WEAK PROJECTIONS OF THE NULL IDEAL

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Abstract. We show that a large class of sigma ideals could be weak projections of the null ideal. Using this, we show that it is consistent that there is a partition of the set of reals into null sets such that the corresponding projected ideal is countably saturated.

1. Introduction

In [1], L. Bukovsky showed that for every partition $\mathcal{F}$ of 2^ω into Lebesgue null sets, there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{G}$ is Lebesgue non-measurable. In [2] (see Problem GC), it was asked if one can improve this to get a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $\mu^\star(\bigcup \mathcal{G}) = \mu^\star(\bigcup(\mathcal{G} \setminus \mathcal{F})) = 1$ (Here, $\mu^\star$ denotes Lebesgue outer measure on 2^ω). Fremlin and Todorcevic [3] showed that if this fails, then there is a partition $\mathcal{F}$ of 2^ω into null sets such that the corresponding projected σ-ideal $\{\mathcal{A} \subseteq \mathcal{F} : \bigcup \mathcal{A} \text{ is null}\}$ is $\aleph_1$-saturated. Although Fremlin's problem remains open, we show here that the existence of such a partition is consistent.

Theorem 1.1. *Suppose κ is a measurable cardinal. Then there is a ccc forcing $\mathbb{P}$ such that the following holds in $V^{\mathbb{P}}$: There exists a partition $\{N_\alpha : \alpha < \kappa\}$ of 2^ω into null sets such that $\mathcal{J} = \{A \subseteq \kappa : \bigcup_{\alpha \in A} N_\alpha$ is null$\}$ is an $\aleph_1$-saturated κ-complete uniform ideal on κ.*

Note that, by a result of Solovay, the use of a measurable cardinal in Theorem 1.1 is necessary. Theorem 1.1 will follow from Theorem 1.4 below about the possible weak projections of the null ideal. To state it precisely, we need the following definitions.

Definition 1.2. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on X, Y respectively. We say that $(X, \mathcal{I})$ is a **weak projection** of $(Y, \mathcal{J})$ iff there exists a sequence $\langle N_i : i \in X \rangle$ of subsets of Y such that $\bigcup_{i \in X} N_i = Y$ and for every $A \subseteq X$, $A \in \mathcal{I}$ iff $\bigcup_{i \in A} N_i \in \mathcal{J}$.*

Definition 1.3. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on X, Y respectively. We say that $(X, \mathcal{I})$ is a **projection** of $(Y, \mathcal{J})$ iff there exists a partition $\{N_i : i \in X\}$ of Y such that for every $A \subseteq X$, $A \in \mathcal{I}$ iff $\bigcup_{i \in A} N_i \in \mathcal{J}$.*

The following theorem says that every σ-ideal can be made a weak projection of the null ideal in some ccc forcing extension. This implies, for example, that it is consistent that the non-stationary ideal on ω_1 is such an ideal.

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Theorem 1.4. *Suppose κ is an uncountable cardinal and $\mathcal{I}$ is a proper σ-ideal on κ. Then there exists a ccc forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, the ideal generated by $\mathcal{I}$ is a weak projection of the null ideal.*

Notation: Let $\mathcal{I}$ be a σ-ideal on X. Define $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$. We say that $\mathcal{I}$ is $\aleph_1$-saturated iff there is no uncountable family of pairwise disjoint members of $\mathcal{I}^+$ (equivalently, $\mathcal{P}(\kappa)/\mathcal{I}$ satisfies ccc). For $\sigma \in 2^{<\omega}$, we write $[\sigma] = \{x \in 2^\omega : \sigma \preceq x\}$. We say that $D \subseteq 2^{<\omega}$ is dense in $2^{<\omega}$ iff for every $\sigma \in 2^{<\omega}$, there exists $\tau \in D$ such that $\sigma \preceq \tau$. For a forcing notion $\mathbb{P}$ and $D \subseteq \mathbb{P}$, we say that D is predense in $\mathbb{P}$ iff every condition in $\mathbb{P}$ is compatible with some condition in D.

2. Forcing

Definition 2.1. *Let $\mathcal{U}$ be the set of all sequences $\bar{\sigma} = \langle \sigma_k : k < \omega \rangle$ such that each $\sigma_k \in 2^{<\omega}$, $|\sigma_k| \geq k$ and $\{\sigma_k : k < \omega\}$ is dense in $2^{<\omega}$. For $\bar{\sigma} \in \mathcal{U}$ and $n < \omega$, define the following.*

(1) $U_{\bar{\sigma},n} = \bigcup\{[\sigma_k] : k \geq n\}$.
(2) $\mathcal{N}_{\bar{\sigma}} = \bigcap\{U_{\bar{\sigma},n} : n < \omega\}$.

Note that for each $\bar{\sigma} \in \mathcal{U}$, $U_{\bar{\sigma},n}$ is an open dense set of measure $\leq 2^{-(n-1)}$ and therefore $\mathcal{N}_{\bar{\sigma}}$ is a dense G_δ set of measure zero.

Suppose $\mathcal{A} \subseteq \mathcal{U}$. We describe a forcing $\mathbb{Q}_{\mathcal{A}}$ which adds a $\bar{\sigma} \in \mathcal{U}$ such that for every $\bar{\tau} \in \mathcal{A}$, $\mathcal{N}_{\bar{\sigma}} \subseteq \mathcal{N}_{\bar{\tau}}$.

Definition 2.2. *For $\mathcal{A} \subseteq \mathcal{U}$, define the forcing $\mathbb{Q} = \mathbb{Q}_{\mathcal{A}}$ as follows: $p \in \mathbb{Q}$ iff $p = (A_p, k_p, \bar{\sigma}_p)$ where*

(a) $A_p \in [\mathcal{A}]^{<\aleph_0}$,
(b) $k_p < \omega$ *and*
(c) $\bar{\sigma}_p = \langle \sigma_{p,k} : k < k_p \rangle$ *where each* $\sigma_{p,k} \in 2^{<\omega}$ *and* $|\sigma_{p,k}| \geq k$.

For $p, q \in \mathbb{Q}$, define $p \leq q$ iff

(i) $A_q \subseteq A_p$, $k_q \leq k_p$,
(ii) $\bar{\sigma}_q = \bar{\sigma}_p \restriction k_q$ *and*
(iii) *for every* $k_q \leq k < k_p$ *and* $\bar{\tau} \in A_q$, $[\sigma_{p,k}] \subseteq U_{\bar{\tau},k}$.

Lemma 2.3. *Suppose $\mathcal{A} \subseteq \mathcal{U}$ and $\mathbb{Q} = \mathbb{Q}_{\mathcal{A}}$. Let $G_{\mathbb{Q}}$ be $\mathbb{Q}$-generic over V. Define $\mathring{\bar{\sigma}} = \bigcup\{\bar{\sigma}_p : p \in G_{\mathbb{Q}}\}$. Then the following hold.*

(1) *$\mathbb{Q}$ is an atomless σ-centered forcing.*
(2) $\bigcup\{A_p : p \in G_{\mathbb{Q}}\} = \mathcal{A}$.
(3) $\Vdash_{\mathbb{Q}} \mathring{\bar{\sigma}} \in \mathcal{U}$.
(4) *For every* $\bar{\tau} \in \mathcal{A}$, $\Vdash_{\mathbb{Q}} \mathcal{N}_{\mathring{\bar{\sigma}}} \subseteq \mathcal{N}_{\bar{\tau}}$.
(5) *For every* $x \in 2^\omega \cap V$, *if* $x \in \bigcap\{\mathcal{N}_{\bar{\tau}} : \bar{\tau} \in \mathcal{A}\}$, *then* $\Vdash_{\mathbb{Q}} x \in \mathcal{N}_{\mathring{\bar{\sigma}}}$.

Proof. (1) Given $p = (A_p, k_p, \bar{\sigma}_p) \in \mathbb{Q}$, observe that $U = \bigcap\{U_{\bar{\tau},k_p} : \tau \in A_p\}$ is open dense. So we can choose $\sigma^0 \neq \sigma^1$ in $2^{<\omega}$ such that for each $j < 2$, $|\sigma^j| \geq k_p + 1$ and $[\sigma^j] \subseteq U$. Define $p_j = (A_p, k_p + 1, \bar{\sigma}_p \cup \{(k_p, \sigma^j)\})$. Then p_0, p_1 are pairwise incompatible extensions of p in $\mathbb{Q}$. So $\mathbb{Q}$ is atomless. Next, note that for each $k_\star < \omega$ and $\bar{\sigma} = \langle \sigma_k : k < k_\star \rangle$, the set of $p \in \mathbb{Q}$ such that $k_p = k_\star$ and $\bar{\sigma}_p = \bar{\sigma}$ is a centered subset of $\mathbb{Q}$. So $\mathbb{Q}$ is σ-centered.

(2) It is easy to see that for each $\bar{\tau} \in \mathcal{A}$, $D_{\bar{\tau}} = \{p \in \mathbb{Q} : \bar{\tau} \in A_p\}$ is dense in $\mathbb{Q}$. It follows that $\bigcup\{A_p : p \in G_{\mathbb{Q}}\} = \mathcal{A}$.

(3) For each $\sigma \in 2^{<\omega}$ and $K < \omega$, define

$$D_{K,\sigma} = \{p \in \mathbb{Q} : (\exists k)(K \leq k < k_p \text{ and } \sigma \preceq \sigma_k)\}$$

Given $p = (A_p, k_p, \bar{\sigma}_p) \in \mathbb{Q}$, since $U = \bigcap\{U_{\bar{\tau},k_p} : \tau \in A_p\}$ is open dense, we can choose $\sigma_\star \in 2^{<\omega}$ such that $\sigma \preceq \sigma_\star$, $|\sigma_\star| \geq K + k_p + 1$ and $[\sigma_\star] \subseteq U$. Define $q = (A_p, K + k_p + 1, \bar{\sigma}_q) \in \mathbb{Q}$ where $\mathsf{dom}(\bar{\sigma}_q) = K + k_p + 1$, $\bar{\sigma}_p \subseteq \bar{\sigma}_q$ and for every $k \in [k_p, K + k_p + 1)$, $\sigma_{q,k} = \sigma_\star$. Then $q \leq p$ and $q \in D_{K,\sigma}$. So $D_{K,\sigma}$ is dense in $\mathbb{Q}$. It follows that $\Vdash_{\mathbb{Q}} \{\sigma_{p,k} : p \in G_{\mathbb{Q}} \text{ and } k < k_p\} \subseteq 2^{<\omega}$ is dense. Hence $\Vdash_{\mathbb{Q}} \mathring{\bar{\sigma}} \in \mathcal{U}$.

(4) Suppose $\bar{\tau} \in \mathcal{A}$ and $p \in \mathbb{Q}$. Choose $q \leq p$ such that $\bar{\tau} \in A_q$. It suffices to show that $q \Vdash_{\mathbb{Q}} \mathcal{N}_{\mathring{\bar{\sigma}}} \subseteq \mathcal{N}_{\bar{\tau}}$. Note that for every $r \leq q$ and $k_q \leq k < k_r$, $[\sigma_{r,k}] \subseteq U_{\bar{\tau},k}$. So $q \Vdash_{\mathbb{Q}} (\forall k \geq k_q)(U_{\bar{\sigma},k} \subseteq U_{\bar{\tau},k})$ and therefore, $q \Vdash_{\mathbb{Q}} \mathcal{N}_{\mathring{\bar{\sigma}}} \subseteq \mathcal{N}_{\bar{\tau}}$.

(5) Fix $x \in 2^\omega \cap V$ such that for every $\bar{\tau} \in \mathcal{A}$, $x \in \mathcal{N}_{\bar{\tau}}$. Let $\bar{\sigma} = \langle \sigma_k : k < \omega \rangle$. We need to show that there are infinitely many $k < \omega$ such that $x \in [\sigma_k]$. For this, it suffices to show that for every $p \in \mathbb{Q}$ and $K < \omega$, there exists $q \leq p$ and $k \in [K, k_q)$ such that $x \in [\sigma_{q,k}]$. But this easily follows from the fact that $U = \bigcap\{U_{\bar{\tau},k_p} : \bar{\tau} \in A_p\}$ is open dense. □

Let $\mathbb{C}$ denote the forcing whose conditions are finite sequences $\bar{\tau} = \langle \tau_k : k < K \rangle$, where each $\tau_k \in 2^{<\omega}$ and $|\tau_k| \geq k$, ordered by end-extension. Note that $\mathbb{C}$ is a countable atomless forcing and hence isomorphic to Cohen forcing. Let $\bar{\tau}_{\mathbb{C}} = \bigcup G_{\mathbb{C}}$ be the generic sequence added by $\mathbb{C}$. It is easy to see that $\Vdash_{\mathbb{C}} \mathsf{range}(\bar{\tau}_{\mathbb{C}})$ is dense in $2^{<\omega}$ and therefore $\Vdash_{\mathbb{C}} \bar{\tau}_{\mathbb{C}} \in \mathcal{U}$.

Definition 2.4. *Let κ be an uncountable cardinal. Let $\mathcal{I}$ be a proper σ-ideal on κ such that $[\kappa]^{\leq \aleph_0} \subseteq \mathcal{I}$.*

(A) *Put $\lambda = |\mathcal{I}|$. Fix a sequence $\bar{X} = \langle X_\alpha : \alpha < \lambda \rangle$ whose range is $\mathcal{I}$. For each $\xi < \kappa$, define $\mathcal{M}_\xi = \{\alpha < \lambda : \xi \in X_\alpha\}$.*

(B) *Define a finite support iteration $\langle (\mathbb{P}_\alpha, \mathring{\mathbb{Q}}_\alpha) : \alpha < \lambda + \kappa \rangle$ with limit $\mathbb{P} = \mathbb{P}_{\lambda+\kappa}$ as follows.*

(1) For each $\alpha < \lambda$, $\Vdash_{\mathbb{P}_\alpha} \mathring{\mathbb{Q}}_\alpha = \mathbb{C}$. Let $\bar{\tau}_\alpha = \bar{\tau}_{\mathbb{C}} \in V^{\mathbb{P}_{\alpha+1}}$ be the generic sequence added by $\mathring{\mathbb{Q}}_\alpha$.

(2) For each $\xi < \kappa$, $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathbb{Q}}_{\lambda+\xi} = \mathbb{Q}_{\mathcal{A}_\xi}$ where $\mathcal{A}_\xi = \{\bar{\tau}_\alpha : \alpha \in \mathcal{M}_\xi\}$. Let $\bar{\sigma}_\xi \in \mathcal{U} \cap V^{\mathbb{P}_{\lambda+\xi+1}}$ be the $\mathring{\mathbb{Q}}_{\lambda+\xi}$-generic sequence (see Lemma 2.3 for the definition of $\bar{\sigma}_\xi$).

Definition 2.5. *For $\eta \leq \kappa$, define $\mathbb{P}'_{\lambda+\eta}$ to be the set of those $p \in \mathbb{P}_{\lambda+\eta}$ that satisfy (a)+(b) below. Define $\mathbb{P}' = \mathbb{P}'_{\lambda+\kappa}$.*

(a) *For each $\alpha \in \mathsf{dom}(p) \cap \lambda$, $p(\alpha)$ is an actual member of $\mathbb{C}$.*

(b) *For each $\xi < \eta$ where $\lambda + \xi \in \mathsf{dom}(p)$, $p(\lambda + \xi) = (A_{p,\xi}, K_{p,\xi}, \bar{\sigma}_{p,\xi})$ where $A_{p,\xi} = \{\bar{\tau}_\alpha : \alpha \in F_{p,\xi}\}$ and $F_{p,\xi} \in [\mathcal{M}_\xi]^{<\aleph_0}$, $K_{p,\xi} < \omega$ and $\bar{\sigma}_{p,\xi} \in \mathbb{C}$ are actual objects. Furthermore, for each $\alpha \in F_{p,\xi}$, $\alpha \in \mathsf{dom}(p)$.*

Claim 2.6. *For each $\eta \leq \kappa$, $\mathbb{P}'_{\lambda+\eta}$ is dense in $\mathbb{P}_{\lambda+\eta}$.*

Proof. By induction on η using the fact that the iteration has finite support to handle the limit stages. $\square$

Remark 2.7. *Note that $\mathbb{P}'$ is isomorphic to the 2-step iteration $\mathbb{R}_0 \star \mathring{\mathbb{R}}_1$ where*

(1) *$\mathbb{R}_0$ is the finite support product of λ copies of $\mathbb{C}$ and*
(2) *$V^{\mathbb{R}_0} \models \mathring{\mathbb{R}}_1$ is the finite support product of $\langle \mathbb{Q}_{\mathcal{A}_\xi} : \xi < \kappa \rangle$.*

Claim 2.8. *Suppose $\eta \leq \kappa$ and $p, q \in \mathbb{P}'_{\lambda+\eta}$ are compatible. Suppose $\alpha < \lambda$, $\alpha \in \mathsf{dom}(p) \setminus \mathsf{dom}(q)$ and $(\forall \xi < \kappa)(\lambda + \xi \in \mathsf{dom}(q) \implies \xi \notin X_\alpha)$. Then there exists $r \in \mathbb{P}'_{\lambda+\eta}$ such that r is a common extension of p, q and $r(\alpha) = p(\alpha)$.*

Proof. Let $s \in \mathbb{P}'_{\lambda+\eta}$ be a common extension of p, q. Define $r \in \mathbb{P}'_{\lambda+\eta}$ as follows.

(a) $\mathsf{dom}(r) = \mathsf{dom}(p) \cup \mathsf{dom}(q)$.
(b) If $\beta \in \mathsf{dom}(r) \cap \lambda$ and $\beta \neq \alpha$, then $r(\beta) = s(\beta)$.
(c) $r(\alpha) = p(\alpha)$.
(d) If $\xi < \kappa$ and $\lambda + \xi \in \mathsf{dom}(p) \setminus \mathsf{dom}(q)$, then $r(\lambda + \xi) = p(\lambda + \xi)$.
(e) If $\xi < \kappa$ and $\lambda + \xi \in \mathsf{dom}(q) \setminus \mathsf{dom}(p)$, then $r(\lambda + \xi) = q(\lambda + \xi)$.
(f) If $\xi < \kappa$, $\lambda + \xi \in \mathsf{dom}(p) \cap \mathsf{dom}(q)$, $p(\lambda + \xi) = (A^p_\xi, K^p_\xi, \bar{\sigma}^p_\xi)$, $q(\lambda + \xi) = (A^q_\xi, K^q_\xi, \bar{\sigma}^q_\xi)$ and $s(\lambda + \xi) = (A^s_\xi, K^s_\xi, \bar{\sigma}^s_\xi)$, then $r(\lambda + \xi) = (A^p_\xi \cup A^q_\xi, K^s_\xi, \bar{\sigma}^s_\xi)$.

By induction on $\xi \leq \eta$, we'll show that $r \restriction (\lambda + \xi)$ extends both $p \restriction (\lambda + \xi)$ and $q \restriction (\lambda + \xi)$ and hence r is as required. If $\xi = 0$ or limit, then this is clear so assume $\xi = \gamma + 1$. We can further assume that $\lambda + \gamma \in \mathsf{dom}(p) \cap \mathsf{dom}(q)$, otherwise this is clear. Since $\lambda + \xi \in \mathsf{dom}(q)$, we have $\xi \notin X_\alpha$ and hence $\alpha \notin \mathcal{A}_\xi$. Now and $s \restriction (\lambda + \xi)$ forces that $s(\lambda + \xi)$ extends both $p(\lambda + \xi)$ and $q(\lambda + \xi)$ and for each $\beta \in \mathcal{A}_\xi$, $r(\beta) = s(\beta)$. It follows that $r \restriction (\lambda + \xi)$ also forces that $r(\lambda + \xi)$ extends both $p(\lambda + \xi)$ and $q(\lambda + \xi)$. $\square$

Lemma 2.9. *Let G be $\mathbb{P}$-generic over V. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V[G]$. For each $\xi < \kappa$, put $\mathcal{N}_\xi = \mathcal{N}_{\bar{\sigma}_\xi}$. Then the following hold in $V[G]$.*

(1) *$\mathcal{J}$ is a proper σ-ideal on κ.*
(2) *Each $\mathcal{N}_\xi$ is a null G_δ set.*
(3) *For each $Y \in \mathcal{J}$, $\bigcup\{\mathcal{N}_\xi : \xi \in Y\}$ is null.*
(4) *For each $Y \in \mathcal{J}^+$, $\bigcup\{\mathcal{N}_\xi : \xi \in Y\} = 2^\omega$.*

Proof. (1) Recall that $\mathcal{J} = \{Y \subseteq \kappa : (\exists X \in \mathcal{I})(Y \subseteq X)\}$. Since $\mathbb{P}$ satisfies ccc, for every countable $\mathcal{W} \subseteq \mathcal{J}$, there exists $X \in \mathcal{I}$ such that $\bigcup \mathcal{W} \subseteq X$. So $\mathcal{J}$ is a proper σ-ideal on κ.

(2) Follows from the definition of $\mathcal{N}_\xi = \mathcal{N}_{\bar{\sigma}_\xi}$.

(3) Fix $Y \in \mathcal{J}$. Since $\mathcal{I}$ is a σ-ideal and $\mathbb{P}$ satisfies ccc, we can find $X \in \mathcal{I}$ (so $X \in V$) such that $\Vdash_{\mathbb{P}} \mathring{Y} \subseteq X$. It suffices to show that $\bigcup\{\mathcal{N}_\xi : \xi \in X\}$ is null. Fix $\alpha < \lambda$ such that $X_\alpha = X$. Applying Lemma 2.3(4), we get $(\forall \beta \in \mathcal{M}_\xi)(\mathcal{N}_\xi \subseteq \mathcal{N}_{\bar{\tau}_\beta})$. Since $\xi \in X_\alpha \implies \alpha \in \mathcal{M}_\xi$, it follows that $(\forall \xi \in X_\alpha)(\mathcal{N}_\xi \subseteq \mathcal{N}_{\bar{\tau}_\alpha})$. Therefore $\bigcup\{\mathcal{N}_\xi : \xi \in X\} \subseteq \mathcal{N}_{\bar{\tau}_\beta}$ is null.

(4) Suppose $V^{\mathbb{P}} \models (\mathring{x} \in 2^\omega \wedge \mathring{Y} \in \mathcal{J}^+)$ and $p \in \mathbb{P}$ is arbitrary. It suffices to find $q \leq p$ and $\xi_\star < \kappa$ such that $q \Vdash_{\mathbb{P}} \xi_\star \in \mathring{Y}$ and $\mathring{x} \in \mathcal{N}_{\xi_\star}$. Since $\mathbb{P}$ satisfies ccc, for each $j < \omega$, we can find $W_j \in [\lambda + \kappa]^{\aleph_0}$ such that

$$D_j = \{r \in \mathbb{P}' : \mathsf{dom}(r) \subseteq W_j \text{ and } r \text{ decides the value of } \mathring{x} \restriction j\}$$

is predense in $\mathbb{P}$. Define

(a) $W = \bigcup\{W_j : j < \omega\}$,
(b) $W' = W \cap \lambda$ and
(c) $W'' = \{\xi < \kappa : \lambda + \xi \in W\}$.

Put $X_\star = \bigcup\{X_\alpha : \alpha \in W'\}$. As W' is countable and $\mathcal{I}$ is a σ-ideal, it follows that $X_\star \in \mathcal{I}$. Since $\Vdash_{\mathbb{P}} \mathring{Y} \in \mathcal{J}^+$, we can choose $q \in \mathbb{P}'$ extending p and $\xi_\star \in \kappa \setminus X_\star$ such that $q \Vdash_{\mathbb{P}} \xi_\star \in \mathring{Y}$. We will show that $q \Vdash_{\mathbb{P}} \mathring{x} \in \mathcal{N}_{\xi_\star}$. But first, we need the following.

Claim 2.10. *For every $\alpha < \lambda$, if $\xi_\star \in X_\alpha$, then $q \Vdash_{\mathbb{P}} \mathring{x} \in \mathcal{N}_{\bar{\tau}_\alpha}$.*

Proof. Fix $\alpha < \lambda$ such that $\xi_\star \in X_\alpha$. Note that $\alpha \notin W'$ since $\xi_\star \notin X_\star$. It suffices to show that for every $q_1 \leq q$ and $i < \omega$, there exist $j > i$ and $q' \leq q_1$ such that $q' \Vdash_{\mathbb{P}} \mathring{x} \in U_{\bar{\tau}_\alpha, j}$. So assume $i < \omega$ and $q_1 \in \mathbb{P}'$ such that $q_1 \leq q$. By possibly extending q_1, we can assume that $\alpha \in \mathsf{dom}(q_1)$. Fix $j > i$ above the domain of $q_1(\alpha)$. Since D_j is predense in $\mathbb{P}$, we can find $r \in \mathbb{P}'$ such that $\mathsf{dom}(r) \subseteq W_j$, r decides $\mathring{x} \restriction j$ and r, q_1 are compatible. Choose $\rho \in 2^{<\omega}$ such that $r \Vdash_{\mathbb{P}} \mathring{x} \restriction j = \rho$. Since $\mathsf{dom}(r) \cap \lambda \subseteq W'$ and $\alpha \notin W'$. So $\alpha \notin \mathsf{dom}(r)$. Using Claim 2.8, it follows that that we can find a common extension q_2 of q_1, r such that $q_2(\alpha) = q_1(\alpha)$. Now $\mathsf{dom}(q_2(\alpha)) < j$, so we can easily extend q_2 to $q' \in \mathbb{P}'$ such that $q'(\alpha)(j) = \rho$. It follows now that $q' \Vdash_{\mathbb{P}} \mathring{x} \in U_{\bar{\tau}_\alpha, j}$. □

By Remark 2.7, we can identify $\mathbb{P}' \cong \mathbb{R}_0 \star \mathring{\mathbb{R}}_1$. Let G be a $\mathbb{P}'$-generic filter over V with $q \in G$. Put $G_\lambda = G \cap \mathbb{P}'_\lambda$ and $H = \{p \restriction W'' : p \in G\}$. Then $\mathring{x}[G] \in 2^\omega \cap V[G_\lambda][H]$. Now apply Lemma 2.3(5) over the model $V[G_\lambda][H]$ with $\mathbb{Q} = \mathbb{Q}_{\xi_\star}$ and use Claim 2.10 to conclude that $V[G] \models \mathring{x}[G] \in \mathcal{N}_{\bar{\sigma}_\xi[G]}$. It follows that $q \Vdash_{\mathbb{P}} \mathring{x} \in \mathcal{N}_{\xi_\star}$. □

Proof of Theorem 1.4: Follows from Lemma 2.9. □

Proof of Theorem 1.1: Let κ be measurable cardinal with a witnessing normal prime ideal $\mathcal{I}$. Let $\mathbb{P}$ be the ccc forcing in Theorem 1.4 for this ideal. Work in $V^{\mathbb{P}}$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. It is well-known (see Theorem 17.1 in [4]) that $\mathcal{J}$ is a normal $\aleph_1$-saturated ideal on κ. Let $\langle M_\alpha : \alpha < \kappa\rangle$ be a sequence of null sets witnessing that $\mathcal{J}$ is a weak projection of the null ideal. So $\bigcup_{\alpha<\kappa} M_\alpha = 2^\omega$ and for every $A \subseteq \kappa$, $A \in \mathcal{J}$ iff $\bigcup_{\alpha\in A} M_\alpha$ is null.

Put $M'_\alpha = M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta$. For each $i < \kappa$, let $\alpha(i)$ be the least $\alpha \geq \sup\{\alpha(j)+1 : j < i\}$ such that $M'_{\alpha(i)} \neq \emptyset$. Note that for any $\alpha_\star < \kappa$, $\bigcup_{\beta<\alpha_\star} M_\beta$ is null and so there exists $\alpha \in [\alpha_\star, \kappa)$ such that $M'_\alpha \neq \emptyset$. So $\langle \alpha(i) : i < \kappa\rangle$ is well defined. Define $N_i = M_{\alpha(i)}$.

We claim that $\{N_i : i < \kappa\}$ is a partition of 2^ω into null sets as required. Put $\mathcal{K} = \{A \subseteq \kappa : \bigcup_{i\in A} N_i \text{ is null}\}$. It is clear that $\mathcal{K}$ is a κ-complete uniform ideal on κ. Towards a contradiction, suppose $\mathcal{K}$ is not $\aleph_1$-saturated and fix a family $\{A_\xi : \xi < \omega_1\}$ of pairwise disjoint members of $\mathcal{K}^+$. Then for each $\xi < \omega_1$, $\bigcup_{i\in A_\xi} N_i$ is non-null. Let $B_\xi = \{\alpha(i) : i \in A_\xi\}$. Since each $N_i \subseteq M_{\alpha(i)}$, it follows that $\{B_\xi : \xi < \omega_1\}$ is an uncountable family of pairwise disjoint sets in $\mathcal{J}^+$ which contradicts the fact that $\mathcal{J}$ is $\aleph_1$-saturated. □

We do not know if an analogous result holds for the meager ideal.

Question 2.11. *Does the analogue of Theorem 1.4 hold for the meager ideal?*

In the case of the null ideal, one can ask if in Theorem 1.4, we can replace "weak projection" by "projection". For example,

Question 2.12. *Is it consistent that the non-stationary ideal on ω_1 is a projection of the null ideal?*

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