TURING INDEPENDENCE AND BAIRE CATEGORY

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Abstract. We show that it is relatively consistent with ZFC that there is a non-meager set of reals X such that for every non-meager $Y \subseteq X$, there exist distinct $x, y, z \in Y$ such that z is computable from the Turing join of x and y.

1. Introduction

Many natural questions about the global structure of Turing degrees $(\mathcal{D}, \leq_T)$ are undecidable in ZFC. For example, in [2] Groszek and Slaman showed that the statement "Every maximal Turing independent set of reals has size continuum" is independent of ZFC. Another such result due to Slaman and Woodin in [5] says that the statement "$(\mathcal{D}, \leq_T)$ is ω-homogeneous" is independent of ZFC. For more examples, see Chapter 9 in [1].

Definition 1.1. *We say that $X \subseteq 2^\omega$ is n-Turing independent iff for every $F \subseteq X$ of size $|F| \leq n$, the Turing join of F does not compute any real in $X \setminus F$. X is Turing independent iff it is n-Turing independent for every $n \geq 1$.*

In [4], we investigated some Ramsey-type problems about Turing independent sets of the following type: Does every large set of reals have a large n-Turing independent subset? In this work, we will deal with this question when large = non-meager. It turns out that when $n = 1$, the answer is yes in ZFC: Every non-meager set of reals has a non-meager 1-Turing independent subset (Theorem 1.4 in [4]). However, the situation is more complicated when $n \geq 2$. Under Martin's axiom, every non-meager set of reals has a non-meager Turing independent subset (Lemma 4.2 in [4]). On the other hand, assuming the consistency of a measurable cardinal, it is consistent that there is a non-meager set of reals all of whose 2-Turing independent subsets are meager (Theorem 1.5 in [4]). The proof given there was based on the following result of Komjáth [3]: Assuming the consistency of a measurable cardinal, it is consistent that there exists a non-meager $Y \subseteq 2^\omega$ such that the graph of every function from Y to Y is meager in $2^\omega \times 2^\omega$. The consistency of this latter fact requires some large cardinals since Komjáth [3] also showed that the existence of such a Y implies that there is an inner model with an inaccessible cardinal. It follows that this approach cannot avoid the use of large cardinals.

Nevertheless, using a new forcing notion, we show the following.

Theorem 1.2. *It is consistent relative to ZFC that there is a non-meager set $X \subseteq 2^\omega$ such that every 2-Turing independent $Y \subseteq X$ is meager.*

This answers Question 4.9 from [4] in the case of Baire category. We expect that the techniques used in the proof of this theorem will be useful for similar problems.

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On Notation: If $\mathcal{J}$ is an ideal on a set X, then $\mathcal{J}^+ = \mathcal{P}(X) \setminus \mathcal{J}$ denotes the family of all $\mathcal{J}$-positive subsets of X. $\omega^{<\omega}$ is the set of all finite sequences in ω. For $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma^\frown\tau$ to denote the concatenation of σ and τ. For $\sigma \in \omega^{<\omega}$, define $[\sigma] = \{x \in \omega^\omega : \sigma \preceq x\}$. We say that $T \subseteq \omega^{<\omega}$ is a nowhere dense subtree iff T is a subtree of $\omega^{<\omega}$ without terminal nodes and $[T] = \{x \in \omega^\omega : (\forall n)(x \restriction n \in T)\}$ is nowhere dense in ω^ω. For $F = \{x_0, x_1, \ldots, x_{n-1}\} \subseteq \omega^\omega$, the join of F, denoted $\bigoplus_{k<n} x_k$, is the real $y \in \omega^\omega$ satisfying $y(nj + k) = x_k(j)$ for every $k < n$ and $j < \omega$. For a set of ordinals X and a function F with $\mathsf{dom}(F) \subseteq [X]^2$, we will sometimes write $F(\alpha, \beta)$ instead of $F(\{\alpha, \beta\})$. Recall that $(\forall^\infty k)$ abbreviates "For all but finitely many k".

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2. Ingredients of the forcing

To simplify the presentation of our forcing, we will work in ω^ω (Baire space) instead of 2^ω (Cantor space). It is easy to lift our result to the Cantor space in view of the following fact.

Fact 2.1. *There exist a countable $D \subseteq 2^\omega$ and a homeomorphism $H : \omega^\omega \to 2^\omega \setminus D$ such that for every $x \in \omega^\omega$, x and $H(x)$ have the same Turing degree.*

Proof. Let $D = \{x \in 2^\omega : (\forall^\infty k)(x(k) = 0)\}$ and define $H : \omega^\omega \to 2^\omega \setminus D$ by

$$H(x) = 0^{x(0)}{}^\frown 1^\frown 0^{x(1)}{}^\frown 1^\frown 0^{x(2)}{}^\frown 1^\frown \cdots$$

$\square$

Fact 2.2. *Let $E \subseteq \omega^\omega$. Suppose there exists $y \in \omega^\omega$ such that*

$$E \subseteq \{x \in \omega^\omega : (\forall^\infty k)(x(k) \neq y(k))\}$$

Then E is meager.

Proof. For each $n < \omega$, define $D_{n,y} = \{x \in \omega^\omega : (\forall k \geq n)(x(k) \neq y(k))\}$. Note that for every $\sigma \in \omega^{<\omega}$, there exists $\tau \in \omega^{<\omega}$ such that $\sigma \preceq \tau$ and $[\tau] \cap D_{n,y} = \emptyset$. So each $D_{n,y}$ is nowhere dense in ω^ω and therefore

$$E \subseteq \bigcup_{n<\omega} D_{n,y} = \{x \in \omega^\omega : (\forall^\infty k)(x(k) \neq y(k))\}$$

is meager. $\square$

2.1. Strong colorings.

For an infinite cardinal μ, the negative square bracket partition relation $\mu \nrightarrow [\mu]^2_\mu$ means the following: There exists a coloring $F : [\mu]^2 \to \mu$ such that for every $X \in [\mu]^\mu$, $\mathsf{range}(F \restriction [X]^2) = \mu$. Todorčević [6] showed that $\omega_1 \nrightarrow [\omega_1]^2_{\omega_1}$. In the same paper (see Section 5), he also generalized this to larger cardinals as follows.

Fact 2.3 ([6]). *Suppose μ is a regular uncountable cardinal that has a non-reflecting stationary subset. Then $\mu \nrightarrow [\mu]^2_\mu$.*

It follows, for example, that if $\mu = \theta^+$ where θ is regular, then $\mu \nrightarrow [\mu]^2_\mu$.

Definition 2.4. *Let $\mu, \lambda, \bar{F}$, $\mathcal{J}_{\lambda,\mu}$ and $\bar{A}$ be as follows.*

(1) *μ is a regular uncountable cardinal satisfying $\mu \nrightarrow [\mu]^2_\mu$ and $\lambda = \mu^+$.*

(2) *For each $1 \leq \xi < \lambda$, fix a function $F_\xi : [\mu]^2 \to \xi \times \mu$ satisfying: For every $X \in [\mu]^\mu$, $\mathsf{range}(F_\xi \restriction [X]^2) = \xi \times \mu$. Such F_ξ's exist because $\mu \nrightarrow [\mu]^2_\mu$. Define*
$$\bar{F} = \langle F_\xi : 1 \leq \xi < \lambda \rangle.$$

(3) *$\mathcal{J}_{\lambda,\mu}$ is the Fubini product of the ideals $[\lambda]^{<\lambda}$ and $[\mu]^{<\mu}$ on λ and μ respectively. So $\mathcal{J}_{\lambda,\mu}$ consists of those $A \subseteq \lambda \times \mu$ that satisfy*
$$|\{\xi < \lambda : |\{i < \mu : (\xi, i) \in A\}| = \mu\}| < \lambda.$$
Note that $\mathcal{J}_{\lambda,\mu}$ is a μ-complete ideal on $\lambda \times \mu$ and $|\mathcal{J}_{\lambda,\mu}| = 2^\lambda$. Let $\bar{A} = \langle A_\alpha : \alpha < 2^\lambda \rangle$ be an injective enumeration of $\mathcal{J}_{\lambda,\mu}$.

Definition 2.5. *We say that a subset $U \subseteq \lambda \times \mu$ is **$\bar{F}$-closed** iff for every $\xi \geq 1$ and $i < j < \mu$, if (ξ, i) and (ξ, j) are in U, then $F_\xi(i, j) \in U$. The **$\bar{F}$-closure** of $U \subseteq \lambda \times \mu$, denoted $cl_{\bar{F}}(U)$ is the intersection of all $\bar{F}$-closed sets that contain U.*

Lemma 2.6. *Let $U \subseteq \lambda \times \mu$ be finite. Then $cl_{\bar{F}}(U)$ is finite.*

Proof. We can assume that $U \neq \emptyset$. By induction on $\gamma = \max(\mathsf{dom}(U))$, we will show that $cl_{\bar{F}}(U) \subseteq (\gamma + 1) \times \mu$ is finite. If $\gamma = 0$, then $cl_{\bar{F}}(U) = U$ and the lemma holds. So assume $1 \leq \gamma < \lambda$. Define
$$V = \{F_\gamma(i, j) : (\gamma, i) \in U \wedge (\gamma, j) \in U \wedge i < j\}.$$

Put $U' = (U \cup V) \cap (\gamma \times \mu)$ and $U'' = (\{\gamma\} \times \mu) \cap U$. By the inductive hypothesis, $cl_{\bar{F}}(U') \subseteq \gamma \times \mu$ is finite. Since $cl_{\bar{F}}(U) = cl_{\bar{F}}(U') \cup U''$, it follows that $cl_{\bar{F}}(U) \subseteq (\gamma + 1) \times \mu$ is also finite. □

2.2. **A sequence of highly surjective Turing functionals.** Before describing the final ingredient of our forcing $\mathbb{Q}$, let us explain how $\mathbb{Q}$ will work.

Remark 2.7. *Let $\mu, \lambda, \mathcal{J}_{\lambda,\mu}$ and $\bar{F}$ be as in Definition 2.4.*

(A) *$\mathbb{Q}$ will add a $(\lambda \times \mu)$-indexed set of reals $X = \{x_{\xi,i} : (\xi, i) \in \lambda \times \mu\} \subseteq \omega^\omega$.*

(B) *In $V^{\mathbb{Q}}$, the meager ideal restricted to X is isomorphic to $\mathcal{J}_{\lambda,\mu}$. More precisely, for every $A \subseteq \lambda \times \mu$,*
$$A \in \mathcal{J}_{\lambda,\mu} \iff \{x_{\xi,i} : (\xi, i) \in A\} \text{ is meager.}$$

(C) *If $\{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq \lambda \times \mu$ and $F_\xi(i, j) = (\zeta, k)$, then $x_{\zeta,k}$ is computable from $x_{\xi,i} \oplus x_{\xi,j}$.*

Note that Clause (C) and the properties of the sequence of colorings $\bar{F}$ imply that for every $\mathcal{J}_{\lambda,\mu}$-positive set $A \subseteq \lambda \times \mu$, the set $\{x_{\xi,i} : (\xi, i) \in A\}$ is not 2-Turing independent. Together with Clause (B), this guarantees that every 2-Turing independent subset of X will be meager. The Turing functionals witnessing Clause (C) will be carefully chosen in order to ensure that Clause (B) is not violated. It turns out that it is enough to ensure that each of these functionals be chosen from a countable family $\{\Phi_n : n < \omega\}$ of "highly surjective" Turing functionals in the sense made precise by Clause (iii) in Definition 2.9. This will become clear in the proof of Clause (B) (Lemma 4.3 below). The following lemma will be used to construct such a family of Turing functionals.

Lemma 2.8. *Let $S = \{(n, \eta_0, \eta_1) : n \in \omega \wedge \{\eta_0, \eta_1\} \subseteq \omega^{<\omega} \wedge |\eta_0| = |\eta_1|\}$. There exists a computable function $F : S \to \omega^{<\omega}$ such that the following hold.*

(0) *For every $n < \omega$, $F(n, \langle\rangle, \langle\rangle) = \langle\rangle$ where $\langle\rangle$ is the empty sequence.*

(1) *For every* $(n, \eta_0, \eta_1) \in S$, $|F(n, \eta_0, \eta_1)| \geq |\eta_0|$.
(2) *If* $(n, \eta_0, \eta_1), (n, \sigma_0, \sigma_1) \in S$, $\eta_0 \preceq \sigma_0$ *and* $\eta_1 \preceq \sigma_1$, *then* $F(n, \eta_0, \eta_1) \preceq F(n, \sigma_0, \sigma_1)$.
(3) *For every* $\eta_0, \eta_1, \rho \in \omega^{<\omega}$, *if* $|\eta_0| = |\eta_1| = |\rho| \geq 1$, *then there exists* $n < \omega$ *such that* $F(n, \eta_0, \eta_1) = \rho$.
(4) *If* $F(n, \eta_0, \eta_1) = \rho$ *and* $|\eta_0| = |\eta_1| = |\rho| \geq 1$, *then* $F(n, \eta_0{}^\frown 0, \eta_1{}^\frown 0) = \rho^\frown 0$.
(5) *Assume that Clauses* $(a) - (d)$ *below hold.*
 (a) $i_\star, m, K, N < \omega$ *and* $N \geq 1$.
 (b) $h : [N+1]^2 \to \omega$.
 (c) $\bar{l} = \langle l_{j,k} : j < k \leq N \rangle$ *where each* $l_{j,k} \in \omega$.
 (d) $\bar{\eta} = \langle \eta_j : j \leq N \rangle$ *and* $\bar{\rho} = \langle \rho_{j,k} : j < k \leq N \rangle$ *are sequences in* ${}^m\omega$.

 Furthermore, suppose for every $j < k \leq N$, $F(h(j,k), \eta_j, \eta_k) = \rho_{j,k}$. *Then there exists an injective sequence* $\bar{i} = \langle i_j : j \leq N \rangle$ *such that*
 (i) $i_0 = i_\star$,
 (ii) for every $1 \leq j \leq N$, $i_j > K$ *and*
 (iii) for every $j < k \leq N$,
$$F(h(j,k), \eta_j{}^\frown i_j, \eta_k{}^\frown i_k) = \rho_{j,k}{}^\frown l_{j,k}.$$

Proof. Let $T = \{(\eta_0, \eta_1, \rho) : \{\eta_0, \eta_1, \rho\} \subseteq \omega^{<\omega} \wedge |\eta_0| = |\eta_1| = |\rho| \geq 1\}$. Let W be the set of all tuples $(i_\star, m, K, N, h, \bar{l}, \bar{\eta}, \bar{\rho})$ satisfying Clauses 5(a)-5(d) above. Fix computable sequences $\bar{d}$, $\bar{t}$ and $\bar{w}$ such that

(i) $\bar{d} = \langle d_k : k < \omega \rangle$ is an injective enumeration of S,
(ii) $\bar{t} = \langle t_k : k < \omega \rangle$ is a an injective enumeration of T and
(iii) $\bar{w} = \langle w_k : k < \omega \rangle$ lists each member of W infinitely often.

Inductively, construct $F = \bigcup_{s<\omega} F_s$ as follows. Note that $\mathsf{dom}(F_{s+1}) \setminus \mathsf{dom}(F_s)$ will be finite for every stage $s \geq 0$.

Stage $s = 0$. Define $F_s(n, \langle\rangle, \langle\rangle) = \langle\rangle$ for every $n < \omega$ where $\langle\rangle$ is the empty sequence. This ensures Clause (0).

Stage $s = 3k$ where $k \geq 1$. Let $d_k = (n, \eta_0, \eta_1)$. If $d_k \in \mathsf{dom}(F_{s-1})$, define $F_s = F_{s-1}$. Otherwise, fix the largest $j < |\eta_0|$ with $(n, \eta_0 \upharpoonright j, \eta_1 \upharpoonright j) \in \mathsf{dom}(F_{s-1})$. Put $F_{s-1}(n, \eta_0 \upharpoonright j, \eta_1 \upharpoonright j) = \rho$ and define $F_s(n, \eta_0 \upharpoonright k, \eta_1 \upharpoonright k) = \rho^\frown 0^{k-j}$ for each $j < k \leq |\eta_0|$. Note that Clauses (1)-(4) are preserved.

Stage $s = 3k+1$ where $k \geq 0$. Let $t_k = (\eta_0, \eta_1, \rho)$ and $m = |\eta_0| = |\eta_1| = |\rho|$. If there exists $n < \omega$ such that $F_{s-1}(n, \eta_0, \eta_1) = \rho$, define $F_s = F_{s-1}$. Otherwise, we must have $m \geq 1$. Choose the least n satisfying
$$(\forall \sigma_0, \sigma_1)[(n, \sigma_0, \sigma_1) \in \mathsf{dom}(F_{s-1}) \implies \sigma_0 = \sigma_1 = \langle\rangle]$$
and define $F_s(n, \eta_0 \upharpoonright k, \eta_1 \upharpoonright k) = \rho$ for every $k \leq m$. Note that Clauses (1)-(4) are again preserved.

Stage $s = 3k+2$ where $k \geq 0$. Let $w_k = (i_\star, m, K, N, h, \bar{l}, \bar{\eta}, \bar{\rho})$. If for some $j < k \leq N$, either $(h(j,k), \eta_j, \eta_k) \notin \mathsf{dom}(F_{s-1})$ or $F_{s-1}((h(j,k), \eta_j, \eta_k)) \neq \rho_{j,k}$, then define $F_s = F_{s-1}$. Otherwise, choose the lexicographically least injective sequence $\bar{i} = \langle i_j : j \leq N \rangle$ such that

(a) $i_0 = i_\star$,
(b) for every $1 \leq j \leq N$, $K < i_j < \omega$ and
(c) for every $j < k \leq N$, $(h(j,k), \eta_j{}^\frown i_j, \eta_k{}^\frown i_k) \notin \mathsf{dom}(F_{s-1})$.

This is possible because $\mathsf{dom}(F_{s-1})\setminus\mathsf{dom}(F_0)$ is finite. Now for every $j < k \leq N$, define

$$F_s(h(j,k), \eta_j{}^\frown i_j, \eta_k{}^\frown i_k) = \rho_{j,k}{}^\frown l_{j,k}.$$

Since $\bar{i}$ is injective, there is no ambiguity here and Clauses (1)-(4) continue to hold.

This concludes the description of F. It should be clear that F is computable since in order to compute $F(n, \eta_0, \eta_1)$, we just have to run the construction described above for $s = 3k$ stages where $d_k = (n, \eta_0, \eta_1)$. It is also easy to check that F satisfies Clauses (0)-(4).

Finally, to see that Clause (5) also holds, suppose $w = (i_\star, m, K, N, h, \bar{l}, \bar{\eta}, \bar{\rho})$ satisfies Clauses (5)(a)-(d) and for every $j < k \leq N$, $F(h(j,k), \eta_j, \eta_k) = \rho_{j,k}$. Choose a stage $s = 3k+2$ large enough so that $w_k = w$ and for every $j < k \leq N$, $(h(j,k), \eta_j, \eta_k) \in \mathsf{dom}(F_{s-1})$. Now observe that an $\bar{i}$ witnessing Clauses (5)(i)-(iii) must exist by the end of stage s. $\square$

Definition 2.9. *Let $\langle \Phi_n : n < \omega\rangle$ be a sequence of Turing functionals in two real variables $x, y \in \omega^\omega$ defined as follows:*

$$\Phi_n(x,y) = \bigcup_{k<\omega} F(n, x \restriction k, y \restriction k).$$

Here, $F : S \to \omega^{<\omega}$ is the computable function from Lemma 2.8. From now on, we will write $\Phi_n(\eta_0, \eta_1) = \rho$ instead of $F(n, \eta_0, \eta_1) = \rho$. So $\langle \Phi_n : n < \omega\rangle$ satisfies the following.

- (i) *For every $\eta_0, \eta_1, \rho \in \omega^{<\omega}$, if $|\eta_0| = |\eta_1| = |\rho| \geq 1$, then there exists $n < \omega$ such that $\Phi_n(\eta_0, \eta_1) = \rho$.*
- (ii) *If $\Phi_n(\eta_0, \eta_1) = \rho$ and $|\eta_0| = |\eta_1| = |\rho| \geq 1$, then $\Phi_n(\eta_0{}^\frown 0, \eta_1{}^\frown 0) = \rho^\frown 0$.*
- (iii) *Assume the following.*
  - *(a) $i_\star, m, K, N < \omega$ and $N \geq 1$.*
  - *(b) $h : [N+1]^2 \to \omega$.*
  - *(c) $\bar{l} = \langle l_{j,k} : j < k \leq N\rangle$ where each $l_{j,k} \in \omega$.*
  - *(d) $\bar{\eta} = \langle \eta_j : j \leq N\rangle$ and $\bar{\rho} = \langle \rho_{j,k} : j < k \leq N\rangle$ are sequences in ${}^m\omega$.*
  - *(e) For every $j < k \leq N$, $\Phi_{h(j,k)}(\eta_j, \eta_k) = \rho_{j,k}$.*

  Then there exists an injective sequence $\bar{i} = \langle i_j : j \leq N\rangle$ such that
  - *(1) $i_0 = i_\star$,*
  - *(2) for every $1 \leq j \leq N$, $i_j > K$ and*
  - *(3) for every $j < k \leq N$, $\Phi_{h(j,k)}(\eta_j{}^\frown i_j, \eta_k{}^\frown i_k) = \rho_{j,k}{}^\frown l_{j,k}$.*

3. Forcing

From now on, we fix μ, λ, $\bar{F}$, $\mathcal{J}_{\lambda,\mu}$ and $\bar{A}$ as in Definition 2.4 and $\langle \Phi_n : n < \omega\rangle$ as in Definition 2.9.

Definition 3.1. *Define the forcing poset $\mathbb{Q}$ as follows. A condition $p \in \mathbb{Q}$ is a tuple $p = (m_p, u_p, v_p, \bar{\eta}_p, \bar{\rho}_p, \bar{n}_p)$ satisfying the following.*

- (i) *$m_p < \omega$.*
- (ii) *$u_p \subseteq \lambda \times \mu$ is finite and $\bar{F}$-closed.*
- (iii) *$v_p \subseteq 2^\lambda$ is finite.*
- (iv) *$\bar{\eta}_p = \langle \eta_{p,\xi,i} : (\xi, i) \in u_p\rangle$ where each $\eta_{p,\xi,i} \in {}^{m_p}\omega$.*
- (v) *$\bar{\rho}_p = \langle \rho_{p,\alpha} : \alpha \in v_p\rangle$ where each $\rho_{p,\alpha} \in {}^{m_p}\omega$.*
- (vi) *$\bar{n}_p = \langle n_{p,\xi,i,j} : (\xi, i) \in u_p \wedge (\xi, j) \in u_p \wedge i < j\rangle$ where each $n_{p,\xi,i,j} < \omega$.*

(vii) *If* $(\xi, i) \in u_p$, $(\xi, j) \in u_p$ *and* $i < j$, *then* $\Phi_n(\eta_{p,\xi,i}, \eta_{p,\xi,j}) = \eta_{p,\zeta,k}$ *where* $n = n_{p,\xi,i,j}$ *and* $(\zeta, k) = F_\xi(i, j)$.

For $p, q \in \mathbb{Q}$, *define* $p \leq q$ *iff the following hold.*

(i) $m_q \leq m_p$.
(ii) $u_q \subseteq u_p$.
(iii) $v_q \subseteq v_p$.
(iv) *For each* $(\xi, i) \in u_q$, $\eta_{q,\xi,i} \preceq \eta_{p,\xi,i}$.
(v) *For each* $\alpha \in v_q$, $\rho_{q,\alpha} \preceq \rho_{p,\alpha}$.
(vi) $\bar{n}_q \subseteq \bar{n}_p$.
(vii) *If* $\alpha \in v_q$ *and* $(\xi, i) \in u_q \cap A_\alpha$, *then for every* $n \in [m_q, m_p)$, $\rho_{p,\alpha}(n) \neq \eta_{p,\xi,i}(n)$.

Recall that, through $\mathbb{Q}$, we are trying to add a set of reals $\{x_{\xi,i} : (\xi, i) \in \lambda \times \mu\}$ satisfying Clauses (A)-(C) in Remark 2.7. To ensure Clause (B), we would like to add another sequence of reals $\langle y_\alpha : \alpha < 2^\lambda \rangle$ such that each y_α is eventually different from every real in $\{x_{\xi,i} : (\xi, i) \in A_\alpha\}$ (see Fact 2.2). So we can think of a condition $p \in \mathbb{Q}$ to be promising the following.

(1) For each $(\xi, i) \in u_p$, $\eta_{p,\xi,i} \preceq x_{\xi,i}$.
(2) For every $\alpha \in v_p$, $\rho_\alpha \preceq y_\alpha$ and if $(\xi, i) \in u_p \cap A_\alpha$, then for every $n \geq m_p$, $x_{\xi,i}(n) \neq y_\alpha(n)$.
(3) Whenever $\{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq u_p$, $F_\xi(i, j) = (\zeta, k)$ and $n = n_{p,\xi,i,j}$, we have $\Phi_n(x_{\xi,i}, x_{\xi,j}) = x_{\zeta,k}$.

Note that in the definition of $\mathbb{Q}$, the requirement that u_p be $\bar{F}$-closed is to ensure that the choices of $\bar{\eta}_p$ and $\bar{n}_p$ are consistent with Definition 3.1 Clause (vii).

Lemma 3.2. *The following sets are dense in* $\mathbb{Q}$.

(1) $\{p \in \mathbb{Q} : m_p \geq m\}$ *for each* $m < \omega$.
(2) $\{p \in \mathbb{Q} : (\xi, i) \in u_p\}$ *for each* $(\xi, i) \in \lambda \times \mu$.
(3) $\{p \in \mathbb{Q} : \alpha \in v_p\}$ *for each* $\alpha < 2^\lambda$.

Proof. (1) Fix $m < \omega$ and $q \in \mathbb{Q}$. We'll construct $p \in \mathbb{Q}$ such that $p \leq q$ and $m_p \geq m$. We can assume that $m_q < m$. Define p as follows: $m_p = m$, $u_q = u_p$, $v_q = v_p$, $\bar{n}_p = \bar{n}_q$, $\bar{\eta}_p = \langle \eta_{p,\xi,i} : (\xi, i) \in u_p \rangle$ where each $\eta_{p,\xi,i} = \eta_{q,\xi,i}{}^\frown 0^{m-m_q}$, $\bar{\rho}_p = \langle \rho_{p,\alpha} : \alpha \in v_p \rangle$ where each $\rho_{p,\alpha} = \rho_{q,\alpha}{}^\frown 1^{m-m_q}$. To see that $p \in \mathbb{Q}$, we invoke Definition 2.9 Clause (ii). It is clear that $p \leq q$.

(2) Suppose $(\xi, i) \in \lambda \times \mu$ and $q \in \mathbb{Q}$. We can assume that $(\xi, i) \notin u_q$. Let $u_\star \subseteq \lambda \times \mu$ be the $\bar{F}$-closure of $u_q \cup \{(\xi, i)\}$. Define $p \in \mathbb{Q}$ as follows: $m_p = m_q$, $u_p = u_\star$, $v_p = v_q$, $\bar{\eta}_p = \langle \eta_{p,\xi,i} : (\xi, i) \in u_\star \rangle$ where $\eta_{p,\xi,i} = \eta_{q,\xi,i}$ if $(\xi, i) \in u_q$ and $\eta_{p,\xi,i} = 0^{m_q}$ if $(\xi, i) \in u_\star \setminus u_q$. Define $\bar{n}_p = \langle n_{p,\xi,i,j} : (\xi, i) \in u_\star \wedge (\xi, j) \in u_\star \wedge i < j \rangle$ where $n_{p,\xi,i,j} = n_{q,\xi,i,j}$ if both (ξ, i) and (ξ, j) are in u_q; otherwise using Definition 2.9 Clause (i), choose $n_{p,\xi,i,j} = n < \omega$ such that $\Phi_n(\eta_{p,\xi,i}, \eta_{p,\xi,j}) = \eta_{p,\zeta,k}$ where $(\zeta, k) = F_\xi(i, j)$. It is easy to see that $p \in \mathbb{Q}$, $p \leq q$ and $(\xi, i) \in u_\star = u_p$.

(3) Fix $\alpha < 2^\lambda$, $q \in \mathbb{Q}$ and assume $\alpha \notin v_q$. Choose $p \in \mathbb{Q}$ such that $m_p = m_q$, $u_p = u_q$, $v_p = v_q \cup \{\alpha\}$, $\bar{n}_p = \bar{n}_q$, $\bar{\eta}_p = \bar{\eta}_q$ and $\bar{\rho}_p = \langle \rho_{p,\beta} : \beta \in v_p \rangle$ where $\rho_{p,\beta} = \rho_{q,\beta}$ if $\beta \in v_q$ and $\rho_{p,\alpha} : m_q \to \omega$ is arbitrary. Then $p \leq q$ and $\alpha \in v_p$. $\square$

The following lemma will be used to show that $\mathbb{Q}$ satisfies ccc (Lemma 3.4).

Lemma 3.3. *Suppose $p, q \in \mathbb{Q}$. Assume the following.*

(1) $m_p = m_q = m$.
(2) *For every* $(\xi, i) \in u_p \cap u_q$, $\eta_{p,\xi,i} = \eta_{q,\xi,i}$.
(3) *For every* $\alpha \in v_p \cap v_q$, $\rho_{p,\alpha} = \rho_{q,\alpha}$.
(4) *If* $(\xi, i), (\xi, j)$ *are in* $u_p \cap u_q$ *and* $i < j$, *then* $n_{p,\xi,i,j} = n_{q,\xi,i,j}$.

Then there exists $r \in \mathbb{Q}$ *such that* $r \leq p$ *and* $r \leq q$.

Proof. Let $u_\star$ be the $\bar{F}$-closure of $u_p \cup u_q$. Define $r \in \mathbb{Q}$ as follows.

(i) $m_r = m$, $u_r = u_\star$, $v_r = v_p \cup v_q$.
(ii) $\bar{\rho}_r = \bar{\rho}_p \cup \bar{\rho}_q$.
(iii) Put $w = u_\star \setminus (u_p \cup u_q)$ and for each $(\xi, i) \in w$, define $\eta_{r,\xi,i} = 0^m$. Define $\bar{\eta}_r = \bar{\eta}_p \cup \bar{\eta}_q \cup \langle \eta_{r,\xi,i} : (\xi, i) \in w \rangle$.
(iv) Finally, define $\bar{n}_r$ as follows. Suppose $(\xi, i) \in u_r$, $(\xi, j) \in u_r$ and $i < j$. If (ξ, i) and (ξ, j) are both in u_p, then $n_{r,\xi,i,j} = n_{p,\xi,i,j}$ and if (ξ, i) and (ξ, j) are both in u_q, then $n_{r,\xi,i,j} = n_{q,\xi,i,j}$. Otherwise, using Definition 2.9 Clause (i), choose $n_{r,\xi,i,j} = n$ such that $\Phi_n(\eta_{r,\xi,i}, \eta_{r,\xi,j}) = \eta_{r,\zeta,k}$ where $(\zeta, k) = F_\xi(i, j)$.

It should be clear that $r \in \mathbb{Q}$ and it is a common extension of p, q. □

Lemma 3.4. $\mathbb{Q}$ *satisfies ccc. Hence forcing with* $\mathbb{Q}$ *preserves all cofinalities.*

Proof. Let $A \subseteq \mathbb{Q}$ be uncountable. Using the Δ-system lemma we can find $\{p_\gamma : \gamma < \omega_1\} \subseteq A$ such that the following hold.

(1) $m_{p_\gamma} = m_\star$ does not depend on $\gamma < \omega_1$.
(2) $\langle u_{p_\gamma} : \gamma < \omega_1 \rangle$ forms a Δ-system with root $u_\star$.
(3) $\langle v_{p_\gamma} : \gamma < \omega_1 \rangle$ forms a Δ-system with root $v_\star$.
(4) For every $(\xi, i) \in u_\star$, $\eta_{p_\gamma,\xi,i} = \eta_{\xi,i}$ does not depend on $\gamma < \omega_1$.
(5) For every $\alpha \in v_\star$, $\rho_{p_\gamma,\alpha} = \rho_\alpha$ does not depend on $\gamma < \omega_1$.
(6) If $(\xi, i), (\xi, j)$ are both in $u_\star$ and $i < j$, then $n_{p_\gamma,\xi,i,j} = n_{\xi,i,j}$ does not depend on $\gamma < \omega_1$.

Now Lemma 3.3 implies that any two conditions in $\{p_\gamma : \gamma < \omega_1\}$ have a common extension in $\mathbb{Q}$. It follows that $\mathbb{Q}$ is ccc. □

A similar argument shows that $\mathbb{Q}$ has ω_1 as a precaliber. We leave the details for the reader as we won't be needing this fact.

4. The model

Let G be a $\mathbb{Q}$-generic filter over V. By Lemma 3.4, all cofinalities and therefore cardinals are preserved in $V[G]$. Next, by Lemma 3.2 Clauses (2)+(3), we must have $\bigcup\{u_p : p \in G\} = \lambda \times \mu$ and $\bigcup\{v_p : p \in G\} = 2^\lambda$. In $V[G]$, define the following.

(a) For each $(\xi, i) \in \lambda \times \mu$, $x_{\xi,i} = \bigcup\{\eta_{p,\xi,i} : p \in G \wedge (\xi, i) \in u_p\}$.
(b) For each $\alpha < 2^\lambda$, define $y_\alpha = \bigcup\{\rho_{p,\alpha} : p \in G \wedge \alpha \in v_p\}$.
(c) For each $\{(\xi, i), (\xi, j)\} \subseteq \lambda \times \mu$ with $i < j$, $n_{\xi,i,j} = n_{p,\xi,i,j}$ where $p \in G$ and $\{(\xi, i), (\xi, j)\} \subseteq u_p$.

By Lemma 3.2 Clause (1), the empty condition forces that both $\mathring{x}_{\xi,i}$ and $\mathring{y}_\alpha$ are in ω^ω. Next, we would like to show that the meager subsets of $\{x_{\xi,i} : (\xi, i) \in \lambda \times \mu\}$ are indexed by members of $\mathcal{J}_{\lambda,\mu}$ (Lemma 4.3 below). The following lemma will be used in its proof.

Lemma 4.1. *Suppose* $p, q \in \mathbb{P}$, $(\xi_\star, i_\star) \in \lambda \times \mu$ *and* $\eta : m_q \to \omega$ *satisfy the following.*

(a) $m_p \leq m_q$.
(b) *Let* $u_p \cap u_q = u_\star$. *For every* $(\xi, i) \in u_\star$, $\eta_{p,\xi,i} \preceq \eta_{q,\xi,i}$.
(c) *Let* $v_p \cap v_q = v_\star$. *For every* $\alpha \in v_\star$, $\rho_{p,\alpha} \preceq \rho_{q,\alpha}$.
(d) *If* $(\xi, i) \in u_p \setminus u_\star$ *and* $(\xi', j) \in u_q$, *then* $\xi' < \xi$.
(e) *If* $\alpha \in v_\star$, $(\xi, i) \in u_\star \cap A_\alpha$ *and* $n \in [m_p, m_q)$, *then* $\eta_{q,\xi,i}(n) \neq \rho_{q,\alpha}(n)$.
(f) *If* $(\xi, i), (\xi, j) \in u_\star$ *and* $i < j$, *then* $n_{p,\xi,i,j} = n_{q,\xi,i,j}$.
(g) $(\xi_\star, i_\star) \in u_p \setminus u_\star$ *and* $\eta_{p,\xi_\star,i_\star} \preceq \eta$.
(h) *If* $\alpha \in v_\star$, *then* $(\xi_\star, i_\star) \notin A_\alpha$.

Then there exists $r \in \mathbb{Q}$ *such that* $r \leq p$, $r \leq q$ *and* $\eta_{r,\xi_\star,i_\star} = \eta$.

Proof. Let $B = \bigcup\{\mathsf{range}(\rho_{q,\alpha}) : \alpha \in v_q\}$ and $K = \max(B) + 1$. Define $r \in \mathbb{Q}$ as follows.

(1) $m_r = m_q$, $u_r = u_p \cup u_q$ and $v_r = v_p \cup v_q$. Note that Clause (d) implies that u_r is $\bar{F}$-closed.
(2) For each $(\xi, i) \in u_q$, define $\eta_{r,\xi,i} = \eta_{q,\xi,i}$.
(3) Choosing $\langle \eta_{r,\xi,i} : (\xi, i) \in u_p \setminus u_\star \rangle$: The main constraint here is that we have to satisfy Definition 3.1 (vii). This is where we use the fact that "Φ_n's are highly surjective". Let $\langle \xi_n : n \leq n_\star \rangle$ be an increasing enumeration of $\{\xi : (\exists i)[(\xi, i) \in (u_p \setminus u_\star)]\}$. For each $n \leq n_\star$, let $I_n = \{i : (\xi_n, i) \in u_p\}$ be the nth "column" of $u_p \setminus u_\star$. We will extend $\eta_{p,\xi,i}$'s columnwise. More precisely, we'll define $\langle \eta_{r,\xi_n,i} : i \in I_n \rangle$ by induction on $n \leq n_\star$. Assume that $\langle \eta_{r,\xi_m,i} : i \in I_m \rangle$ has been defined for every $m < n$ and we would like to define $\langle \eta_{r,\xi_n,i} : i \in I_n \rangle$. We have the following cases.

(i) If $|I_n| = 1$ and $\xi_n = \xi_\star$ (so $I_n = \{i_\star\}$), then define $\eta_{r,\xi_n,i_\star} = \eta$.

(ii) If $|I_n| = 1$ and $\xi_n \neq \xi_\star$, then define $\eta_{r,\xi_n,i} \in {}^{m_q}\omega$ (where $I_n = \{i\}$) by $\eta_{p,\xi_n,i} \preceq \eta_{r,\xi_n,i}$ and for every $m_p \leq m < m_q$, $\eta_{r,\xi_n,i}(m) = K$.

(iii) Now assume $|I_n| = N + 1 \geq 2$ (so $N \geq 1$). By induction on $m_p \leq l < m_q$, define $\langle \eta_{r,\xi_n,i} \restriction l : i \in I_n \rangle$ as follows. Start by defining $\eta_{r,\xi_n,i} \restriction m_p = \eta_{p,\xi_n,i}$ for each $i \in I_n$. Assume that $\langle \eta_{r,\xi_n,i} \restriction l : i \in I_n \rangle$ has been defined for some $l \in [m_p, m_q)$ such that for every $i < j$ in I_n,

$$\Phi_{n_{p,\xi_n,i,j}}(\eta_{r,\xi_n,i} \restriction l, \eta_{r,\xi_n,j} \restriction l) = \eta_{r,\xi',i'} \restriction l \tag{$\dagger$}$$

where $(\xi', i') = F_{\xi_n}(i, j)$ (so $\xi' < \xi_n$). Note that this condition is satisfied at $l = m_p$ because p is a condition and therefore satisfies Definition 3.1 (vii).

We will now choose $\langle \eta_{r,\xi_n,i}(l) : i \in I_n \rangle$ while ensuring that the above condition ($\dagger$) continues to hold at $l + 1$. This will be done using Definition 2.9 Clause (iii) as follows.

Let $I_n = \{i_j : j \leq N\}$ be an injective enumeration of I_n where $i_0 = i_\star$ if $\xi_n = \xi_\star$ (and hence $i_\star \in I_n$). Recall that for every $\alpha \in v_\star$, $K > \max(\mathsf{range}(\rho_{q,\alpha}))$. Let $h : [N + 1]^2 \to \omega$ be defined by $h(j, k) = n_{p,\xi_n,i_j,i_k}$. Define $\bar{\rho} = \langle \rho_{j,k} : j < k \leq N \rangle$ and $\bar{l} = \langle l_{j,k} : j < k \leq N \rangle$ by $\rho_{j,k} = \eta_{r,\xi',i'} \restriction l$ and $l_{j,k} = \eta_{r,\xi',i'}(l)$ where $(\xi', i') = F_{\xi_n}(i_j, i_k)$.

Now observe that, by Definition 2.9 Clause (iii), we can choose an injective sequence $\langle m_j : j \leq N \rangle$ such that

(α) $\xi_n = \xi_\star \implies m_0 = \eta(l)$,
(β) $\xi_n \neq \xi_\star \implies m_0 > K$,
(γ) for every $1 \leq j \leq N$, $m_j > K$ and
(δ) for every $j < k \leq N$,

$$\Phi_{h(j,k)}((\eta_{r,\xi_n,i_j} \restriction l)^\frown m_j, (\eta_{r,\xi_n,i_k} \restriction l)^\frown m_k) = \rho_{j,k}{}^\frown l_{j,k}.$$

Define $\eta_{r,\xi_n,i_j}(l) = m_j$ for each $j \leq N$. It is clear that (†) continues to hold at $l+1$. Together with (2), this concludes the description of $\bar{\eta}_r = \langle \eta_{r,\xi,i} : (\xi,i) \in u_r \rangle$.

(4) Choosing $\langle \rho_\alpha : \alpha \in v_r \rangle$: First define $\rho_{r,\alpha} = \rho_{q,\alpha}$ for $\alpha \in v_q$. Next, choose $S \in [\omega]^{<\omega}$ such that S contains the ranges of $\eta_{r,\xi,i}$ for all $(\xi,i) \in u_r$. Now choose $\langle \rho_{r,\alpha} : \alpha \in v_p \setminus v_\star \rangle$ such that $\rho_{p,\alpha} \preceq \rho_{r,\alpha}$ and $\rho_{r,\alpha}(l) \in \omega \setminus S$ for every $l \in [m_p, m_q)$ and $\alpha \in v_p \setminus v_\star$.

(5) Choosing $n_{r,\xi,i,j}$'s: Suppose $(\xi,i),(\xi,j)$ are in u_r and $i<j$. Note that Clause (d) implies that (ξ,i) and (ξ,j) are either both in u_q or both in u_p. Define $n_{r,\xi,i,j} = n_{q,\xi,i,j}$ in the former case and $n_{r,\xi,i,j} = n_{p,\xi,i,j}$ in the latter. There is no ambiguity by Clause (f).

Let us check that Definition 3.1 Clause (vii) holds for r. Suppose (ξ,i) and (ξ,j) are both in u_r and $i<j$. Put $(\zeta,k) = F_\xi(i,j)$ and $n = n_{r,\xi,i,j}$. We must show that $\Phi_n(\eta_{r,\xi,i}, \eta_{r,\xi,j}) = \eta_{r,\zeta,k}$. If $\{(\xi,i),(\xi,j)\} \subseteq u_q$, then this is clear as q satisfies Definition 3.1 Clause (vii). By Clause (d), the only other possibility is that $\{(\xi,i),(\xi,j)\} \subseteq u_p \setminus u_\star$. In this case, the construction of $\eta_{r,\xi,i}$ and $\eta_{r,\xi,j}$ in Clause (3)(iii) guarantees this.

It follows that $r \in \mathbb{Q}$. It should also be clear that $r \leq q$ and $\eta_{r,\xi_\star,i_\star} = \eta$. Finally, to see that $r \leq p$, it suffices to check the following: If $\alpha \in v_p$, $(\xi,i) \in u_p \cap A_\alpha$ and $n \in [m_p, m_r)$, then $\eta_{r,\xi,i}(n) \neq \rho_{r,\alpha}(n)$. Fix $\alpha \in v_p$, $(\xi,i) \in u_p \cap A_\alpha$ and $n \in [m_p, m_r)$. We have the following cases.

Case 1. $\alpha \in v_\star$. By Clause (h), $(\xi,i) \neq (\xi_\star, i_\star)$. Next, if $(\xi,i) \in u_\star$, then Clause (e) ensures that $\eta_{r,\xi,i}(n) \neq \rho_{r,\alpha}(n)$. Finally, if $(\xi,i) \in u_p \setminus (u_\star \cup \{(\xi_\star,i_\star)\})$, then the choice of $\eta_{r,\xi,i}(n)$ in Clauses 3(ii)-(iii) ensures that $\eta_{r,\xi,i}(n) \geq K > \rho_{r,\alpha}(n)$.

Case 2. $\alpha \in v_p \setminus v_\star$. In this case, the choice of $\rho_{r,\alpha}$ in Clause (4) guarantees that $\eta_{r,\xi,i}(n) \neq \rho_{r,\alpha}(n)$. □

Remark 4.2. *With the possible exception of Clause (d), it should be clear that all the hypotheses of Lemma 4.1 are necessary. To see why Clause (d) cannot be dropped, consider the following situation.*

(i) $\zeta_3 < \zeta_2 < \zeta_1 < \xi < \lambda$.
(ii) $i_1 < i_2 < i_3 < \mu$ *and* $j < \mu$.
(iii) $F_\xi(i_1,i_2) = (\zeta_1,j)$, $F_\xi(i_1,i_3) = (\zeta_2,j)$ *and* $F_\xi(i_2,i_3) = (\zeta_3,j)$.
(iv) $\{(\xi,i_1),(\xi,i_2),(\zeta_1,j),(\zeta_2,j),(\zeta_3,j)\} \subseteq u_q$.
(v) $\{(\xi,i_1),(\xi,i_2),(\xi,i_3),(\zeta_1,j),(\zeta_2,j),(\zeta_3,j)\} \subseteq u_p$.
(vi) $n_{p,\xi,i_1,i_3} = n_{p,\xi,i_2,i_3}$.
(vii) $m_q = m_p + 1$, $\eta_{q,\xi,i_1} = \eta_{q,\xi,i_2}$ *and* $\eta_{q,\zeta_2,j} \neq \eta_{q,\zeta_3,j}$.
(viii) $v_q = v_p = \emptyset$.

Now use Clauses (vi)+(vii) to conclude that p,q must be incompatible. Note that the $(\xi_\star,i_\star)$ and η in Clauses (g) and (h) do not play a role here.

We are now ready to show the following.

Lemma 4.3. *For every $\mathring{A} \in \mathcal{P}(\lambda \times \mu) \cap V^{\mathbb{P}}$,*

$$V^{\mathbb{P}} \models \{\mathring{x}_{\xi,i} : (\xi,i) \in \mathring{A}\} \textit{ is meager iff } \mathring{A} \in \mathcal{J}_{\lambda,\mu}.$$

Proof. First suppose that $p \in \mathbb{Q}$, $\mathring{A} \in V^{\mathbb{P}} \cap \mathcal{P}(\lambda \times \mu)$ and $p \Vdash \mathring{A} \in \mathcal{J}_{\lambda,\mu}$. We'll show that $p \Vdash \{\mathring{x}_{\xi,i} : (\xi,i) \in \mathring{A}\}$ is meager. Recalling the definition of $\mathcal{J}_{\lambda,\mu}$, we have

$$p \Vdash |\{\xi < \lambda : |\{i < \mu : (\xi,i) \in \mathring{A}\}| = \mu\}| < \lambda.$$

Put $\mathring{W} = \{\xi < \lambda : |\{i < \mu : (\xi, i) \in \mathring{A}\}| = \mu\}$. Since λ is regular, $p \Vdash \sup(\mathring{W}) < \lambda$. Choose $\{(q_n, \xi_n) : n < \omega\}$ such that $(\forall n)(q_n \Vdash \sup(\mathring{W}) = \xi_n)$ and $\{q_n : n < \omega\}$ is a maximal antichain below p. Put $\xi_\star = \sup(\{\xi_n : n < \omega\})$. Then $\xi_\star < \lambda$ and $p \Vdash \sup(\mathring{W}) \leq \xi_\star$.

As μ is regular, for each $\xi_\star < \xi < \lambda$, $p \Vdash \sup(\{i < \mu : (\xi, i) \in \mathring{A}\}) < \mu$. So by repeating the previous argument, we can find $j(\xi) < \mu$ such that $p \Vdash (\forall i < \mu)((\xi, i) \in \mathring{A} \implies i < j(\xi))$. Define $B = \{(\xi, i) \in \lambda \times \mu : (\xi \leq \xi_\star) \text{ or } (\xi > \xi_\star \wedge i < j(\xi))\}$ and note that $p \Vdash \mathring{A} \subseteq B$. Since $B \in V \cap \mathcal{J}_{\lambda,\mu}$, we can choose $\alpha < 2^\lambda$ such that $B = A_\alpha$. Now observe that for every $(\xi, i) \in A_\alpha$, $p \Vdash (\forall^\infty n)(\mathring{x}_{\xi,i}(n) \neq \mathring{y}_\alpha(n))$ and hence $p \Vdash \{\mathring{x}_{\xi,i} : (\xi, i) \in \mathring{A}\}$ is meager (Fact 2.2).

For the other direction, towards a contradiction, assume that for some $p' \in \mathbb{Q}$ and $\mathring{S} \in V^{\mathbb{P}} \cap \mathcal{P}(\lambda \times \mu)$,

$$p' \Vdash \mathring{S} \in \mathcal{J}^+_{\lambda,\mu} \text{ and } \{\mathring{x}_{\xi,i} : (\xi, i) \in \mathring{S}\} \text{ is meager.}$$

Choose a $\mathbb{P}$-generic filter G over V with $p' \in G$. Working in $V[G]$, choose a sequence $\langle C_n : n < \omega\rangle$ such that each C_n is a closed nowhere dense subset of ω^ω and $\{x_{\xi,i} : (\xi, i) \in S\} \subseteq \bigcup\{C_n : n < \omega\}$. Put $S_n = \{(\xi, i) : x_{\xi,i} \in C_n\}$. Since $\mathcal{J}_{\lambda,\mu}$ is a σ-ideal, $S \in \mathcal{J}^+_{\lambda,\mu}$ and $S \subseteq \bigcup\{S_n : n < \omega\}$, we can fix some $n_\star < \omega$ such that $S_{n_\star} \in \mathcal{J}^+_{\lambda,\mu}$. Define $T = \{y \restriction n : (y \in C_{n_\star}) \wedge (n < \omega)\}$ and $A = S_{n_\star}$. It follows that $V[G] \models T \subseteq \omega^{<\omega}$ is a nowhere dense subtree, $A \in \mathcal{J}^+_{\lambda,\mu}$ and $(\forall (\xi, i) \in A)(x_{\xi,i} \in [T])$. Therefore we can choose a condition $p \in G$ that forces

$$\mathring{A} \in \mathcal{J}^+_{\lambda,\mu} \text{ and } \mathring{T} \subseteq \omega^{<\omega} \text{ is a nowhere dense subtree and } (\forall (\xi, i) \in \mathring{A})(\mathring{x}_{\xi,i} \in [\mathring{T}]).$$

Define $W = \{(\xi, i) \in \lambda \times \mu : (\exists q \leq p)(q \Vdash (\xi, i) \in \mathring{A})\}$. Since $p \Vdash \mathring{A} \subseteq W$, we get $W \in \mathcal{J}^+_{\lambda,\mu}$. Define $B = \{\xi < \lambda : |\{i < \mu : (\xi, i) \in W\}| = \mu\}$. Clearly, $|B| = \lambda$. For each $\xi \in B$, let $C_\xi = \{i < \mu : (\xi, i) \in W\}$. So $|C_\xi| = \mu$. For each $\xi \in B$ and $i \in C_\xi$, fix $p_{\xi,i} \leq p$ such that $p_{\xi,i} \Vdash (\xi, i) \in \mathring{A}$ **and** $(\xi, i) \in u_{p_{\xi,i}}$.

Since μ is regular uncountable, for each $\xi \in B$, using the Δ-system lemma, we can choose $D_\xi \in [C_\xi]^\mu$ such that the following hold.

(i) $m_{p_{\xi,i}} = m_\xi$ does not depend on $i \in D_\xi$.
(ii) $\langle u_{p_{\xi,i}} : i \in D_\xi\rangle$ forms a Δ-system with root u_ξ. Note that u_ξ is $\bar{F}$-closed.
(iii) $\langle v_{p_{\xi,i}} : i \in D_\xi\rangle$ forms a Δ-system with root v_ξ.
(iv) For every $(\zeta, j) \in u_\xi$, $\eta_{p_{\xi,i},\zeta,j} = \eta_{\xi,\zeta,j}$ does not depend on $i \in D_\xi$.
(v) For every $\alpha \in v_\xi$, $\rho_{p_{\xi,i},\alpha} = \rho_{\xi,\alpha}$ does not depend on $i \in D_\xi$.
(vi) If $(\zeta, j), (\zeta, k)$ are both in u_ξ and $j < k$, then $n_{p_{\xi,i},\zeta,j,k} = n_{\xi,\zeta,j,k}$ does not depend on $i \in D_\xi$.
(vii) $\eta_{p_{\xi,i},\xi,i} = \eta_\xi$ does not depend on $i \in D_\xi$.

For each $\xi \in B$, define $p_\xi \in \mathbb{Q}$ as follows.

(a) $m_{p_\xi} = m_\xi$, $u_{p_\xi} = u_\xi$ and $v_{p_\xi} = v_\xi$.
(b) For every $(\zeta, j) \in u_\xi$, $\eta_{p_\xi,\zeta,j} = \eta_{\xi,\zeta,j}$.
(c) For every $\alpha \in v_\xi$, $\rho_{p_\xi,\alpha} = \rho_{\xi,\alpha}$.
(d) If (ζ, j) and (ζ, k) are both in u_ξ and $j < k$, then $n_{p_\xi,\zeta,j,k} = n_{\xi,\zeta,j,k}$.

It is easy to check that $p_\xi \leq p$ and for every $i \in D_\xi$, $p_{\xi,i} \leq p_\xi$. Next, choose $B_\star \in [B]^\lambda$ such that the following hold.

(i) $m_\xi = m_\star$ does not depend on $\xi \in B_\star$.
(ii) $\langle u_{p_\xi} : \xi \in B_\star\rangle$ forms a Δ-system with root $u_\star$. Note that $u_\star$ is $\bar{F}$-closed.

(iii) $\langle v_{p_\xi} : \xi \in B_\star\rangle$ forms a Δ-system with root $v_\star$.
(iv) For every $(\zeta, j) \in u_\star$, $\eta_{p_\xi,\zeta,j} = \eta_{\zeta,j}$ does not depend on $\xi \in B_\star$.
(v) For every $\alpha \in v_\star$, $\rho_{p_\xi,\alpha} = \rho_\alpha$ does not depend on $\xi \in B_\star$.
(vi) If $(\zeta, j), (\zeta, k)$ are both in $u_\star$ and $j < k$, then $n_{p_\xi,\zeta,j,k} = n_{\zeta,j,k}$ does not depend on $\xi \in B_\star$.
(vii) $\eta_\xi = \eta_\star$ does not depend on $\xi \in B_\star$.

Define $p_\star \in \mathbb{Q}$ as follows.

(a) $m_{p_\star} = m_\star$, $u_{p_\star} = u_\star$ and $v_{p_\star} = v_\star$.
(b) For every $(\zeta, j) \in u_\star$, $\eta_{p_\star,\zeta,j} = \eta_{\zeta,j}$.
(c) For every $\alpha \in v_\star$, $\rho_{p_\star,\alpha} = \rho_\alpha$.
(d) If (ζ, j) and (ζ, k) are both in $u_\star$ and $j < k$, then $n_{p_\star,\zeta,j,k} = n_{\zeta,j,k}$.

It is clear that for every $\xi \in B_\star$ and $i \in D_\xi$, $p_{\xi,i} \le p_\xi \le p_\star \le p$. Since $p_\star \le p$, it follows that $p_\star \Vdash \mathring{T} \subseteq \omega^{<\omega}$ is a nowhere dense subtree. So we can choose $\langle (q_n, \eta_n) : n < \omega\rangle$ such that the following hold.

(i) $\{q_n : n < \omega\}$ is a maximal antichain below $p_\star$.
(ii) For every $n < \omega$, $\eta_n \in \omega^{<\omega}$ and $\eta_\star \preceq \eta_n$.
(iii) For every $n < \omega$, $q_n \Vdash [\eta_n] \cap [\mathring{T}] = \emptyset$.

Put $U_\star = \bigcup\{u_{q_n} : n < \omega\}$ and $V_\star = \bigcup\{v_{q_n} : n < \omega\}$ and note that $U_\star \in [\lambda \times \mu]^{<\omega_1}$ and $V_\star \in [2^\lambda]^{<\omega_1}$. Define $A_\star = \bigcup\{A_\alpha : \alpha \in V_\star\}$. Since $\mu \ge \omega_1$ and $\mathcal{J}_{\lambda,\mu}$ is a μ-complete ideal, we must have $A_\star \in \mathcal{J}_{\lambda,\mu}$. So we can choose $\xi_1 < \lambda$ such that for every $\xi_1 \le \xi < \lambda$, $|\{i < \mu : (\xi, i) \in A_\star\}| < \mu$. Since $\langle u_\xi \setminus u_\star : \xi \in B_\star\rangle$ and $\langle v_\xi \setminus v_\star : \xi \in B_\star\rangle$ consist of pairwise disjoint sets and $|B_\star| = \lambda$ is regular uncountable, we can fix $\xi_\star \in B_\star$ such that

(a) $\xi_\star > \xi_1$,
(b) $(\xi, i) \in U_\star \implies \xi_\star > \xi$,
(c) $(u_{\xi_\star} \setminus u_\star) \cap U_\star = \emptyset$ and
(d) $(v_{\xi_\star} \setminus v_\star) \cap V_\star = \emptyset$.

As $|\{i < \mu : (\xi_\star, i) \in A_\star\}| < \mu$, we can choose $i_1 < \mu$ such that for every $i_1 \le i < \mu$, $(\xi_\star, i) \notin A_\star$. Since the sequences $\langle u_{p_{\xi_\star,i}} \setminus u_{\xi_\star} : i \in D_{\xi_\star}\rangle$ and $\langle v_{p_{\xi_\star,i}} \setminus v_{\xi_\star} : i \in D_{\xi_\star}\rangle$ consist of pairwise disjoint sets and $|D_{\xi_\star}| = \mu$ is regular uncountable, we can fix $i_\star \in D_{\xi_\star}$ such that

(a) $i_\star > i_1$,
(b) $(\forall \alpha \in V_\star)((\xi_\star, i_\star) \notin A_\alpha)$,
(c) $(u_{p_{\xi_\star,i_\star}} \setminus u_{\xi_\star}) \cap U_\star = \emptyset$ and
(d) $(v_{p_{\xi_\star,i_\star}} \setminus v_{\xi_\star}) \cap V_\star = \emptyset$.

Note that clause (b) follows from clause (a) and the choice of i_1. Furthermore, $U_\star \cap u_{p_{\xi_\star,i_\star}} = u_\star$ and $V_\star \cap v_{p_{\xi_\star,i_\star}} = v_\star$. Since $\{q_n : n < \omega\}$ is dense below $p_\star$ and $p_{\xi_\star,i_\star} \le p_\star$, one of the q_n's is compatible with $p_{\xi_\star,i_\star}$. By reindexing q_n's we can assume that q_0 and $p_{\xi_\star,i_\star}$ are compatible. Now we come to our key claim.

Claim 4.4. *There exists $r \in \mathbb{Q}$ such that $r \le q_0$, $r \le p_{\xi_\star,i_\star}$* ***and*** *$\eta_0 \preceq \eta_{r,\xi_\star,i_\star}$.*

Let us first see why this is enough to get a contradiction. Fix a common extension r of q_0 and $p_{\xi_\star,i_\star}$ such that $\eta_0 \preceq \eta_{r,\xi_\star,i_\star}$. Since $r \le p_{\xi_\star,i_\star}$, it also forces that $(\xi_\star, i_\star) \in \mathring{A}$. Furthermore, $r \le q_0 \le p_\star \le p$ and $p \Vdash (\forall (\xi, i) \in \mathring{A})(\mathring{x}_{\xi,i} \in [\mathring{T}])$. It follows that $r \Vdash \eta_0 \preceq \mathring{x}_{\xi_\star,i_\star} \in [\mathring{T}]$ and therefore $r \Vdash [\eta_0] \cap [\mathring{T}] \ne \emptyset$. But $r \le q_0$ and $q_0 \Vdash [\eta_0] \cap [\mathring{T}] = \emptyset$. A contradiction.

Proof of Claim 4.4: Fix a common extension $r_\star$ of q_0 and $p_{\xi_\star,i_\star}$. By possibly extending η_0 and $r_\star$, we can assume that $\eta_0 \in {}^{m_{r_\star}}\omega$. For if $|\eta_0| < m_{r_\star}$, then we extend η_0 to any $\eta_0' \in {}^{m_q}\omega$. This is okay because $q_0 \Vdash [\eta_0'] \cap [\mathring{T}] = \emptyset$. On the other hand, if $|\eta_0| = m > m_{r_\star}$, then we extend $r_\star$ to a condition in $r_\star' \in \mathbb{Q}$ such that $m_{r_\star'} = m$ (as in the proof of Lemma 3.2(1)). So we can assume $|\eta_0| = m_{r_\star}$.

Now there is no reason for $\eta_{r_\star,\xi_\star,i_\star}$ to extend η_0 but we are going to correct $r_\star$ using Lemma 4.1 as follows. First, shrink $r_\star$ to a condition $q \in \mathbb{Q}$ defined as follows.

(1) $m_q = m_{r_\star}$, $u_q = u_{r_\star} \cap (\xi_\star \times \mu)$ and $v_q = v_{q_0}$.
(2) For each $(\xi, i) \in u_q$, $\eta_{q,\xi,i} = \eta_{r_\star,\xi,i}$.
(3) For each $\alpha \in v_q$, $\rho_{q,\alpha} = \rho_{r_\star,\alpha}$.
(4) If $(\xi, i), (\xi, j)$ are both in u_q and $i < j$, then $n_{q,\xi,i,j} = n_{r_\star,\xi,i,j}$.

Now $u_{q_0} \subseteq U_\star \subseteq \xi_\star \times \mu$ and $r_\star \le q_0$. So $q \le q_0$. We claim that q, $p = p_{\xi_\star,i_\star}$ and $\eta = \eta_0$ satisfy all the hypotheses of Lemma 4.1. To see this, note that Clauses (a)-(c) and (e)-(f) follow from the fact that $r_\star \le p_{\xi_\star,i_\star}$. Clause (d) holds since $u_{p_{\xi_\star,i_\star}} \cap (\xi_\star \times \mu) \subseteq u_q \subseteq \xi_\star \times \mu$. Clause (g) holds as $(\xi_\star, i_\star) \notin u_q \cap u_{p_{\xi_\star,i_\star}}$ and $\eta_{p_{\xi_\star,i_\star}} = \eta_\star \preceq \eta_0$. Finally, Clause (h) follows from the fact that $v_q \cap v_{p_{\xi_\star,i_\star}} = v_\star \subseteq V_\star$ and $(\forall \alpha \in V_\star)(\xi_\star, i_\star) \notin A_\alpha$. Therefore, we can apply Lemma 4.1 to get a common extension $r \in \mathbb{Q}$ of q and $p_{\xi_\star,i_\star}$ such that $\eta_{r,\xi_\star,i_\star} = \eta_0$. This establishes Claim 4.4 and the proof of Lemma 4.3 is complete. $\square$

Lemma 4.5. *Suppose $\{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq \lambda\times\mu$ where $i < j$ and $F_\xi(i, j) = (\zeta, k)$. Then $V^{\mathbb{P}} \models \Phi_{\mathring{n}_{\xi,i,j}}(\mathring{x}_{\xi,i}, \mathring{x}_{\xi,j}) = \mathring{x}_{\zeta,k}$.*

Proof. Fix $\{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq \lambda \times \mu$ such that $i < j$ and $F_\xi(i, j) = (\zeta, k)$. Let $p \in \mathbb{P}$. Choose $q \le p$ such that $\{(\xi, i), (\xi, j)\} \subseteq u_q$ and hence $(\zeta, k) \in u_q$. Put $n = n_{q,\xi,i,j}$. By Definition 3.1 Clause (vii), $q \Vdash \mathring{n}_{\xi,i,j} = n$ and $\Phi_n(\mathring{x}_{\xi,i}, \mathring{x}_{\xi,j}) = \mathring{x}_{\eta,k}$. $\square$

Lemma 4.6. *For every $\mathring{A} \in \mathcal{P}(\lambda \times \mu) \cap V^{\mathbb{P}}$,*

$$V^{\mathbb{P}} \models \textit{ If } \{\mathring{x}_{\xi,i} : (\xi, i) \in \mathring{A}\} \textit{ is 2-Turing independent, then it is meager.}$$

Proof. In view of Lemma 4.3, it suffices to show the following: If $p \in \mathbb{Q}$, $\mathring{A} \in \mathcal{P}(\lambda \times \mu) \cap V^{\mathbb{Q}}$ and $p \Vdash \mathring{A} \in \mathcal{J}^+_{\lambda,\mu}$, then there exist $r \le p$ and $(\xi, i), (\xi, j) \in \lambda \times \mu$ with $i < j$ such that letting $F_\xi(i, j) = (\zeta, k)$, we have that

$$r \Vdash \{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq \mathring{A}$$

For then, Lemma 4.5 implies that $x_{\zeta,k}$ is computable from $x_{\xi,i} \oplus x_{\xi,j}$ via the Turing functional $\Phi_{n_{\xi,i,j}}$. Hence $r \Vdash \{\mathring{x}_{\xi,i} : (\xi, i) \in \mathring{A}\}$ is not 2-Turing independent.

So fix $p \in \mathbb{Q}$, and $\mathring{A} \in \mathcal{P}(\lambda \times \mu) \cap V^{\mathbb{Q}}$ such that $p \Vdash \mathring{A} \in \mathcal{J}^+_{\lambda,\mu}$. Then

$$p \Vdash |\{\xi < \lambda : |\{i < \mu : (\xi, i) \in \mathring{A}\}| = \mu\}| = \lambda.$$

By repeating the Δ-system argument in the proof of Lemma 4.3, we can find $B_\star \in [\lambda]^\lambda$, $\langle D_\xi : \xi \in B_\star\rangle$, $\langle p_{\xi,i} : \xi \in B_\star$ and $i \in D_\xi\rangle$ such that $p_{\xi,i} \Vdash (\xi, i) \in \mathring{A}$, m_ξ, u_ξ, v_ξ, p_ξ, $m_\star$, $u_\star$, $v_\star$ and $p_\star$ are as there.

Fix any $\zeta \in B_\star$ and $k \in D_\zeta$. Recall that $\langle u_\xi : \xi \in B_\star\rangle$ and $\langle v_\xi : \xi \in B_\star\rangle$ form Δ-systems with roots $u_\star$ and $v_\star$ respectively. So we can choose $\xi \in B_\star$ such that $\xi > \zeta$, $u_\zeta \cap u_\xi = u_{p_{\zeta,k}} \cap u_\xi = u_\star$ and $v_\zeta \cap v_\xi = v_{p_{\zeta,k}} \cap v_\xi = v_\star$. Next, recall that $\langle u_{p_{\xi,i}} : i \in D_\xi\rangle$ and $\langle v_{p_{\xi,i}} : i \in D_\xi\rangle$ are Δ-systems with roots u_ξ and v_ξ respectively. It follows that for all but finitely many $i \in D_\xi$, we have $u_\zeta \cap u_{p_{\xi,i}} = u_{p_{\zeta,k}} \cap u_{p_{\xi,i}} = u_\star$ and $v_\zeta \cap v_{p_{\xi,i}} = v_{p_{\zeta,k}} \cap v_{p_{\xi,i}} = v_\star$. Let X_ξ be the set of all such $i \in D_\xi$. Then

$|X_\xi| = \mu$. As $\mathsf{range}(F_\xi \restriction [X_\xi]^2) = \xi \times \mu$, we can fix $i < j$ in X_ξ such that $F_\xi(i, j) = (\zeta, k)$.

Claim 4.7. *There exists a common extension $r \in \mathbb{Q}$ of $p_{\xi,i}$, $p_{\xi,j}$ and $p_{\zeta,k}$.*

Proof. Define r as follows.

(1) $m_r = m_\star$, $u_r = cl_{\bar{F}}(u_{p_{\xi,i}} \cup u_{p_{\xi,j}} \cup u_{p_{\zeta,k}})$, $v_r = v_{p_{\xi,i}} \cup v_{p_{\xi,j}} \cup v_{p_{\zeta,k}}$.
(2) $\bar{\rho}_r = \bar{\rho}_{p_{\xi,i}} \cup \bar{\rho}_{p_{\xi,j}} \cup \bar{\rho}_{p_{\zeta,k}}$.
(3) Let $w = u_r \setminus (u_{p_{\xi,i}} \cup u_{p_{\xi,j}} \cup u_{p_{\zeta,k}})$ and for each $(\epsilon, l) \in w$, put $\eta_{r,\epsilon,l} = 0^{m_\star}$. Define $\bar{\eta}_r = \bar{\eta}_{p_{\xi,i}} \cup \bar{\eta}_{p_{\xi,j}} \cup \bar{\eta}_{p_{\zeta,k}} \cup \langle \eta_{r,\epsilon,l} : (\epsilon, l) \in w \rangle$.
(4) Finally, choose $\bar{n}_r$ as follows. Suppose $\{(\epsilon, i_1), (\epsilon, i_2)\} \subseteq u_r$ and $i_1 < i_2$. If (ϵ, i_1) and (ϵ, i_2) are both in u_q for some $q \in \{p_{\xi,i}, p_{\xi,j}, p_{\zeta,k}\}$, then define $n_{r,\epsilon,i_1,i_2} = n_{q,\epsilon,i_1,i_2}$. Otherwise, using Definition 2.9 Clause (i), choose $n_{r,\epsilon,i_1,i_2} = n$ such that $\Phi_n(\eta_{r,\epsilon,i_1}, \eta_{r,\epsilon,i_2}) = \eta_{r,\gamma,i_3}$ where $(\gamma, i_3) = F_\epsilon(i_1, i_2)$.

It is easy to see that $r \in \mathbb{Q}$ and r extends each one of $p_{\xi,i}$, $p_{\xi,j}$ and $p_{\zeta,k}$. □

It follows now that $r \Vdash \{(\xi, i), (\xi, j), (\zeta, k)\} \subseteq \mathring{A}$. This completes the proof of Lemma 4.6. □

Remark 4.8. *The referee made the following observation. The standard colorings witnessing $\mu \nrightarrow [\mu]^2_\mu$ are quite absolute in the sense that they remain so in any forcing extension that preserves all stationary subsets of μ. Therefore, if we use such colorings to obtain $\bar{F}$ in Definition 2.4, then Lemma 4.6 easily follows. In fact, we get the following: If $\{x_{\xi,i} : (\xi, i) \in A\}$ is 2-Turing independent, then $|\{\xi < \lambda : |\{i < \mu : (\xi, i) \in A\}| = \mu\}| \leq 2$.*

Now our proof of Lemma 4.6 does not depend of the choice of $\bar{F}$ and one should note here that not all witnesses to $\mu \nrightarrow [\mu]^2_\mu$ remain so in any ccc extension. For example, let $c : [\mu]^2 \to \mu$ satisfy $\mathsf{range}(c \restriction [X]^2) = \mu$ for every $X \in [\mu]^\mu$. Let $\mathbb{Q}$ be the forcing for adding $d : [\mu]^2 \to 2$ using finite conditions (so $\mathbb{Q}$ is the forcing for adding μ Cohen reals). Observe that in $V^{\mathbb{Q}}$, the product $cd : [\mu]^2 \to \mu$ continues to satisfy $\mathsf{range}(cd \restriction [X]^2) = \mu$ for every $X \in [\mu]^\mu$. In $V^{\mathbb{Q}}$, let $\mathbb{R}$ be the forcing whose conditions are finite $s \subseteq \mu$ satisfying $\mathsf{range}(cd \restriction [s]^2) = \{0\}$, ordered by reverse inclusion. In $V^{\mathbb{Q}\star\mathbb{R}}$, define $H = \bigcup G_{\mathbb{R}}$. Then $\mathbb{Q} \star \mathbb{R}$ is ccc and $V^{\mathbb{Q}\star\mathbb{R}} \models H \in [\mu]^\mu$ and $\mathsf{range}(cd \restriction [H]^2) = \{0\}$. It follows that $V^{\mathbb{Q}\star\mathbb{R}}$ is a ccc extension of $V^{\mathbb{Q}}$ in which cd is no longer a witness for $\mu \nrightarrow [\mu]^2$.

Proof of Theorem 1.2: Let G be a $\mathbb{Q}$-generic filter over V where $\mathbb{Q}$ is as in Definition 3.1. In $V[G]$, define $X = \{x_{\xi,i} : (\xi, i) \in \lambda \times \mu\}$. By Lemma 4.3, $X \subseteq \omega^\omega$ is non-meager. By Lemma 4.6, for every non-meager $Y \subseteq X$, Y is not 2-Turing independent. To obtain such a subset of 2^ω, take the image $X' = H[X]$ of X under the map H in Fact 2.1. Since H is a homeomorphism from ω^ω to a co-countable subspace of 2^ω, X' must be a non-meager subset of 2^ω. To see that every 2-Turing independent subset of X' is meager, use the fact that x and $H(x)$ are computable from each other. □

5. Lebesgue measure

For $n \geq 1$, let $(\star)_n$ be the following statement: Every Lebesgue non-null set of reals has a Lebesgue non-null n-Turing independent subset. In [4], the following was shown. $(\star)_1$ is a theorem of ZFC, $(\star)_2$ is consistent relative to ZFC (as it follows from Martin's axiom) and $\neg(\star)_2$ is consistent relative to ZFC plus "there exists a measurable cardinal". So we ask the following.

Question 5.1. *Can we prove the consistency of the following statement without assuming the consistency of any large cardinals: There is a non-null set of reals all of whose* 2*-Turing independent subsets are null.*

Another open question is the following. Its category analogue was proved in [4] (Theorem 1.2).

Question 5.2. *Can we prove the existence of a Lebesgue non-null Turing independent set of reals in ZFC?*

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