurable cardinal below continuum and there is a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is countably additive. We now sketch a proof of this.

Definition 3. *A strictly positive finitely additive probability measure (SPFAM) on a Boolean algebra B is a function $m : B \to [0,1]$ satisfying the following:*

- *For every $b \in B$, $m(b) = 0 \Leftrightarrow b = 0_B$ and $m(1_B) = 1$*
- *For every $a, b \in B$, if $a \cap b = 0$ then $m(a \cup b) = m(a) + m(b)$*

Lemma 6. *Let B be a complete Boolean algebra (cBa) with a SPFAM m and let μ, $\mathring{C}$ be B-names such that $[[\mathring{C}$ is a cBa and μ is a SPFAM on $\mathring{C}]]_B = 1$. Then $B \star \mathring{C}$ admits a SPFAM extending m, identifying B with a complete subalgebra of $B \star \mathring{C}$.*

Proof: We begin by reviewing $B \star \mathring{C}$ - See [14] for details. Let $S = \{\tau \in V^B : [[\tau \in \mathring{C}]]_B = 1\}$. Define an equivalence relation on S: $\sigma \sim \tau$ iff $[[\sigma = \tau]]_B = 1$. Let D be a complete set of representatives. Then D is a cBa under Boolean operations induced from $\mathring{C}$. In particular, for any $c_1, c_2 \in D$, $c_1 \leq_D c_2$ iff $[[c_1 \leq_C c_2]]_B = 1$. We let $B \star \mathring{C} = D$. The map $e : B \to D$ defined by setting $e(b)$ to be the unique $\tau \in D$ such that $[[\tau = 1_C]]_B = b$ and $[[\tau = 0_C]]_B = -b$ is a complete embedding of B into D and we identify the image $e[B]$ with B.

Now define $\phi : D \to [0,1]$ as follows: Let $\tau \in D$. For each $n \geq 1$, let $\langle I_0, I_1, \ldots, I_{2^n-1} \rangle$ be the dyadic partition of $[0,1]$ into intervals of length $1/2^n$. Let

$$\phi_n(\tau) = \sum_{k=0}^{2^n-1} m([[\mu(\tau) \in I_k]]_B) k/2^n$$

Then $0 \leq \phi_n(\tau) \leq \phi_{n+1}(\tau) \leq 1$ for every $n \geq 1$. Let $\phi(\tau) = \sup_n \phi_n(\tau)$.

Claim 2.5. *ϕ is a SPFAM on D, extending m.*

Clearly, $\phi(0_D) = 0$, $\phi(1_D) = 1$. If $\sigma, \tau \in D$ are disjoint then $[[\sigma \cap \tau = 0]]_B = 1$. Hence $[[\mu(\sigma \cup \tau) = \mu(\sigma) + \mu(\tau)]]_B = 1$ and it follows that $\phi(\sigma \cup \tau) = \phi(\sigma) + \phi(\tau)$. ϕ is strictly positive because $[[\mu \text{ is strictly positive}]]_B = 1$. Finally if $b \in B$, then $[[e(b) = \{\langle 1_C, b \rangle, \langle 0_C, -b \rangle\}]]_B = 1$. Hence, $\phi(b) = 1 \cdot m(b) + 0 \cdot m(-b) = m(b)$. ☕

Theorem 12. *Suppose κ is measurable and I is a witnessing normal ideal. Let G be a generic filter for finite support iteration of random focring of length κ. Let $\hat{I}$ be the induced ideal in $V[G]$. Then $\mathcal{P}(\kappa)/\hat{I}$ is a cBa that admits a SPFAM m. Furthermore one can identify the random algebra R_ω with a complete subalgebra of $\mathcal{P}(\kappa)/\hat{I}$ on which m agrees with the Lebegsue measure.*

Proof: Let $j : V \to M$ be the ultrapower embedding. Let $\langle B_\alpha, \mathring{C}_\alpha : \alpha < \kappa \rangle$ be the finite support iteration of random forcing; i.e.,

- $B_0 = R_\omega$
- $B_{\alpha+1} = B_\alpha \star \mathring{C}_\alpha$ where $[[\mathring{C}_\alpha = R_\omega]]_{B_\alpha} = 1$
- When λ is limit, B_λ is the Boolean completion of the direct limit of $\langle B_\alpha : \alpha < \lambda \rangle$.

So we have $B_0 \lessdot B_1 \lessdot \cdots \lessdot B = B_\kappa$ where B_κ is the completion of the direct limit of this iteration.

By elementarity plus the fact that M is countably closed, $j(B)$ is the finite support iteration of random forcings of length $j(\kappa)$ in M. Notice that $B \lessdot j(B)$ since $j(B)$ extends B through a longer iteration. Let G be B-generic over V. Then $j \restriction G$ is identity and hence $j[G] = G$. Let $\hat{I}$ be the ideal generated by I in $V[G]$. Consider the map $\phi : \mathcal{P}(\kappa)/\hat{I} \to j(B)/G$ defined by $\phi([X]) = [[\kappa \in j(\mathring{X})]]_{j(B)}/G$. Then ϕ is a Boolean isomorphism. Now in $M[G]$, $j(B)/G$ is a finite support iteration of random forcing indexed by $j(\kappa)\backslash\kappa$. By previous lemma, we can construct an increasing sequence of SPFAMs $\langle m_\alpha : \kappa + 1 \leq \alpha < j(\kappa) \rangle$ on this iteration. Let $m : j(B)/G \to [0,1]$ be their union. Since $M[G]$ and $V[G]$ have same reals, we can also assume that the measure algebra of $m_{\kappa+1}$ is the random algebra in $V[G]$. Now in $V[G]$, we can lift m to a SPFAM on $\mathcal{P}(\kappa)/\hat{I}$ via the isomorphism ϕ and this finishes the proof.

To lift m to an extension of Lebesgue measure on 2^ω, create a tree $\langle X_\sigma : \sigma \in 2^{<\omega} \rangle$ of subsets of κ such that

- $X_\phi = \kappa$
- For every $\sigma \in 2^{<\omega}$, X_σ is a disjoint union of $X_{\sigma 0}$ and $X_{\sigma 1}$
- $m(X_\sigma) = 2^{-|\sigma|}$, where $|\sigma|$ is the length of σ

Furthermore, m restricted to the sigma algebra generated by $\{X_\sigma : \sigma \in 2^{<\omega}\}$ is isomorphic to Lebesgue measure on R_ω under an isomorphism that takes X_σ to $[\sigma] \in 2^\omega$.

Let $f : \kappa \to 2^\omega$ be such that $f(\alpha) = y$ iff $\forall n(\alpha \in X_{y\restriction n})$. Define $\nu : \mathcal{P}(2^\omega) \to [0,1]$ by $\nu(Y) = m(f^{-1}[Y])$. Then, ν is a finitely additive total extension of Lebesgue measure whose null ideal is 2^ω-additive. To finish note that since Cohen reals are added cofinally often, every set of reals of size less than κ is Lebesgue null. It is well known that that if there a real valued measurable cardinal below the continuum then there is a Sierpinski set of size ω_1. Hence ν is nowhere countably additive. ☕

We now address the question:

Question 8. *Let $m : \mathcal{P}([0,1]) \to [0,1]$ be a finitely additive total extension of Lebesgue measure whose null ideal is countably additive. Can the density topology of m on $[0,1]$ provide an example of a ccc Hausdorff space without isolated points such that every partition of $[0,1]$ into countably many sets contains a set with non empty interior?*

It turns out that the answer is no. In fact, we'll show the following:

Theorem 13. *Let $m : \mathcal{P}([0,1]) \to [0,1]$ be a finitely additive total extension of Lebesgue measure. Denote the m-density topology by $\mathcal{T}$. Then there is a countable partition of $[0,1]$ into $\mathcal{T}$-interior free sets.*

Proof: Call $S \subseteq [0,1]$ *small* if S can be covered by countably many m-null sets. Let $\{S_n : n \in \omega\}$ be a maximal collection of pairwise disjoint small sets of positive measure. Then $A = [0,1] \setminus \bigcup_{n\in\omega} S_n$ does not contain any small set of positive measure. Hence the null ideal of m restricted to A is countably additive. It is now enough to split A into countably many $\mathcal{T}$-interior free sets.

Lemma 7. *Let m be as above. Suppose $X \subset [0,1]$ is open in the m-density topology $\mathcal{T}$. Then there is a G_δ set G (in usual topology) such that $m(G) = m(X)$ and $X \setminus G$ is Lebesgue null.*

Proof: Fix an arbitrary $\varepsilon > 0$. Let $\mathcal{V}$ be the collection of all closed intervals in which the fractional measure of X is more than $1-\varepsilon$. Since every point of X is a density one point of X, $\mathcal{V}$ is a Vitali covering of X; i.e., for each $\delta > 0$, and $y \in X$ there is an interval $I \in \mathcal{V}$ that contains y and has length less than δ. By Vitali covering theorem, there is a disjoint subcollection $\{I_n : n \in \omega\} \subset \mathcal{F}$ which covers all but a Lebesgue null part of X. Setting $U_\varepsilon = \bigcup\{I_n : n \in \omega\}$, we get $m(X) \geq \sum_{n\in\omega} m(X \cap I_n) \geq (1-\varepsilon)\sum_{n\in\omega} m(I_n) = (1-\varepsilon)m(U_\varepsilon)$. Let G be the intersection of U_ε's where ε runs over positive rationals.

Lemma 8. *Let m, A be as above. Then there is a partition of A into countably many $\mathcal{T}$-interior free sets.*

We first show that there is no positive measure $X \subseteq A$ all of whose positive measure subsets have non empty interior in the m-density topology. Suppose otherwise. Let $\{Y_n : n \in \omega\}$ be a maximal collection of pairwise disjoint $\mathcal{T}$-open subsets of X. Then $Y = \bigcup\{Y_n : n \in \omega\}$ covers all but an m-null part of X. Hence there is a G_δ set G such that $X\Delta G$ is m-null. The same holds of any positive measure subset of X. It follows that $m \restriction \mathcal{P}(X)$ is countably additive but its measure algebra is separable. But this is impossible by Gitik Shelah theorem.

Now let $\{W_n : n \in \omega\}$ be a maximal family of pairwise disjoint, $\mathcal{T}$-interior free, positive measure subsets of A. Let $X = A \setminus \bigcup_{n\in\omega} W_n$. Then every positive measure subset of X has non empty interior and hence X must be m-null and we are done. ☕

Chapter 3

On a hierarchy of Borel additive subgroups of reals

3.1 Introduction

In his "A course of pure mathematics", G. H. Hardy considers the following limit (Example XXIV.14 in [12]):

$$\lim_{n\to\infty} \sin(n!\pi x)$$

He remarks that when x is rational, this limit is 0. Let G be the set of reals where this limit is 0. It is easily verified that G is an additive subgroup of $\mathbb{R}$. It is also not hard to see that Euler's constant e is also in G. Going a little further, one can show the following.

Lemma 9. *Suppose $x \in [0,1]$. Then x has a unique representation of the form*

$$x = \sum_{n\geq 2} \frac{x_n}{n!}$$

where $x_n \in \{0, 1, \ldots, n-1\}$. Under this representation, $x \in G$ iff $\lim_{n\to\infty} \frac{x_n}{n}$ is either 0 or 1.

Proof: Let $d(x, \mathbb{Z})$ denote the distance of x from the set of integers. First notice that for any sequence of reals $\langle x_n : n \geq 1\rangle$, $\lim_{n\to\infty} \sin(\pi x_n) = 0$ iff $\lim_{n\to\infty} d(x_n, \mathbb{Z}) = 0$. For any $x \in [0,1)$ with x_n's as above, $d(n!x,\mathbb{Z}) = d(b_n,\mathbb{Z})$, where $b_n = \dfrac{x_{n+1}}{n+1} + \varepsilon_n$ where $0 \leq \varepsilon_n < 1/n$. Hence if $\dfrac{x_n}{n} \to 0$ or 1, then $d(b_n,\mathbb{Z}) \to 0$. Conversely suppose, along some subsequence n_k, $\lim\limits_{k\to\infty} \dfrac{x_{n_k}}{n_k} = a$ where $0 < a < 1$. Then for all large enough k, $d((n_k-1)!x,\mathbb{Z})$ is arbitrarily close to a so that $\sin(n!\pi x)$ is bounded away from zero on this subsequence.

3.2 A true $\mathbf{\Pi}^0_3$ group

As we remarked above, G can also be described as follows:

$$G = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x,\mathbb{Z}) = 0\}$$

In this section, we'll show that G is a true $\mathbf{\Pi}^0_3$ additive subgroup of $\mathbb{R}$. Recall the definition of the pointclasses $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$:

- $\mathbf{\Sigma}^0_1$ is the family of open subsets of $\mathbb{R}$, $\mathbf{\Pi}^0_1$ is the family of closed sets.
- for each $1 < \alpha < \omega_1$, a set X is in $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) iff it is the union (resp. intersection) of a countable subfamily of $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$ (resp. $\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$).

One can also consider these pointclasses over other Polish spaces (separable completely metrizable spaces) like 2^ω (Cantor space), ω^ω (Baire space). A set in $\mathbf{\Pi}^0_\alpha$ is a true $\mathbf{\Pi}^0_\alpha$ set if it is not in $\mathbf{\Sigma}^0_\alpha$. A true $\mathbf{\Sigma}^0_\alpha$ set is defined similarly. We begin by recalling some basic facts about Wadge reductions. In what follows, all Polish spaces are uncountable and therefore the Borel hierarchies on such spaces do not terminate at any countable level.

Let $f : X \to Y$ be a continuous map between Polish spaces and let $A \subseteq X$ and $B \subseteq Y$. We write $f : (X,A) \xrightarrow{\text{Wadge}} (Y,B)$ (read "f Wadge reduces A to B") if

$f^{-1}[B] = A$. The following are easily proved.

Lemma 10. *If $f : (X, A) \xrightarrow{Wadge} (Y, B)$ where B is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y then A is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in X.*

Lemma 11. *If $f : (X, A) \xrightarrow{Wadge} (Y, B)$ and A is true $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in X, and if B is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y then B is true $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) in Y.*

Let A be a $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set in X. We say A is $\mathbf{\Sigma}^0_\alpha$-complete (resp. $\mathbf{\Pi}^0_\alpha$-complete) if for every $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set B in any Polish space Y there is a Wadge reduction $f : (Y, B) \xrightarrow{\text{Wadge}} (X, A)$. The following result in Wadge's thesis shows that any two true Σ^0_α (resp. Π^0_α) sets in Cantor space Wadge reduce to each other.

Theorem 14 (Wadge)**.** *A subset A of 2^ω is Σ^0_α-complete if A is true Σ^0_α in 2^ω.*

Some examples follow.

- Any countable dense subset D of an uncountable Polish space X is $\mathbf{\Sigma}^0_2$-complete. E.g., $Q = \{x \in 2^\omega : \lim_{n\to\infty} x(n) = 0\}$. The trueness of D follows by Baire category theorem. Hence by Wadge's theorem, D is $\mathbf{\Sigma}^0_2$-complete.
- Let $\langle , \rangle : \omega^2 \to \omega$ be a pairing function; for example, $\langle m, n\rangle = \frac{1}{2}(m+n+1)(m+n)+n$. Let $P = \{x \in 2^\omega : \forall m(\lim_{n\to\infty} x(\langle m, n\rangle) = 0\}$. Then P is $\mathbf{\Pi}^0_3$-complete.

Proof: It is clear that P is $\mathbf{\Pi}^0_3$. To show that P is $\mathbf{\Pi}^0_3$-complete, fix an arbitrary $\mathbf{\Pi}^0_3$ set A in a Polish space X. Let $A = \bigcap_{n\in\omega} A_n$, where A_n's are $\mathbf{\Sigma}^0_2$. Since Q is $\mathbf{\Sigma}^0_2$-complete, there are Wadge reductions $f_m : (X, A_n) \xrightarrow{\text{Wadge}} (2^\omega, Q)$. Let $F : (X, A) \xrightarrow{\text{Wadge}} (2^\omega, P)$ be given by $F(x)(\langle m, n\rangle) = f_m(n)$.

Theorem 15. *The set $G = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z})) = 0\}$ is a true $\mathbf{\Pi}^0_3$ additive subgroup of reals.*

Proof: It is clear that G is an additive subgroup of reals and that it is $\mathbf{\Pi}^0_3$:

$$x \in G \Leftrightarrow \forall \varepsilon > 0(\exists n_0(\forall n \geq n_0(d(n!x, \mathbb{Z}) \leq \varepsilon)))$$

It suffices to construct a Wadge reduction from $P = \{x \in 2^\omega : \forall m(\lim_{n\to\infty} x(\langle m, n\rangle) = 0\}$ to G. Let $f : 2^\omega \to \mathbb{R}$ be defined as follows. Given $x \in 2^\omega$, let $y_x : \omega \to \omega$ be defined by letting $y_x(n)$ to be the least index $m < n$ such that $x(\langle m, n\rangle) = 1$. In case no such $m < n$ exists, we let $y_x(n) = n$. It is clear that the function $x \mapsto y_x$ is continuous and for every $x \in 2^\omega$, $x \in P \Leftrightarrow \lim_{n\to\infty} y_x(n) = \infty$. Set $f(x) = \sum_{n\geq 2} \frac{a_n}{n!}$, where $a_n = \lfloor\left(\frac{n}{2 + y_x(n)}\right)\rfloor$, and $\lfloor x\rfloor$ denotes the greatest integer not greater than x. Now if $\lim_{n\to\infty} y_x(n) = \infty$, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$, hence $f(x) \in G$. On the other hand, if $x \notin P$, then along some subsequence $\langle n_k : k \in \omega\rangle$, $y_x(n_k)$ is constant so that $\frac{a_{n_k}}{n_k}$ does not go to either 0 or 1. ☕

3.3 A few more groups

Let $G_0 = G$, $G_{k+1} = \{x \in \mathbb{R} : \lim_{n\to\infty} \mathrm{frac}(n!x) \in G_k\}$. Then one can easily check that, for each $k \in \omega$, G_k is an additive subgroup of $\mathbb{R}$. Next we show that

Lemma 12. *G_k is $\mathbf{\Pi}^0_{k+3}$.*

Proof: Let $W = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z})$ exists$\}$. Then W is $\mathbf{\Pi}^0_3$ since

$$x \in W \Leftrightarrow \forall \varepsilon > 0(\exists n_0(\forall m, n \geq n_0(|d(m!x, \mathbb{Z}) - d(n!x, \mathbb{Z})| \leq \varepsilon)))$$

Let $h : W \to \mathbb{R}$ be defined by $h(x) = \lim_{n\to\infty} d(n!x, \mathbb{Z})$. For every open interval (a, b), and for every $x \in W$,

$$h(x) \in (a, b) \Leftrightarrow \exists n_0 \forall n \geq n_0(d(n!x, \mathbb{Z}) \in [a + 1/n, b - 1/n])$$

This implies that for every open set U, $h^{-1}[U]$ is the intersection of a $\mathbf{\Sigma}^0_2$ set with W. This implies that, if $G_k \in \mathbf{\Pi}^0_{k+3}$, then $G_{k+1} = h^{-1}[G_k]$ is the intersection of a $\mathbf{\Pi}^0_{k+4}$ set with W hence is also $\mathbf{\Pi}^0_{k+4}$. ☕

In the remaining part of this section we will show that G_k is a true $\mathbf{\Pi}^0_{k+3}$ set. First we need a nice family of $\mathbf{\Pi}^0_k$-complete sets for $k \geq 3$. The following construction appears in [21].

Let $\phi : 2^\omega \to 2^\omega$ be defined by $\phi(x)(m) = 1$ iff $\forall n(x(\langle m, n\rangle)) = 0$. Extend ϕ to $2^{\leq\omega}$ by defining $\phi(\sigma) = \phi(\sigma\overline{0})$, where $\sigma \in 2^{<\omega}$ and $\sigma\overline{0}$ is σ followed by 0's. Note that although ϕ is not continuous, (e.g., $0^n\overline{1}$ converges to $\overline{0}$, $\phi(0^n\overline{1}) = \overline{0}$ does not converge to $\phi(\overline{0}) = \overline{1}$), $\phi(x \restriction n)$ does converge to $\phi(x)$. Let $H_1 = \{\overline{0}\}$, $H_{k+1} = \phi^{-1}[H_k]$. Then H_k is $\mathbf{\Pi}^0_k$-complete.

Theorem 16. *For every $k \in \omega$, G_k is a true $\mathbf{\Pi}^0_{k+3}$ additive subgroup of reals.*

Proof: When $k = 0$, this was proved above. Suppose $f : (2^\omega, H_{k+3}) \xrightarrow{\text{Wadge}} (\mathbb{R}, G_k)$, where H_{k+3} is the $\mathbf{\Pi}^0_{k+3}$-complete set defined above. For $x \in 2^\omega$, let $a_n = f(\phi(x \restriction n))$ and $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(\phi(x \restriction n)) = f(\lim_{n\to\infty} \phi(x \restriction n)) = f(\phi(x))$. Put $b_n = \lfloor n(d(a_n, \mathbb{Z}))\rfloor$ and define $g : 2^\omega \to \mathbb{R}$ by $g(x) = \sum_{n\geq 2} \frac{b_n}{n!}$. Then g is continuous. Also $g(x) \in G_{k+1}$ iff $\lim_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} d(a_n, \mathbb{Z}) = d(a, \mathbb{Z}) \in G_k$. Hence $x \in H_{k+4} \Leftrightarrow \phi(x) \in H_{k+3} \Leftrightarrow f(\phi(x)) \in G_k \Leftrightarrow a \in G_k \Leftrightarrow d(a, \mathbb{Z}) \in G_k \Leftrightarrow g(x) \in G_{k+1}$.

As a corollary, $G_\omega = \bigcup\{G_k : k \in \omega\}$ is in $\mathbf{\Sigma}^0_\omega$ but not in $\mathbf{\Sigma}^0_k$ for any $k \in \omega$ since $G_\omega = \{x \in \mathbb{R} : \lim_{n\to\infty} d(n!x, \mathbb{Z}) \in G_\omega\}$. ☕

Chapter 4

On definability of free maximal ideals

4.1 Introduction

Let $C^{\star}(\mathbb{R})$ denote the ring of bounded continuous functions on $\mathbb{R}$. Let I be the ideal of those functions in $C^{\star}(\mathbb{R})$ that go to zero as $|x| \to \infty$. Madhuresh has asked (email conversation) if one can explicitly define a maximal ideal extending I. Under the axiom of constructibility, every set is definable so the interesting question is: Is there a model of set theory where there is no definable maximal extension of I? In this note we show that one can pass in a definable fashion, from any maximal ideal extending I in $C^{\star}(\mathbb{R})$ to a non principal ultrafilter on the set of natural numbers. It is well known that any such ultrafilter corresponds to a set of reals which is both Lebesgue non measurable and does not have the Baire property and Shelah [26] has shown that any model of ZFC has a generic extension in which every ordinal definable set of reals has the Baire property. Hence it is consistent to assume that there is no ordinal definable free maximal ideal in $C^{\star}(\mathbb{R})$.

4.2 Free maximal ideals in $C(\mathbb{R})$

Let $C(\mathbb{R})$ be the ring of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. A free maximal ideal M in $C(\mathbb{R})$ is a maximal ideal that satisfies: For every real x, there is some $f \in M$ such that $f(x) \neq 0$.

Theorem 17. *Let M be a free maximal ideal in $C(\mathbb{R})$. Then there is a non principal ultrafilter on ω which is definable by a formula using M as a parameter.*

Proof: For each $f \in M$ let $Z_f = \{x : f(x) = 0\}$. Let $\mathcal{F} = \{Z_f : f \in M\}$. Then the following are easily verified:

- $\phi \notin \mathcal{F}$, $\mathbb{R} \in \mathcal{F}$.
- If A and B are in $\mathcal{F}$ then $A \cap B$ is also in $\mathcal{F}$.
- If $A \in \mathcal{F}$ and B is a closed set containing A then $B \in \mathcal{F}$.

Moreover, the maximality of M implies that $\mathcal{F}$ is a maximal family of closed sets with these properties: If C is a non empty closed set not in $\mathcal{F}$ then the distance function $g(x) = d(x, C)$ is not in M. So for some $f \in M$ and $h \in C(\mathbb{R})$, $f + gh = 1$. Hence Z_f is a closed set in $\mathcal{F}$ disjoint with C. In the literature on rings of continuous functions, $\mathcal{F}$ is called a Z-ultrafilter - See, for example [8]. Note further that for every $x \in \mathbb{R}$, there is a closed set $C \in \mathcal{F}$ that does not contain x. Let us call a Z-ultrafilter free if it has this additional property. We can now forget about M and construct a non principal ultrafilter on the set of integers $\mathbb{Z}$ directly from $\mathcal{F}$. First suppose that $\mathbb{Z} \in \mathcal{F}$. Let $\mathcal{U} = \{C \cap \mathbb{Z} : C \in \mathcal{F}\}$. Then $\mathcal{U}$ is a filter over $\mathbb{Z}$ that contains all cofinite subsets of $\mathbb{Z}$ as $\mathcal{F}$ is free. To see that $\mathcal{U}$ is maximal, suppose $A \subset \mathbb{Z}$ is not in $\mathcal{U}$. Then, since A is closed in $\mathbb{R}$, for some $C \in \mathcal{F}$, $A \cap C = \phi$. But then, $\mathbb{Z} \cap C$ is a set in $\mathcal{U}$ disjoint with

A. Next suppose $\mathbb{Z} \notin \mathcal{F}$. Then there is a closed set $C \in \mathcal{F}$ disjoint with $\mathbb{Z}$. It follows that every closed set in $\mathcal{F}$ contains infinitely many non integer points. For each $C \in \mathcal{F}$, let $Z_C = \{n \in \mathbb{Z} : C \cap (n, n+1) \neq \phi\}$. Let $\mathcal{U} = \{Z_C : C \in \mathcal{F}\}$. We have $\phi \notin \mathcal{U}$ and $\mathbb{Z} \in \mathcal{U}$. Also if $C \in \mathcal{F}$ and $X \subseteq \mathbb{Z}$ contains Z_C then there is a closed set D containing C that meets $(n, n+1)$ iff $n \in X$. Next suppose $C, D \in \mathcal{F}$ then there is a closed set C' containing C such that $Z_C \cap Z_D = Z_{C' \cap D}$. Hence $\mathcal{U}$ is a filter. Finally if $X \subset \mathbb{Z}$ is not in $\mathcal{U}$ then the closed set $C = \bigcup\{[n, n+1] : n \in X\}$ is not in $\mathcal{F}$. Hence there is a $D \in \mathcal{F}$ disjoint with C. But then Z_D is in $\mathcal{U}$ and is disjoint with X. Thus $\mathcal{U}$ is a free ultrafilter over $\mathbb{Z}$. ☕

4.3 Free maximal ideal in $C^\star(\mathbb{R})$

Theorem 18. *Let M be a free maximal ideal in $C^\star(\mathbb{R})$. Then there is a non principal ultrafilter on ω which is definable by a formula using M as a parameter.*

Proof: Recall that a maximal ideal M in $C^\star(\mathbb{R})$ is free if for every $x \in \mathbb{R}$, there is an $f \in M$ such that $f(x) \neq 0$. It is enough to construct a free Z-ultrafilter definable in M as above. For each $\varepsilon > 0$, $f \in M$, let $Z_{f,\varepsilon} = \{x : |f(x)| \leq \varepsilon\}$. Consider the family $\mathcal{F} = \{Z_{f,\varepsilon} : f \in M, \varepsilon > 0\}$. Then $\phi \notin \mathcal{F}$ and $\mathbb{R} \in \mathcal{F}$. Suppose $C \in \mathcal{F}$ and D is a closed set containing C. Let $C = Z_{f,\varepsilon}$ where $f \in M$, $\varepsilon > 0$. We can also assume that f is non negative. Let $g \in C^\star(\mathbb{R})$ be defined as follows:

$$g(x) = \begin{cases} \dfrac{d(x,D)}{1 + d(x,D)} + \dfrac{1}{f(x)} & \text{if } x \notin C \\ 1/\varepsilon & \text{if } x \in C \end{cases}$$

Here $d(x, D)$ denotes the distance of x from D. Then $gf \in M$ and $Z_{gf,1} = D \in \mathcal{F}$.

Next suppose $C, D \in \mathcal{F}$. We can assume that $C = Z_{f,1}$ and $D = Z_{g,1}$ for some $f, g \in M$. Let $h = f^2 + g^2$. Then $Z_{h,1} \subseteq C \cap D$. It follows $\mathcal{F}$ is closed under finite intersection. Next suppose C is a non empty closed set not in $\mathcal{F}$. Suppose no set in $\mathcal{F}$ is disjoint with C. Let $\mathcal{G}$ be the proper filter generated by $\mathcal{F} \cup \{C\}$. Let N be the collection of all $f \in C^\star(\mathbb{R})$ such that for some $\varepsilon > 0$, $Z_{f,\varepsilon} \in \mathcal{G}$. Then N is an ideal in $C^\star(\mathbb{R})$ that contains M. Moreover the function $g(x) = \dfrac{d(x, C)}{1 + d(x, C)} + 1$ is in $N \backslash M$. Since M is maximal, $1 \in N$ which means $\phi \in \mathcal{G}$ - a contradiction as $\mathcal{G}$ was assumed to be proper. Finally note that since M is free, for every $x \in \mathbb{R}$ there is a function $f \in M$ such that $f(x) \neq 0$. Hence for sufficiently small $\varepsilon > 0$, the set $Z_{f,\varepsilon}$ is in $\mathcal{F}$ and $x \notin Z_{f,\varepsilon}$. Hence $\mathcal{F}$ is a free Z-ultrafilter and we can proceed as before. ☕

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