Shape equations of the axisymmetric vesicles

Kumar Gaurav

Roll number:
gmail id: kumgaurav5620@gmail.com
IIT Kanpur

Sunday 20th November, 2022

Number of hours spent in papers/books collection	1 hrs.
Number of hours spent in reading and understanding the relevant papers/books/topics	35 hrs.
Number of hours spent in typing the report	20 hrs.
Number of hours spent in proof-reading your report before final submission and making it error free	3 hrs.

1 Introduction

All cells have a membrane made of phospholipids which separates the interior of the cell from the exterior and helps transport the molecules in and out of the cell. Cell organelles like the endoplasmic reticulum, nucleus, and Golgi complex also contain cell membranes. These cell membranes are usually a few nanometres thick. Phospholipids are amphipathic. Therefore, the hydrophilic phosphate group lies on the surface, and the hydrophobic lipid lies inside. They form a bilayer structure, as shown in Fig. 1.

These membranes are flexible and undergo bending deformations upon applying weak forces by external agents or internal structures such as cytoskeleton and proteins. However, the cell's surface area and volume remain constant even during the large deformation. Calculating the membrane shape is, therefore, a significant biological problem. Some examples of different membrane shapes for bacteria are shown in Fig. 2. The spirochetes have a corkscrew shape, which favors penetration and packing in the host cells. The biconcave disc-like shape of the RBC gives it a large surface area to volume ratio and thus helps it undergo large deformation while traveling through narrow blood vessels. The cell organelles like the endoplasmic reticulum and Golgi apparatus are a system of interconnected tubules, cylinders, and discs.

The qualitative relationship between liquid crystals and biological membranes has been recognized as early as the 1850s. However, the first quantitative theory for the shape of the membranes was given by Prof. W. Helfrich in 1973 using the curvature elasticity theory of liquid crystals.

2 Helfrich's Shape equation

The biconcave-disc-like shape of the RBC was always a puzzle for researchers. In 1970, Canham proposed that the shape of the RBC can be determined solely by the bending energy. However, the resultant shape from the model was more like a dumbell than the original discocyte shape of the RBC. Another approach proposed was based on the shell theory, which failed to define the state of zero stress and the caterpillar motion of the RBC. Helfrich proposed that the phospholipid is analogous to the director of the uniaxial liquid crystal. Hence, the bilayer membrane can be treated like a homeotropic nematic liquid crystal cell with twice the thickness of a

single lipid molecule. He used the frank energy density function for uniaxial liquid crystals to write the bending energy in the form

$$F_b = \frac{1}{2}k_c \oint (c_1 + c_2 - c_o)^2 dA + \bar{K} \oint c_1 c_2 dA \tag{1}$$

where k_c is the bending rigidity, $\bar{K}$ is the Gaussian curvature modulus, c_1 and c_0 are the two principal curvatures and c_0 is the spontaneous curvature. The volume and the surface area of the vesicle remain constant, and hence these two constraints should be considered while finding the first variation. The last term in the bending energy is neglected due to the Gauss-Bonnet theorem, which says that the integration should be a constant for a given topology. Ou-Yang and Helfrich [3] derived a general shape equation by considering the variation of the form

$$\delta \boldsymbol{r} = \psi \boldsymbol{n}, \tag{2}$$

where $\boldsymbol{r}$ is the positon vector and $\boldsymbol{n}$ is the normal vector to the surface. The shape equations are obtained from the first variation,

$$\delta(F_b + \lambda A + \Delta P V) = 0, \tag{3}$$

where A is the total area, V is the total volume, λ and ΔP are corresponding Lagrange multipliers. Substituting (1) and (2) in (3), [4] obtained the shape equation

$$\Delta P - 2\lambda H + k_c(2H + c_0)(2H^2 - 2K - c_0 H) + 2k_c \nabla^2 H = 0 \tag{4}$$

Note that there is a typo in the shape equation given in the [1] where "$+2K$" is written instead of "$-2K$".

3 Axisymmetric Vesicles

[1] described three ways of deriving shape equations for the axisymmetric vesicles. These are:

1. Use (4) directly for the axisymmetric case
2. Writing energy functional for the axisymmetric case and then deriving the Euler-Lagrange equation using the contour arc length s as the parameter
3. Similar to the second except that we use distance from the symmetry axis ρ as the parameter (see Fig. 3).

3.1 Mean and Gaussian curvatures

The mean and Gaussian curvatures appear in the bending energy (1) and in the shape equation (4). The second method uses the parameter space (ϕ, s) while the third method uses the parameter space (ϕ, ρ), where ϕ is the azimuthal angle. Therefore to simplify the shape equations, we need to find the mean and Gaussian curvature as a function of both ρ and s. From the Fig. 3, we can write the position vector in these parameter spaces as

$$\begin{aligned} \boldsymbol{r}(\phi, s) &= \rho(s)\cos\phi \boldsymbol{e}_1 + \rho(s)\sin\phi \boldsymbol{e}_2 + z(s)\boldsymbol{e}_3 \\ \text{and} \qquad \boldsymbol{r}(\phi, \rho) &= \rho\cos\phi \boldsymbol{e}_1 + \rho\sin\phi \boldsymbol{e}_2 + z(\rho)\boldsymbol{e}_3. \end{aligned} \tag{5}$$

Also, from the Fig. 3, we find that

$$\begin{aligned} dz &= ds\sin\psi \quad \text{and} \quad d\rho = ds\cos\psi \\ dz &= d\rho\tan\psi. \end{aligned} \tag{6}$$

Using (5) and (6), we can get the tangent to the surface as

$$
\begin{aligned}
\boldsymbol{r}_1(\phi,s) &= \frac{\partial \boldsymbol{r}(\phi,s)}{\partial \boldsymbol{\phi}} = -\rho(s)\sin\phi\boldsymbol{e}_1 + \rho(s)\cos\phi\boldsymbol{e}_2 \\
\boldsymbol{r}_1(\phi,\rho) &= \frac{\partial \boldsymbol{r}(\phi,\rho)}{\partial \boldsymbol{\phi}} = -\rho\sin\phi\boldsymbol{e}_1 + \rho\cos\phi\boldsymbol{e}_2 \\
\boldsymbol{r}_2(\phi,s) &= \frac{\partial \boldsymbol{r}(\phi,s)}{\partial s} = \frac{d\rho(s)}{ds}\cos\phi\boldsymbol{e}_1 + \frac{d\rho(s)}{ds}\sin\phi\boldsymbol{e}_2 + \sin\psi(s)\boldsymbol{e}_3 \\
&= \cos\psi(s)\cos\phi\boldsymbol{e}_1 + \cos\psi(s)\sin\phi\boldsymbol{e}_2 + \sin\psi(s)\boldsymbol{e}_3 \\
\boldsymbol{r}_2(\phi,\rho) &= \frac{\partial \boldsymbol{r}(\phi,\rho)}{\partial s} = \cos\phi\boldsymbol{e}_1 + \sin\phi\boldsymbol{e}_2 + \tan\psi(\rho)\boldsymbol{e}_3.
\end{aligned} \tag{7}
$$

Using the above relations we find the components of metric tensor

$$
\begin{aligned}
g_{11}(s) = \boldsymbol{r}_1\cdot\boldsymbol{r}_1 = \rho(s)^2, \quad g_{12}(s) = \boldsymbol{r}_1\cdot\boldsymbol{r}_2 = 0 \quad \text{and} \quad g_{22}(s) = 1, \\
g_{11}(\rho) = \boldsymbol{r}_1\cdot\boldsymbol{r}_1 = \rho^2, \quad g_{12}(\rho) = \boldsymbol{r}_1\cdot\boldsymbol{r}_2 = 0 \quad \text{and} \quad g_{22}(\rho) = \sec^2\psi(\rho),
\end{aligned}
$$

and hence the components of inverse of metric tensor becomes

$$
\begin{aligned}
&g^{11}(s) = \frac{g_{22}(s)}{g(s)} = \frac{g_{22}(s)}{g_{11}(s)g_{22}(s)} = \frac{1}{g_{11}(s)} = \frac{1}{\rho(s)^2}, \qquad g^{11}(\rho) = \frac{g_{22}(\rho)}{g(\rho)} = \frac{g_{22}(\rho)}{g_{11}(\rho)g_{22}(\rho)} = \frac{1}{g_{11}(\rho)} = \frac{1}{\rho^2}, \\
&g^{12}(s) = -\frac{g_{12}(s)}{g(s)} = 0, \qquad g^{12}(\rho) = -\frac{g_{12}(\rho)}{g(\rho)} = 0 \\
&g^{22}(s) = \frac{g_{11}(s)}{g(s)} = \frac{g_{11}(s)}{g_{11}(s)g_{22}(s)} = \frac{1}{g_{22}(s)} = 1, \qquad g^{22}(\rho) = \frac{g_{11}(\rho)}{g(\rho)} = \frac{g_{11}(\rho)}{g_{11}(\rho)g_{22}(\rho)} = \frac{1}{g_{22}(\rho)} = \cos^2\psi(\rho), \\
&\text{and} \quad g(s) = g_{11}g_{22} = \rho(s)^2, \qquad g(\rho) = \rho^2\cos^2\psi(\rho).
\end{aligned} \tag{8}
$$

Using (7), we obtain the derivatives of the tangent vectors

$$
\begin{aligned}
\boldsymbol{r}_{11}(\psi,s) &= \frac{\partial \boldsymbol{r}_1(\psi,s)}{\partial \boldsymbol{\phi}} = -\rho(s)\cos\phi\boldsymbol{e}_1 - \rho(s)\sin\phi\boldsymbol{e}_2, \\
\boldsymbol{r}_{11}(\psi,\rho) &= \frac{\partial \boldsymbol{r}_1(\psi,\rho)}{\partial \boldsymbol{\phi}} = -\rho\cos\phi\boldsymbol{e}_1 - \rho\sin\phi\boldsymbol{e}_2, \\
\boldsymbol{r}_{12}(\psi,s) &= \frac{\partial \boldsymbol{r}_1(\psi,s)}{\partial s} = -\frac{d\rho(s)}{ds}\sin\phi\boldsymbol{e}_1 + \frac{d\rho(s)}{ds}\cos\phi\boldsymbol{e}_2, \\
&= -\cos\psi(s)\sin\phi\boldsymbol{e}_1 + \cos\psi(s)\cos\phi\boldsymbol{e}_2, \\
\boldsymbol{r}_{12}(\psi,\rho) &= \frac{\partial \boldsymbol{r}_1(\psi,\rho)}{\partial \rho} = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2, \\
\boldsymbol{r}_{22}(\psi,s) &= \frac{\partial \boldsymbol{r}_2(\psi,s)}{\partial s} = -\sin\psi(s)\cos\phi\frac{d\psi(s)}{ds}\boldsymbol{e}_1 - \sin\psi(s)\sin\phi\frac{d\psi(s)}{ds}\boldsymbol{e}_2 + \cos\psi\frac{d\psi(s)}{ds}\boldsymbol{e}_3, \\
\boldsymbol{r}_{22}(\psi,\rho) &= \frac{\partial \boldsymbol{r}_2(\psi,\rho)}{\partial \rho} = \sec^2\psi\frac{d\psi(\rho)}{d\rho}\boldsymbol{e}_3.
\end{aligned} \tag{9}
$$

The normal to the surface at any point can be expressed as

$$
\boldsymbol{n} = \frac{\boldsymbol{r}_1\times\boldsymbol{r}_2}{|\boldsymbol{r}_1\times\boldsymbol{r}_2|},
$$

which upon using (7) simplifies to

$$
\begin{aligned}
\boldsymbol{n}(\phi,s) &= \sin\psi(s)\cos\phi\boldsymbol{e}_1 + \sin\psi(s)\sin\phi\boldsymbol{e}_2 - \cos\psi(s)\boldsymbol{e}_3 \\
\boldsymbol{n}(\phi,\rho) &= \sin\psi(\rho)\cos\phi\boldsymbol{e}_1 + \sin\psi(\rho)\sin\phi\boldsymbol{e}_2 - \cos\psi(\rho)\boldsymbol{e}_3.
\end{aligned} \tag{10}
$$

Now using (9) and (10), we obtain

$$
\begin{aligned}
& L_{11}(s) = \boldsymbol{n}\cdot\boldsymbol{r}_{11} = -\rho(s)\sin\psi(s) \qquad L_{11}(\rho) = \boldsymbol{n}\cdot\boldsymbol{r}_{11} = -\rho\sin\psi(\rho), \\
& L_{12}(s) = \boldsymbol{n}\cdot\boldsymbol{r}_{12} = 0 \qquad L_{12}(\rho) = \boldsymbol{n}\cdot\boldsymbol{r}_{12} = 0 \\
\text{and} \quad & L_{22}(s) = \boldsymbol{n}\cdot\boldsymbol{r}_{22} = -\frac{d\psi(s)}{ds} \qquad L_{22}(\rho) = \boldsymbol{n}\cdot\boldsymbol{r}_{22} = -\sec\psi(\rho)\frac{d\psi(\rho)}{d\rho}.
\end{aligned} \tag{11}
$$

The mean curvature, H,and the Gaussian curvature, K, can now be obtained using (8) and (11)

$$
\begin{aligned}
H(s) &= \frac{1}{2}g^{ij}(s)L_{ij}(s) = \frac{1}{2}\left(g^{11}(s)L_{11}(s) + g^{22}(s)L_{22}(s)\right) = -\frac{1}{2}\left[\frac{\sin\psi(s)}{\rho(s)} + \frac{d\psi(s)}{ds}\right] \\
H(\rho) &= \frac{1}{2}g^{ij}(\rho)L_{ij}(\rho) = \frac{1}{2}\left(g^{11}(\rho)L_{11}(\rho) + g^{22}(\rho)L_{22}(\rho)\right) = -\frac{1}{2}\left[\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho} + \frac{\sin\psi}{\rho}\right] \\
K(s) &= \frac{L(s)}{g(s)} = \frac{L_{11}(s)L_{22}(s)}{g_{11}(s)g_{22}(s)} = \frac{\sin\psi(s)}{\rho(s)}\frac{d\psi(s)}{ds} \\
K(\rho) &= \frac{L(\rho)}{g(\rho)} = \frac{L_{11}(\rho)L_{22}(\rho)}{g_{11}(\rho)g_{22}(\rho)} = \cos\psi(\rho)\frac{\sin\psi(\rho)}{\rho}\frac{d\psi(\rho)}{d\rho}.
\end{aligned} \tag{12}
$$

3.2 Shape equation from Helfrich's equation

Substitution of H and K in (4) will give the required shape equation for the angle ψ. Equation (4) contains Laplacian of the mean curvature, which needs to be evaluated. The Laplacian in the parameter space (ϕ, ρ) can be expressed as

$$
\nabla^2 = \frac{1}{\sqrt{g(\rho)}}\frac{\partial}{\partial_i}\left(g^{ij}(\rho)\sqrt{g(\rho)}\frac{\partial}{\partial_j}\right)
$$

which upon using (8) becomes

$$
\begin{aligned}
\nabla^2 &= \frac{1}{\rho\sec\psi(\rho)}\frac{\partial}{\partial\phi}\left[\left(g^{11}(\rho)\frac{\partial}{\partial\phi} + g^{12}(\rho)\frac{\partial}{\partial\rho}\right)\rho\sec\psi(\rho)\right] + \frac{1}{\rho\sec\psi(\rho)}\frac{\partial}{\partial\rho}\left[\left(g^{21}(\rho)\frac{\partial}{\partial\phi} + g^{22}(\rho)\frac{\partial}{\partial\rho}\right)\rho\sec\psi(\rho)\right], \\
&= \frac{\cos\psi(\rho)}{\rho}\frac{\partial}{\partial\phi}\left[\frac{\sec\psi(\rho)}{\rho}\frac{\partial}{\partial\phi}\right] + \frac{\cos\psi(\rho)}{\rho}\frac{\partial}{\partial\rho}\left[\left(\rho\cos\psi(\rho)\frac{\partial}{\partial\rho}\right)\right], \\
&= \frac{\cos\psi(\rho)}{\rho}\frac{\partial}{\partial\phi}\left[\frac{\sec\psi(\rho)}{\rho}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\rho}\left(\rho\cos\psi(\rho)\frac{\partial}{\partial\rho}\right)\right].
\end{aligned} \tag{13}
$$

Since, $H(\rho)$ is independent of the azimuthal angle ϕ, the Laplacian of $H(\rho)$ becomes

$$
\begin{aligned}
\nabla^2 H(\rho) &= \frac{\cos\psi(\rho)}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cos\psi(\rho)\frac{\partial}{\partial\rho}\right)H(\rho) \\
&= \frac{\cos^2\psi(\rho)}{\rho}\frac{dH(\rho)}{d\rho} - \cos\psi(\rho)\sin\psi(\rho)\frac{d\psi(\rho)}{d\rho}\frac{dH(\rho)}{d\rho} + \cos^2\psi(\rho)\frac{d^2H(\rho)}{d\rho^2}.
\end{aligned}
$$

Using $H(\rho)$ from (12), we derive the following relations

$$
\begin{aligned}
\frac{dH(\rho)}{d\rho} &= -\frac{1}{2}\left[-\sin\psi\left(\frac{d\psi}{d\rho}\right)^2 + \cos\psi\frac{d^2\psi}{d\rho^2} - \frac{\sin\psi}{\rho^2} + \frac{\cos\psi}{\rho}\frac{d\psi}{d\rho}\right], \\
\frac{d^2H(\rho)}{d\rho^2} &= -\frac{1}{2}\left[-\cos\psi\left(\frac{d\psi}{d\rho}\right)^3 - 3\sin\psi\frac{d\psi}{d\rho}\frac{d^2\psi}{d\rho^2} + \cos\psi\frac{d^3\psi}{d\rho^3} + 2\frac{\sin\psi}{\rho^3} - 2\frac{\cos\psi}{\rho^2}\frac{d\psi}{d\rho} - \frac{\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + \frac{\cos\psi}{\rho}\frac{d^2\psi}{d\rho^2}\right],
\end{aligned} \tag{14}
$$

where $\psi = \psi(\rho)$. By substituting the above expressions, we get the final form of the Laplacian of $H(\rho)$

$$
\begin{aligned}
\nabla^2 H(\rho) =& -\frac{1}{2}\frac{\cos^2\psi}{\rho}\left[-\sin\psi\left(\frac{d\psi}{d\rho}\right)^2 + \cos\psi\frac{d^2\psi}{d\rho^2} - \frac{\sin\psi}{\rho^2} + \frac{\cos\psi}{\rho}\frac{d\psi}{d\rho}\right] \\
&+ \frac{1}{2}\cos\psi\sin\psi\frac{d\psi}{d\rho}\left[-\sin\psi\left(\frac{d\psi}{d\rho}\right)^2 + \cos\psi\frac{d^2\psi}{d\rho^2} - \frac{\sin\psi}{\rho^2} + \frac{\cos\psi}{\rho}\frac{d\psi}{d\rho}\right] \\
&- \frac{1}{2}\cos^2\psi\left[-\cos\psi\left(\frac{d\psi}{d\rho}\right)^3 - 3\sin\psi\frac{d\psi}{d\rho}\frac{d^2\psi}{d\rho^2} + \cos\psi\frac{d^3\psi}{d\rho^3} + 2\frac{\sin\psi}{\rho^3} - 2\frac{\cos\psi}{\rho^2}\frac{d\psi}{d\rho} - \frac{\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + \frac{\cos\psi}{\rho}\frac{d^2\psi}{d\rho^2}\right] \\
=& \frac{1}{2}\Bigg[-\cos^3\psi\frac{d^3\psi}{d\rho^3} + 4\sin\psi\cos^2\psi\frac{d^2\psi}{d\rho^2}\frac{d\psi}{d\rho} - \cos\psi(\sin^2\psi - \cos^2\psi)\left(\frac{d\psi}{d\rho}\right)^3 + \frac{3\sin\psi\cos^2\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 \\
&- \frac{2\cos^3\psi}{\rho}\frac{d^2\psi}{d\rho^2} - \frac{\sin^2\psi - \cos^2\psi}{\rho^2}\cos\psi\frac{d\psi}{d\rho} - \frac{\sin\psi\cos^2\psi}{\rho^3}\Bigg]. \qquad (15)
\end{aligned}
$$

The Helfrich's shape equation (4), can be simplified as

$$
\begin{aligned}
\Delta P - 2\lambda H + k_c(2H + c_0)(2H^2 - 2K - c_0 H) + 2k_c\nabla^2 H = 0 \\
\Delta P - 2\lambda H + k_c(4H^3 - 4KH - 2c_0 K - c_0^2 H) + 2k_c\nabla^2 H = 0. \qquad (16)
\end{aligned}
$$

From (12), we have

$$
\begin{aligned}
H^3(\rho) &= -\frac{1}{8}\left[\cos^3\psi\left(\frac{d\psi}{d\rho}\right)^3 + \frac{\sin^3\psi}{\rho^3} + 3\frac{\cos^2\psi\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + 3\frac{sin^2\psi\cos\psi}{\rho^2}\frac{d\psi}{d\rho}\right], \\
K(\rho)H(\rho) &= -\frac{1}{2}\left[\frac{cos^2\psi\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + \frac{\sin^2\psi\cos\psi}{\rho^2}\frac{d\psi}{d\rho}\right]. \qquad (17)
\end{aligned}
$$

Finally, substituting (12),(15) and (17) in (16), we get the final equation of the shape

$$
\begin{aligned}
\Delta P + \lambda\left[\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho} + \frac{\sin\psi}{\rho}\right] + k_c\Bigg[&-\frac{1}{2}\left\{\cos^3\psi\left(\frac{d\psi}{d\rho}\right)^3 + \frac{\sin^3\psi}{\rho^3} + 3\frac{\cos^2\psi\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + 3\frac{sin^2\psi\cos\psi}{\rho^2}\frac{d\psi}{d\rho}\right\} \\
&- 2\left\{\frac{cos^2\psi\sin\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 + \frac{\sin^2\psi\cos\psi}{\rho^2}\frac{d\psi}{d\rho}\right\} - 2c_0\cos\psi\frac{\sin\psi}{\rho}\frac{d\psi}{d\rho} + \frac{c_0^2}{2}\left\{\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho} + \frac{\sin\psi}{\rho}\right\} \\
&- \cos^3\psi\frac{d^3\psi}{d\rho^3} + 4\sin\psi\cos^2\psi\frac{d^2\psi}{d\rho^2}\frac{d\psi}{d\rho} - \cos\psi(\sin^2\psi - \cos^2\psi)\left(\frac{d\psi}{d\rho}\right)^3 + \frac{3\sin\psi\cos^2\psi}{\rho}\left(\frac{d\psi}{d\rho}\right)^2 \\
&- \frac{2\cos^3\psi}{\rho}\frac{d^2\psi}{d\rho^2} - \frac{\sin^2\psi - \cos^2\psi}{\rho^2}\cos\psi\frac{d\psi}{d\rho} - \frac{\sin\psi\cos^2\psi}{\rho^3}\Bigg] = 0,
\end{aligned}
$$

which upon rearranging becomes

$$
\begin{aligned}
\cos^3\psi\frac{d^3\psi}{d\rho^3} =& 4\sin\psi\cos^2\psi\frac{d^2\psi}{d\rho^2}\frac{d\psi}{d\rho} - \cos\psi(\sin^2\psi - \frac{1}{2}\cos^2\psi)\left(\frac{d\psi}{d\rho}\right)^3 + \frac{7\sin\psi\cos^2\psi}{2\rho}\left(\frac{d\psi}{d\rho}\right)^2 \\
&- \frac{2\cos^3\psi}{\rho}\frac{d^2\psi}{d\rho^2} + \left[\frac{c_0^2}{2} - \frac{2c_0\sin\psi}{\rho} + \frac{\sin^2\psi}{2\rho^2} + \frac{\lambda}{k_c} - \frac{\sin^2\psi - \cos^2\psi}{\rho^2}\right]\cos\psi\frac{d\psi}{d\rho} \\
&\left[\frac{\Delta P}{k_c} + \frac{\lambda\sin\psi}{k_c\rho} - \frac{\sin^3\psi}{2\rho^3} + \frac{c_0^2\sin\psi}{2\rho} - \frac{\sin\psi\cos^2\psi}{\rho^3}\right]. \qquad (18)
\end{aligned}
$$

The term in the red doesn't match with [1]. Instead of $\sin^3\psi$ the equation 7 in [1] contains $\sin^2\psi$. However, the term matches with equation 4.30 of [2].

3.3 Shape equation using s parameter

The second way to find the shape equation is to find Lagrangian for the axisymmetric shape shown in Fig. 3. From the Fig. 3, we can write the surface area by considering it is made of rings of infinitesimal width ds and calculate volume by considering discs of infinitesimal thickness dz. This will give us the following relations upon using (6)

$$\begin{aligned} dA &= 2\pi\rho(s)ds, \\ dV &= \pi\rho^2(s)dz = \pi\rho^2(s)\sin\psi(s)ds. \end{aligned} \tag{19}$$

Using the above relations and (12), the energy functional becomes

$$\begin{aligned} F_s = F_b + \lambda A + \Delta PV &= \frac{1}{2}k_c \oint (c_1 + c_2 - c_o)^2 dA + \lambda \oint dA + \Delta P \oint dV \\ &= \pi k_c \int_{s_0}^{s_1} \rho(s)\left(\frac{d\psi(s)}{ds} + \frac{\sin\psi(s)}{\rho(s)} - c_0\right)^2 ds + 2\pi\lambda \int_{s_0}^{s_1} \rho(s)ds + \pi\Delta P \int_{s_0}^{s_1} \rho^2(s)\sin\psi(s)ds \\ &= 2\pi k_c \int_{s_0}^{s_1} \left[\frac{\rho(s)}{2}\left(\frac{d\psi(s)}{ds} + \frac{\sin\psi(s)}{\rho(s)} - c_0\right)^2 + \frac{\lambda\rho(s)}{k_c} + \frac{\Delta P}{2k_c}\rho^2(s)\sin\psi(s)\right] ds. \end{aligned} \tag{20}$$

The equation (6) also gives the following relation between $\psi(s)$ and $\rho(s)$

$$\frac{d\rho(s)}{ds} = \cos\psi(s). \tag{21}$$

The modified Lagrangian for the functional incorporating the above non-holonomic constraint becomes

$$L\left(\rho(s), \frac{d\rho(s)}{ds}, \psi(s), \frac{d\psi(s)}{ds}, \gamma(s)\right) = \frac{\rho(s)}{2}\left(\frac{d\psi(s)}{ds} + \frac{\sin\psi(s)}{\rho(s)} - c_0\right)^2 + \frac{\lambda\rho(s)}{k_c} + \frac{\Delta P}{2k_c}\rho^2(s)\sin\psi(s) + \gamma(s)\left(\frac{d\rho}{ds} - \cos\psi(s)\right), \tag{22}$$

and energy functional becomes

$$\tilde{F}_s = 2\pi k_c \int_{s_0}^{s_1} L\left(\rho(s), \dot{\rho}(s), \psi(s), \dot{\psi}(s), \gamma(s)\right) ds,$$

where $\dot{()} = d()/ds$. The term in the red color is not matching with equation 9 of [1]. There is a factor of 1/2 missing in [1]. However, it is matching with equation 4.93 of [2].

To find the extremal, let us consider the variation of the form

$$\begin{aligned} \psi_\varepsilon(s) &= \psi(s) + \epsilon\alpha(s), \\ \rho_\varepsilon(s) &= \rho(s) + \epsilon\beta(s), \\ \gamma_\varepsilon(s) &= \gamma(s) + \epsilon\tau(s), \\ s_{0\varepsilon} &= s_0 + \varepsilon\zeta_0, \\ s_{1\varepsilon} &= s_1 + \varepsilon\zeta_1 \end{aligned} \tag{23}$$

Calculating the first variation of the functional $\tilde{F}_s$

$$\frac{d}{d\varepsilon}\tilde{F}_s\left(\psi_\varepsilon, \dot{\psi}_\varepsilon, \rho_\varepsilon, \dot{\rho}_\varepsilon, \gamma_\varepsilon\right)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{s_0}^{s_1} L\left(\rho_\varepsilon(s), \dot{\rho}_\varepsilon(s), \psi_\varepsilon(s), \dot{\psi}_\varepsilon(s), \gamma_\varepsilon(s)\right) ds\Big|_{\varepsilon=0},$$

which upon using the Taylor's series expansion becomes

$$\begin{aligned} \frac{d}{d\varepsilon}\tilde{F}_s\left(\psi_\varepsilon, \dot{\psi}_\varepsilon, \rho_\varepsilon, \dot{\rho}_\varepsilon, \gamma_\varepsilon\right)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{s_0}^{s_1}\left[L\left(\rho, \dot{\rho}, \psi, \dot{\psi}, \gamma\right) + \varepsilon\left(\frac{\partial L}{\partial\psi}\alpha(s) + \frac{\partial L}{\partial\dot{\psi}}\dot{\alpha}(s) + \frac{\partial L}{\partial\rho}\beta(s) + \frac{\partial L}{\partial\dot{\rho}}\dot{\beta}(s) + \frac{\partial L}{\partial\gamma}\tau(s)\right)\right] ds\Big|_{\varepsilon=0} \\ &= \int_{s_0}^{s_1}\left(\frac{\partial L}{\partial\psi}\alpha(s) + \frac{\partial L}{\partial\dot{\psi}}\dot{\alpha}(s) + \frac{\partial L}{\partial\rho}\beta(s) + \frac{\partial L}{\partial\dot{\rho}}\dot{\beta}(s) + \frac{\partial L}{\partial\gamma}\tau(s)\right) ds + \left[\zeta_1 L\Big|_{s_1} - \zeta_0 L\Big|_{s_0}\right]. \end{aligned}$$

Using integral by parts, the above equation simplifies to

$$\frac{d}{d\varepsilon}\tilde{F}_s\left(\psi_\varepsilon,\dot{\psi}_\varepsilon,\rho_\varepsilon,\dot{\rho}_\varepsilon,\gamma_\varepsilon\right)\Bigg|_{\varepsilon=0}=\int_{s_0}^{s_1}\left[\left\{\frac{\partial L}{\partial\psi}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\psi}}\right)\right\}\alpha(s)+\left\{\frac{\partial L}{\partial\rho}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\rho}}\right)\right\}\beta(s)+\frac{\partial L}{\partial\gamma}\tau(s)\right]ds$$
$$+\left[\frac{\partial L}{\partial\dot{\psi}}\alpha(s)\right]_{s_0}^{s_1}+\left[\frac{\partial L}{\partial\dot{\rho}}\beta(s)\right]_{s_0}^{s_1}+\left[\zeta_1 L\Big|_{s_1}-\zeta_0 L\Big|_{s_0}\right].$$

Making the first variation zero gives the equation of the shape of the axisymmetric vesicles. The last four terms give the natural boundary condition. Since $\alpha(s)$, $\beta(s)$, and $\tau(s)$ are independent variations, the integral going to zero means all terms multiplying these variations should go to zero. This gives these three shape equations

$$\begin{aligned}\frac{\partial L}{\partial\psi}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\psi}}\right)&=0,\\ \frac{\partial L}{\partial\rho}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\rho}}\right)&=0,\\ \frac{\partial L}{\partial\gamma}&=0.\end{aligned}\tag{24}$$

Substitute L from (22) to obtain the final form of the shape equation. To derive the final form of the shape equation we need following relations

$$\begin{aligned}\frac{\partial L}{\partial\psi}&=\rho(s)\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)\frac{\cos\psi(s)}{\rho(s)}+\frac{\Delta P}{2k_c}\rho^2(s)\cos\psi(s)+\gamma(s)\sin\psi(s)\\ \frac{\partial L}{\partial\dot{\psi}}&=\rho(s)\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)\\ \frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\psi}}\right)&=\frac{d\rho(s)}{ds}\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)+\rho(s)\left(\frac{d^2\psi(s)}{ds^2}+\frac{\cos\psi(s)}{\rho(s)}\frac{d\psi(s)}{ds}-\frac{\sin\psi(s)}{\rho^2(s)}\frac{d\rho(s)}{ds}\right)\\ \frac{\partial L}{\partial\rho}&=\frac{1}{2}\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)^2-\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)\frac{\sin\psi(s)}{\rho(s)}+\frac{\lambda}{k_c}+\frac{\Delta P}{k_c}\rho(s)\sin\psi(s)\\ \frac{\partial L}{\partial\dot{\rho}}&=\gamma(s)\\ \frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\rho}}\right)&=\frac{d\gamma(s)}{ds}\\ \frac{\partial L}{\partial\gamma}&=\frac{d\rho}{ds}-\cos\psi(s).\end{aligned}\tag{25}$$

Using above relations, we can simplify the shape equations (24) as

$$\frac{\partial L}{\partial\psi}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\psi}}\right)=0$$
$$\rho(s)\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)\frac{\cos\psi(s)}{\rho(s)}+\frac{\Delta P}{2k_c}\rho^2(s)\cos\psi(s)+\gamma(s)\sin\psi(s)$$
$$-\frac{d\rho(s)}{ds}\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)-\rho(s)\left(\frac{d^2\psi(s)}{ds^2}+\frac{\cos\psi(s)}{\rho(s)}\frac{d\psi(s)}{ds}-\frac{\sin\psi(s)}{\rho^2(s)}\frac{d\rho(s)}{ds}\right)=0$$
$$\frac{\Delta P}{2k_c}\rho(s)\cos\psi(s)+\frac{\gamma(s)\sin\psi(s)}{\rho(s)}+\frac{\sin 2\psi(s)}{2\rho^2}-\frac{\cos\psi(s)}{\rho}\frac{d\psi(s)}{ds}=\frac{d^2\psi(s)}{ds^2}\tag{26}$$

$$\frac{\partial L}{\partial\rho}-\frac{d}{ds}\left(\frac{\partial L}{\partial\dot{\rho}}\right)=0$$
$$\frac{1}{2}\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)^2-\left(\frac{d\psi(s)}{ds}+\frac{\sin\psi(s)}{\rho(s)}-c_0\right)\frac{\sin\psi(s)}{\rho(s)}+\frac{\lambda}{k_c}+\frac{\Delta P}{k_c}\rho(s)\sin\psi(s)=\frac{d\gamma(s)}{ds}$$
$$\frac{1}{2}\left(\frac{d\psi(s)}{ds}-c_0\right)^2-\frac{\sin^2\psi(s)}{2\rho^2(s)}+\frac{\lambda}{k_c}+\frac{\Delta P}{k_c}\rho(s)\sin\psi(s)=\frac{d\gamma(s)}{ds}\tag{27}$$

$$\frac{\partial L}{\partial \gamma} = \frac{d\rho}{ds} - \cos\psi(s) = 0$$
$$\frac{d\rho}{ds} = \cos\psi(s). \tag{28}$$

To compare with the solution of Helfrich's equation (18), we change the parameter from s to ρ and combine these equations into one equation. For this purpose we use the following relations

$$\begin{aligned}
\frac{d\psi(s)}{ds} &= \frac{d\psi(\rho)}{d\rho}\frac{d\rho(s)}{ds} = \cos\psi(\rho)\frac{d\psi(\rho)}{d\rho} \\
\frac{d}{ds}\left(\frac{d\psi(s)}{ds}\right) &= \frac{d}{d\rho}\left(\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho}\right)\frac{d\rho(s)}{ds} = \cos\psi(\rho)\frac{d}{d\rho}\left(\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho}\right) \\
&= \cos\psi(\rho)\left[-\sin\psi(\rho)\left(\frac{d\psi(\rho)}{d\rho}\right)^2 + \cos\psi(\rho)\frac{d^2\psi(\rho)}{d\rho^2}\right] \\
\frac{d}{ds}\left(\frac{d^2\psi(s)}{ds^2}\right) &= \cos\psi(\rho)\frac{d}{d\rho}\left\{\cos\psi(\rho)\left[-\sin\psi(\rho)\left(\frac{d\psi(\rho)}{d\rho}\right)^2 + \cos\psi(\rho)\frac{d^2\psi(\rho)}{d\rho^2}\right]\right\} \\
&= \cos\psi(\rho)\Bigg\{-\sin\psi(\rho)\frac{d\psi(\rho)}{d\rho}\left[-\sin\psi(\rho)\left(\frac{d\psi(\rho)}{d\rho}\right)^2 + \cos\psi(\rho)\frac{d^2\psi(\rho)}{d\rho^2}\right] \\
&+ \cos\psi(\rho)\left[-\cos\psi(\rho)\left(\frac{d\psi(\rho)}{d\rho}\right)^3 - 3\sin\psi(\rho)\frac{d^2\psi(\rho)}{d\rho^2}\frac{d\psi(\rho)}{d\rho} + \cos\psi(\rho)\frac{d^3\psi(\rho)}{d\rho^3}\right]\Bigg\} \\
&= \cos\psi(\rho)\left\{(\sin^2\psi(\rho) - \cos^2\psi(\rho))\left(\frac{d\psi(\rho)}{d\rho}\right)^3 - 4\cos\psi(\rho)\sin\psi(\rho)\frac{d^2\psi(\rho)}{d\rho^2}\frac{d\psi(\rho)}{d\rho} + \cos^2\psi(\rho)\frac{d^3\psi(\rho)}{d\rho^3}\right\}.
\end{aligned} \tag{29}$$

To combine these equations, first rewrite (26) in the form

$$\gamma = \rho\csc\psi\frac{d^2\psi}{ds^2} + \cot\psi\frac{d\psi}{ds} - \frac{\cos\psi}{\rho} - \frac{\Delta P\rho^2}{2k_c}\cot\psi$$

and take derivative w.r.t s, which results in

$$\begin{aligned}
\frac{d\gamma}{ds} =& \cot\psi\frac{d^2\psi}{ds^2} - \rho\csc\psi\cot\psi\frac{d^2\psi}{ds^2}\frac{d\psi}{ds} + \rho\csc\psi\frac{d^3\psi}{ds^3} - \csc^2\psi\left(\frac{d\psi}{ds}\right)^2 + \cot\psi\frac{d^2\psi}{ds^2} + \frac{\sin\psi}{\rho}\frac{d\psi}{ds} + \frac{\cos^2\psi}{\rho^2} \\
&- \frac{\Delta P\rho}{k_c}\cos\psi\cot\psi + \frac{\Delta P\rho^2}{2k_c}\csc^2\psi\frac{d\psi}{ds},
\end{aligned}$$

where γ, ρ and ψ are function of s. Finally, use (27),(28) and (29) in the above equation and rearrange to get the second shape equation

$$\begin{aligned}
\cos^3\psi\frac{d^3\psi}{d\rho^3} =& \left(3\sin\psi\cos^2\psi + \frac{\cos^2\psi}{\sin\psi}\right)\frac{d^2\psi}{d\rho^2}\frac{d\psi}{d\rho} - \cos\psi\sin^2\psi\left(\frac{d\psi}{d\rho}\right)^3 + \frac{(2+5\sin^2\psi)\cos^2\psi}{2\rho\sin\psi}\left(\frac{d\psi}{d\rho}\right)^2 \\
&- 2\frac{\cos^3\psi}{\rho}\frac{d^2\psi}{d\rho^2} - \left(\frac{c_0\sin\psi}{\rho} + \frac{\sin^2\psi}{\rho^2} + \frac{\Delta P\rho}{2k_c\sin\psi}\right)\cos\psi\frac{d\psi}{d\rho} \\
&+ \left(\frac{\Delta P}{k_c} + \frac{\lambda\sin\psi}{k_c\rho} + \frac{c_0^2\sin\psi}{2\rho} - \frac{\sin\psi(1+\cos^2\psi)}{2\rho^3}\right).
\end{aligned} \tag{30}$$

3.4 Shape equation using ρ as parameter

We can redo the derivation of the shape equation done in previous section by replacing s with ρ. To avoid the repetition, I have omitted some intermediate steps. The area element dA and the volume element dV using (6) becomes

$$dA = 2\pi\rho d\rho/\cos\psi(\rho),$$
$$dV = \pi\rho^2(s)dz = \pi\rho^2\tan\psi(\rho)d\rho. \tag{31}$$

Using the above relations and (12), the energy functional becomes

$$F_\rho = 2\pi k_c \int_{\rho_0}^{\rho_1} L\left(\rho, \psi(\rho), \frac{d\psi(\rho)}{d\rho}\right) d\rho, \tag{32}$$

where

$$L\left(\rho, \psi(\rho), \frac{d\psi(\rho)}{d\rho}\right) = \frac{\rho}{2\cos\psi(\rho)}\left(\cos\psi(\rho)\frac{d\psi(\rho)}{d\rho} + \frac{\sin\psi(\rho)}{\rho} - c_0\right)^2 + \frac{\lambda\rho}{k_c\cos\psi(\rho)} + \frac{\Delta P}{2k_c\cos\psi(\rho)}\rho^2(s)\sin\psi(\rho). \tag{33}$$

Similar to (23), let us consider the variation of the form

$$\begin{aligned} \psi_\varepsilon(\rho) &= \psi(\rho) + \epsilon\alpha(\rho), \\ \rho_{0\varepsilon} &= \rho_0 + \varepsilon\zeta_0, \\ \rho_{1\varepsilon} &= \rho_1 + \varepsilon\zeta_1 \end{aligned} \tag{34}$$

Calculating the first variation of the functional F_ρ and using the Taylor series expansion and integration by parts as in the previous section, we obtain

$$\frac{d}{d\varepsilon}F_\rho\left(\psi_\varepsilon, \dot{\psi}_\varepsilon, \rho_\varepsilon\right)\Big|_{\varepsilon=0} = \int_{\rho_0}^{\rho_1}\left\{\frac{\partial L}{\partial\psi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial\dot{\psi}}\right)\right\}\alpha(\rho))d\rho + \left[\frac{\partial L}{\partial\dot{\psi}}\alpha(\rho)\right]_{\rho_0}^{\rho_1} + \left[\zeta_1 L\Big|_{\rho_1} - \zeta_0 L\Big|_{\rho_0}\right].$$

Making variation go to zero results in the shape equation

$$\frac{\partial L}{\partial\psi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial\dot{\psi}}\right) = 0 \tag{35}$$

Using the following relations

$$\begin{aligned} \frac{\partial L}{\partial\psi} =& \frac{\rho\tan\psi\sec\psi}{2}\left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right)^2 + \rho\sec\psi\left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right)\left(-\sin\psi\frac{d\psi}{d\rho} + \frac{\cos\psi}{\rho}\right) \\ &+ \frac{\lambda\sec\psi\tan\psi\rho}{k_c} + \frac{\Delta P}{2k_c}\rho^2(s)\sec^2\psi(\rho) \\ \frac{\partial L}{\partial\dot{\psi}} =& \rho\left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right) \\ \frac{d}{d\rho}\left(\frac{\partial L}{\partial\dot{\psi}}\right) =& \left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right) + \rho\left[-\sin\psi\left(\frac{d\psi}{d\rho}\right)^2 + \cos\psi\frac{d^2\psi}{d\rho^2} + \frac{\cos\psi}{\rho}\frac{d\psi}{d\rho} - \frac{\sin\psi}{\rho^2}\right]. \end{aligned} \tag{36}$$

in (35), we obtain the third shape equation

$$\begin{aligned} &\frac{\rho\tan\psi\sec\psi}{2}\left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right)^2 + \rho\sec\psi\left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right)\left(-\sin\psi\frac{d\psi}{d\rho} + \frac{\cos\psi}{\rho}\right) + \frac{\lambda\sec\psi\tan\psi\rho}{k_c} \\ &+ \frac{\Delta P}{2k_c}\rho^2(s)\sec^2\psi(\rho) - \left(\cos\psi\frac{d\psi}{d\rho} + \frac{\sin\psi}{\rho} - c_0\right) - \rho\left[-\sin\psi\left(\frac{d\psi}{d\rho}\right)^2 + \cos\psi\frac{d^2\psi}{d\rho^2} + \frac{\cos\psi}{\rho}\frac{d\psi}{d\rho} - \frac{\sin\psi}{\rho^2}\right] = 0, \end{aligned}$$

which upon rearranging becomes

$$\cos^2\psi\frac{d^2\psi}{d\rho^2} = \frac{\sin\psi\cos\psi}{2}\left(\frac{d\psi}{d\rho}\right)^2 - \frac{\cos^2\psi}{\rho}\left(\frac{d\psi}{d\rho}\right) + \frac{\sin 2\psi}{2\rho^2} + \frac{\Delta P}{2k_c\cos\psi} + \frac{\lambda\sin\psi}{k_c\cos\psi} + \frac{\sin\psi}{2\cos\psi}\left(\frac{\sin\psi}{\rho} - c_0\right)^2. \tag{37}$$

4 Shape equations for special cases

The equations (18),(30) and (37) are three shape equations derived from three different approaches. These equations are different for the general case. The equation (4) is derived using the variation of the form $\delta\boldsymbol{r} = \psi\boldsymbol{n}$, which is not guaranteed for the case of variation considered in section 3.3 and 3.4. However, these equations come out to be the same for special cases.

4.1 Sphere

For the sphere

$$\rho = r_0 \sin\psi \implies \frac{d\psi}{d\rho} = \frac{\sec\psi}{r_0}, \tag{38}$$

where r_0 is the radius of the sphere. The above relations can be used to obtain the following results

$$\begin{aligned}\frac{d^2\psi}{d\rho^2} &= \frac{d}{d\rho}\left(\frac{\sec\psi}{r_0}\right) = \frac{\sec\psi\tan\psi}{r_0}\frac{d\psi}{d\rho} = \frac{\sec^2\psi\tan\psi}{r_0^2},\\ \frac{d^3\psi}{d\rho^3} &= \frac{d}{d\rho}\left(\frac{\sec^2\psi\tan\psi}{r_0^2}\right) = \frac{\sec^5\psi}{r_0^3} + \frac{2\sec^3\psi\tan^2\psi}{r_0^3}.\end{aligned} \tag{39}$$

Substituting, (38) and (39), in (18), (30) and (37) and after applying trigonometric identities, we obtain the same shape equation for all three cases

$$\Delta P r_0^3 + 2\lambda r_0^2 + k_c c_0 r_0^2 - 2k_c c_0 r_0 = 0. \tag{40}$$

This a cubic equation in r_0 and has three solutions

$$r_0 = 0, \qquad r_0 = \frac{-(2\lambda + k_c c_o^2) \pm \sqrt{(2\lambda + k_c c_0^2)^2 + 8k_c \Delta P c_0}}{2\Delta P} \tag{41}$$

4.2 Cylinder

For the cylinder

$$\rho = r_0 \quad \text{and} \quad \psi = \frac{\pi}{2} \implies \frac{d\psi}{d\rho} = \frac{d^2\psi}{d\rho^2} = \frac{d^3\psi}{d\rho^3} = 0. \tag{42}$$

Substituting the above relations in the (18) and (30), results in the same equation

$$\Delta P r_0^3 + \lambda r_0^2 + \frac{k_c}{2}(c_0^2 r_0^2 - 1) = 0. \tag{43}$$

However, substituting (42) in (37) gives

$$\Delta P r_0^3 + 2\lambda r_0^2 + k_c(c_0 r_0 - 1)^2 = 0. \tag{44}$$

[1] and[2] gives the above equation for the cylinder and claims that it is different from the shape equation derived from other methods (equation (43) and (44) are different). However, the equation (37), which results from variation in ρ is not valid for a given cylinder for which $\rho = r_0$ is constant. It can also be observed from (37) where $\cos\psi = 0$ appears in the denominator. Furthermore, dA and dV in (31) is undefined for $\psi = \pi/2$. We can derive the shape equation for the cylinder by restarting from equation (12). From (12), we get

$$H = -\frac{1}{2\rho},$$

using which

$$\begin{aligned}F_\rho = F_b + \lambda A + \Delta P V &= \frac{1}{2}k_c \oint (c_1 + c_2 - c_o)^2 dA + \lambda \oint dA + \Delta P \oint dV\\ &= \frac{k_c}{2}\oint \left(\frac{1}{\rho} - c_0\right)^2 dA + \int \Delta P dV + \oint \lambda dA\\ &= \frac{k_c}{2}\left(\frac{1}{\rho} - c_0\right)^2 2\pi\rho L + \Delta P \psi \rho^2 L + \lambda 2\pi L.\end{aligned}$$

For the F_b to be minimum,

$$\frac{dF_b}{d\rho} = 0 \implies \Delta P r_0^3 + \lambda r_0^2 + \frac{k_c}{2}(c_0^2 r_0^2 - 1) = 0, \tag{45}$$

which is exactly equal to (43).

Solving the three shape equations for the torus gives the same ratio of generating radii equal to $1/\sqrt{2}$ which is a Clifford torus. However λ, and ΔP comes out to be different for all these three equations.

5 Shape equation with different parameters

For the parameter ρ, the functional is of the form given in (32). Now change the parameter from ρ to s. Then the functional modifies to

$$F_s = 2\pi k_c \int_{s_0}^{s_1} L'\left(\rho(s), \frac{d\rho(s)}{ds}, \psi(s), \frac{d\psi(s)}{ds}\right) ds, \tag{46}$$

where

$$L'\left(\rho(s), \frac{d\rho(s)}{ds}, \psi(s), \frac{d\psi(s)}{ds}\right) = L\left(\rho, \psi(\rho), \frac{d\psi(\rho)}{d\rho}\right)\frac{d\rho}{ds}. \tag{47}$$

The Euler-Lagrange equation can be obtained by replacing L by L' and ρ by s in (35). Now, substituting L' from (47), we obtain the final form of Euler-Lagrange equation

$$\frac{d}{ds}\left[\frac{\partial\left(L\frac{d\rho}{ds}\right)}{\partial\left(\frac{d\psi(s)}{ds}\right)}\right] - \frac{\partial\left(L\frac{d\rho}{ds}\right)}{\partial\psi(s)} = 0$$

$$\frac{d}{ds}\left[\frac{\partial L}{\partial\left(\frac{d\psi(s)}{ds}\right)}\frac{d\rho}{ds} + \frac{\partial\left(\frac{d\rho}{ds}\right)}{\partial\left(\frac{d\psi(s)}{ds}\right)}L\right] - \frac{\partial L}{\partial\psi(s)}\frac{d\rho}{ds} + \frac{\partial\left(\frac{d\rho}{ds}\right)}{\partial\psi(s)}L = 0. \tag{48}$$

In cases where $d\rho/ds$ is independent of ψ, the equation reduces to

$$\frac{d}{ds}\left[\frac{\partial L}{\partial\left(\frac{d\psi(s)}{ds}\right)}\frac{d\rho}{ds}\right] - \frac{\partial L}{\partial\psi(s)}\frac{d\rho}{ds} = 0,$$

$$\frac{d}{d\rho}\left[\frac{\partial L}{\partial\left(\frac{d\psi(\rho)}{d\rho}\right)}\frac{\partial\left(\frac{d\psi(\rho)}{d\rho}\right)}{\partial\left(\frac{d\psi(s)}{ds}\right)}\frac{d\rho}{ds}\right]\frac{d\rho}{ds} - \frac{\partial L}{\partial\psi(s)}\frac{d\rho}{ds} = 0,$$

$$\frac{d}{d\rho}\left[\frac{\partial L}{\partial\left(\frac{d\psi(\rho)}{d\rho}\right)}\frac{\partial\left(\frac{d\psi(s)}{ds}\frac{ds}{d\rho}\right)}{\partial\left(\frac{d\psi(s)}{ds}\right)}\frac{d\rho}{ds}\right]\frac{d\rho}{ds} - \frac{\partial L}{\partial\psi(s)}\frac{d\rho}{ds} = 0,$$

$$\frac{d}{d\rho}\left[\frac{\partial L}{\partial\left(\frac{d\psi(\rho)}{d\rho}\right)}\frac{ds}{d\rho}\frac{d\rho}{ds}\right]\frac{d\rho}{ds} - \frac{\partial L}{\partial\psi(\rho)}\frac{d\rho}{ds} = 0,$$

$$\frac{d}{d\rho}\left[\frac{\partial L}{\partial\left(\frac{d\psi(\rho)}{d\rho}\right)}\right] - \frac{\partial L}{\partial\psi(\rho)} = 0, \tag{49}$$

which is exactly equal to the Euler-Lagrange equation (35) for the parameter ρ. However, in our case $d\rho/ds = \cos\psi$ and hence $d\rho/ds$ depends on ψ. This results in different Euler-Lagrange equations for parameters s and ρ.

The Lagrangian L' given in (47) is from equation (25) from [1]. However, when $\rho(s)$ and $\psi(s)$ are not independent, it will result in a modified Lagrangian, which should incorporate the relation between $\rho(s)$ and $\psi(s)$ as in (22). Therefore the above analysis should not be valid when the variation in $\rho(s)$ and $\psi(s)$ are not independent.

6 Variation in the tangent plane

Any general variation of the position vector, $\delta\boldsymbol{r}$ can be decomposed into three directions as

$$\delta\boldsymbol{r}(s,\phi) = \psi(s,\phi)\boldsymbol{n} + \alpha_1(s,\phi)\frac{\partial}{\partial s}\boldsymbol{r}(s,\phi) + \alpha_2(s,\phi)\frac{\partial}{\partial \phi}\boldsymbol{r}(s,\phi).$$

If we consider variation only in the tangent plane then the above relation can be expressed as

$$\boldsymbol{r}'(s,\phi) = \boldsymbol{r}(s,\phi) + \alpha_1(s,\phi)\frac{\partial}{\partial s}\boldsymbol{r}(s,\phi) + \alpha_2(s,\phi)\frac{\partial}{\partial \phi}\boldsymbol{r}(s,\phi). \tag{50}$$

Now if we choose a second set of parameters

$$s' = s + \alpha_1(s,\phi) \qquad \text{and} \qquad \phi' = \phi + \alpha_2(s,\phi),$$

and express $\boldsymbol{r}$ in this new parameter space using Taylor series then we obtain

$$\begin{aligned}
\boldsymbol{r}(s',\phi') &= \boldsymbol{r}(s,\phi) + (s'-s)\frac{\partial \boldsymbol{r}(s',\phi')}{\partial s'}\bigg|_{\substack{s'=s\\ \phi'=\phi}} + (\phi'-\phi)\frac{\partial \boldsymbol{r}(s',\phi')}{\partial \phi'}\bigg|_{\substack{s'=s\\ \phi'=\phi}} + o(\alpha_1,\alpha_2)\\
&= \boldsymbol{r}(s,\phi) + (s'-s)\left(\frac{\partial \boldsymbol{r}(s',\phi')}{\partial s}\frac{\partial s}{\partial s'} + \frac{\partial \boldsymbol{r}(s',\phi')}{\partial \phi}\frac{\partial \phi}{\partial s'}\right)\bigg|_{\substack{s'=s\\ \phi'=\phi}} + (\phi'-\phi)\left(\frac{\partial \boldsymbol{r}(s',\phi')}{\partial s}\frac{\partial s}{\partial \phi'}\right.\\
&\left. + \frac{\partial \boldsymbol{r}(s',\phi')}{\partial \phi}\frac{\partial \phi}{\partial \phi'}\right)\bigg|_{\substack{s'=s\\ \phi'=\phi}} + o(\alpha_1,\alpha_2)\\
&= \boldsymbol{r}(s,\phi) + \alpha_1(s,\phi)\left[\frac{\partial \boldsymbol{r}(s',\phi')}{\partial s}\left(1 - \frac{\partial \alpha_1(s,\phi)}{\partial s'}\right) - \frac{\partial \boldsymbol{r}(s',\phi')}{\partial \phi}\frac{\partial \alpha_2(s,\phi)}{\partial s'}\right]\bigg|_{\substack{s'=s\\ \phi'=\phi}} +\\
&\alpha_2(s,\phi)\left[-\frac{\partial \boldsymbol{r}(s',\phi')}{\partial s}\frac{\partial \alpha_2(s,\phi)}{\partial \phi'} + \frac{\partial \boldsymbol{r}(s',\phi')}{\partial \phi}\left(1 - \frac{\partial \alpha_2(s,\phi)}{\partial \phi'}\right)\right]\bigg|_{\substack{s'=s\\ \phi'=\phi}} + o(\alpha_1,\alpha_2)\\
&= \boldsymbol{r}(s,\phi) + \alpha_1(s,\phi)\frac{\partial \boldsymbol{r}(s,\phi)}{\partial s} + \alpha_2(s,\phi)\frac{\partial \boldsymbol{r}(s,\phi)}{\partial \phi} + o(\alpha_1,\alpha_2).
\end{aligned} \tag{51}$$

From comparing (50) and (51), we get

$$\boldsymbol{r}'(s,\phi) = \boldsymbol{r}(s',\phi') + o(\alpha_1,\alpha_2),$$

when variation is considered only in the tangent plane. Therefore, any variation in the tangent plane is equivalent to the reparameterization of the surface (up to the linear order in variation). This happens because of the fluid nature of the membrane. The phospholipid molecules are allowed to move along the surface. Hence variations only along the normal direction play a role in deriving the shape equation.

7 Axisymmetric energy functional

Helfrich's equation is derived using the general energy functional with the variation of the form $\delta\boldsymbol{r} = \psi\boldsymbol{n}$. To find the shape equation for the axisymmetric vesicles, we later substituted the Gaussian and mean curvatures of the axisymmetric vesicles. A simpler way to derive the shape equation for the axisymmetric vesicles is to use axisymmetric energy functional instead of general energy functional as done in 3.3 and3.4 and then use the variation of the form $\delta\boldsymbol{r} = f\boldsymbol{n}$.

Figure 4 shows the original curve and the curve after the variation δf in the normal direction. The figure shows the normal at the arc length s, which intersects the new curve at arc length $s' = s + ds$. The coordinates of points A, B, C and D are

$$\boldsymbol{r}_A = [\rho(s), z(s)], \quad \boldsymbol{r}_B = [\rho(s) + \cos\psi(s)ds, z(s) - \sin\psi(s)ds], \quad \boldsymbol{r}_c = [\rho(s) + \delta f(s)\sin\psi(s), z(s) + \delta f(s)\cos\psi(s)]$$
$$\text{and} \qquad \boldsymbol{r}_D = [\rho(s+ds) + \delta f(s+ds)\sin\psi(s+ds), z(s+ds) + \delta f(s+ds)\cos\psi(s+ds)] \tag{52}$$

If ρ' is the distance from z axis for new curve then

$$\rho'(s+ds) = \rho(s) + \delta f \sin\psi(s).$$

Therefore the variation of ρ becomes

$$\begin{aligned}\delta\rho(s+ds) = \rho'(s+ds) - \rho(s+ds) &= \rho(s) + \delta f(s)\sin\psi(s) - \rho(s) - \cos\psi(s)ds + o(ds)\\ &= \delta f(s)\sin\psi(s) - \cos\psi(s)ds + o(ds).\end{aligned}$$

For $ds \to 0$, the above equation becomes

$$\delta\rho(s) = \delta f(s)\sin\psi(s). \tag{53}$$

Similarly, from Fig. 4

$$\tan(\psi(s) + \delta\psi(s)) = \frac{z(s) + \delta f(s)\cos\psi(s) - z(s+ds) - \delta f(s+ds)\cos\psi(s+ds)}{\rho(s+ds) + \delta f(s+ds)\sin\psi(s+ds) - \rho(s) - \delta f(s)\sin\psi(s)},$$

which upon using Taylor series reduces to

$$\tan\psi(s) + \sec^2\psi(s)\delta\psi(s) = \frac{\sin\psi(s)ds + \delta f(s)\sin\psi(s)d\psi(s) - \cos\psi(s)d(\delta f(s)}{\cos\psi(s)ds + \delta f(s)\cos\psi(s)d\psi(s) + \sin\psi(s)d(\delta f(s)} + o(ds\delta f(s)) + o(ds),$$

which simplifies to

$$\delta\psi(s) = -\frac{d(\delta f(s))}{ds} + h.o.t. \tag{54}$$

The variation in the infinitesimal arc length ds is given by

$$\begin{aligned}\delta(ds) &= \sqrt{(\boldsymbol{r}_C - \boldsymbol{r}_D)\cdot(\boldsymbol{r}_C - \boldsymbol{r}_D)} - \sqrt{(\boldsymbol{r}_A - \boldsymbol{r}_B)\cdot(\boldsymbol{r}_A - \boldsymbol{r}_B)},\\ &= \sqrt{(\boldsymbol{r}_C - \boldsymbol{r}_D)\cdot(\boldsymbol{r}_C - \boldsymbol{r}_D)} - ds,\end{aligned}$$

which upon using (52) and Taylor series expansion becomes

$$\begin{aligned}\delta(ds) &= \sqrt{[\sin\psi(s)ds + \delta f(s)\sin\psi(\psi) - \cos\psi(s)d(\delta f(s))]^2 + [\cos\psi(s)ds + \delta f(s)\cos\psi(\psi) + \sin\psi(s)d(\delta f(s))]^2}\\ &\qquad - ds + h.o.t\\ &= ds\left(1 + \delta f(s)\frac{d\psi(s)}{ds}\right) - ds + h.o.t = d\psi(s)\delta f(s) + h.o.t.\end{aligned} \tag{55}$$

To find the first variation of axisymmetric energy functional, we introduce a parameter t such that the energy functional becomes

$$\begin{aligned}E &= 2\pi k_c \int_{s_0}^{s_1} L(\rho(s), \psi(s), \dot{\psi}(s))ds,\\ &= 2\pi k_c \int_{t_0}^{t_1} L(\rho(s), \psi(s), \dot{\psi}(s))\frac{ds}{dt}dt\\ &= 2\pi k_c \int_{t_0}^{t_1} L(\rho(t), \psi(t), \dot{\psi}(t), \dot{s}(t))dt\end{aligned} \tag{56}$$

where dot represents derivative w.r.t argument of the function and

$$L(\rho(t),\psi(t),\dot{\psi}(t),\dot{s}(t)) = \left[\frac{\rho(t)}{2}\left(\frac{d\psi(t)}{ds(t)}+\frac{\sin\psi(t)}{\rho(t)}-c_0\right)^2+\frac{\lambda\rho(t)}{k_c}+\frac{\Delta P}{2k_c}\rho^2(t)\sin\psi(t)\right]\frac{ds(t)}{dt}$$
$$= \left[\frac{\rho(t)}{2}\left(\frac{\dot{\psi}(t)}{\dot{s}(t)}+\frac{\sin\psi(t)}{\rho(t)}-c_0\right)^2+\frac{\lambda\rho(t)}{k_c}+\frac{\Delta P}{2k_c}\rho^2(t)\sin\psi(t)\right]\dot{s}(t). \tag{57}$$

Setting the first variation of E to zero, we get

$$\delta\int_{t_0}^{t_1} L(\rho(t),\psi(t),\dot{\psi}(t),\dot{s}(t))dt = 0$$
$$\int_{t_0}^{t_1}\delta L(\rho(t),\psi(t),\dot{\psi}(t),\dot{s}(t))dt + \left(\delta t_1 L\big|_{t=t_1}-\delta t_0 L\big|_{t=t_0}\right)=0$$
$$\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial\rho}\delta\rho+\frac{\partial L}{\partial\psi}\delta\psi+\frac{\partial L}{\partial\dot{\psi}}\delta\dot{\psi}+\frac{\partial L}{\partial\dot{s}}\delta\dot{s}\right]dt+\left(\delta t_1 L\big|_{t=t_1}-\delta t_0 L\big|_{t=t_0}\right)=0 \tag{58}$$

From the equations (53),(54) and (55), we have

$$\delta\rho=\delta f\sin\psi,\quad \delta\psi=-\frac{d\delta f}{ds}=-\frac{d\delta f}{dt}\frac{dt}{ds}=-\frac{\delta\dot{f}}{\dot{s}},\quad \delta\dot{\psi}=\frac{d\delta\psi}{dt}=-\frac{\delta\ddot{f}}{\dot{s}}+\frac{\delta\dot{f}}{\ddot{s}}\quad\text{and}\quad \delta\dot{s}=\dot{\psi}\delta f. \tag{59}$$

Substituting the above relations in (58) and neglecting the boundary conditions we obtain

$$\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial\rho}\delta f\sin\psi-\frac{\partial L}{\partial\psi}\frac{\delta\dot{f}}{\dot{s}}+\frac{\partial L}{\partial\dot{\psi}}\left(\frac{-\delta\ddot{f}}{\dot{s}}+\frac{\delta\dot{f}}{\ddot{s}}\right)+\frac{\partial L}{\partial\dot{s}}\delta f\dot{\psi}\right]dt=0,$$
$$\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial\rho}\delta f\sin\psi+\frac{d}{dt}\left(\frac{1}{\dot{s}}\frac{\partial L}{\partial\psi}\right)\delta f+\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right)\left(\frac{\delta\dot{f}}{\dot{s}}\right)+\frac{\partial L}{\partial\dot{s}}\delta f\dot{\psi}\right]dt-\left[\frac{\delta f}{\dot{s}}\frac{\partial L}{\partial\psi}\right]_{t_0}^{t_1}-\left[\left(\frac{\partial L}{\partial\dot{\psi}}\right)\left(\frac{\delta\dot{f}}{\dot{s}}\right)\right]_{t_0}^{t_1}=0,$$
$$\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial\rho}\sin\psi+\frac{d}{dt}\left(\frac{1}{\dot{s}}\frac{\partial L}{\partial\psi}\right)-\frac{d}{dt}\left(\frac{1}{\dot{s}}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right)\right)+\frac{\partial L}{\partial\dot{s}}\dot{\psi}\right]\delta f dt+\left[\frac{\delta f}{\dot{s}}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right)-\frac{\partial L}{\partial\psi}\right)\right]_{t_0}^{t_1}-\left[\left(\frac{\partial L}{\partial\dot{\psi}}\right)\left(\frac{\delta\dot{f}}{\dot{s}}\right)\right]_{t_0}^{t_1}=0.$$

Therefore, the shape equation becomes

$$\frac{\partial L}{\partial\rho}\sin\psi+\frac{d}{dt}\left[\frac{1}{\dot{s}}\left(\frac{\partial L}{\partial\psi}-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right)\right)\right]+\frac{\partial L}{\partial\dot{s}}\dot{\psi}=0. \tag{60}$$

Finally, using the following relations

$$\frac{\partial L}{\partial\psi}=\left[\rho\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)\frac{\cos\psi}{\rho}+\frac{\Delta P}{2k_c}\rho^2\cos\psi\right]\dot{s}$$
$$\frac{\partial L}{\partial\dot{\psi}}=\rho\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)$$
$$\frac{\partial L}{\partial\rho}=\left[\frac{1}{2}\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)^2-\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)\frac{\sin\psi}{\rho}+\frac{\lambda}{k_c}+\frac{\Delta P}{k_c}\rho\cos\psi\right]\dot{s}$$
$$\frac{\partial L}{\partial\dot{s}}=\left[-\rho\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)\frac{\dot{\psi}}{\dot{s}^2}+\frac{\rho}{2}\left(\frac{\dot{\psi}}{\dot{s}}+\frac{\sin\psi}{\rho}-c_0\right)^2+\frac{\lambda\rho}{k_c}+\frac{\Delta P}{2k_c}\rho^2\sin\psi\right], \tag{61}$$

in (60) and rearranging, we get the final shape equation

$$\begin{aligned}\frac{k_c\rho}{\dot{s}^2}\dddot{\psi} =& \frac{3k_c\rho\ddot{s}}{\dot{s}^3}\ddot{\psi} - \frac{2k_c\dot{\rho}}{\dot{s}^2}\ddot{\psi} + \frac{k_c\sin\psi}{2\dot{s}}\dot{\psi}^2 - \frac{k_c\rho}{2\dot{s}^2}\dot{\psi}^3 + \left(\frac{k_c(2-3\sin^2\psi)}{2\rho} - \frac{k_c\ddot{\rho}}{\dot{s}^2} + \frac{k_c\rho\dddot{s}}{\dot{s}^3} - \frac{3k_c\rho\ddot{s}^2}{\dot{s}^4}\right.\\ &\left. + \frac{3k_c\dot{\rho}\ddot{s}}{\dot{s}^3} - c_0k_c\sin\psi + \frac{c_0^2k_c}{2}\rho + \lambda\rho\right)\dot{\psi} + \left(\Delta P\rho\cos\psi\dot{\rho} - \frac{k_c\sin 2\psi}{2\rho^2}\dot{\rho} + c_0k_c\frac{\ddot{\rho}}{\dot{s}} - c_0k_c\frac{\dot{\rho}\ddot{s}}{\dot{s}}\right)\\ &+ \dot{s}\left(-\frac{k_c\sin^2\psi}{2\rho^2} + \frac{c_0^2k_c}{2} + \lambda + \Delta P\rho\sin\psi\right)\sin\psi. \end{aligned} \tag{62}$$

The above equation is true for any parameter t. By substituting ρ as t and using the following relations

$$\dot{\rho} = 1, \quad \dot{s} = \frac{ds}{d\rho} = \sec\psi, \tag{63}$$

we obtain the original shape equation (18) from Helfrich's model. Therefore, the equations are the same whether we use the axisymmetric energy or use energy for general shape and put axisymmetric conditions later.

8 Conclusion

Deriving shape equations using different variations and parameters results in different equations for the axisymmetric case. For the sphere, all these equations come out to be the same. For cylinders, two equations are the same. Finally, we also arrive at the same equation using the axisymmetric energy functional instead of the general energy functional in Helfrich's model.

References

[1] Hu Jian-Guo and Ou-Yang Zhong-Can. Shape equations of the axisymmetric vesicles. *Phys. Rev. E*, 47:461–467, Jan 1993.

[2] Zhanchun Tu, Zhongcan Ou-Yang, Jixing Liu, and Yuzhang Xie. *Geometric Methods in Elastic Theory of Membranes in Liquid Crystal Phases*. WORLD SCIENTIFIC, 2nd edition, 2018.

[3] Ou-Yang Zhong-can and W. Helfrich. Instability and deformation of a spherical vesicle by pressure. *Phys. Rev. Lett.*, 59:2486–2488, Nov 1987.

[4] Ou-Yang Zhong-can and Wolfgang Helfrich. Bending energy of vesicle membranes: General expressions for the first, second, and third variation of the shape energy and applications to spheres and cylinders. *Phys. Rev. A*, 39:5280–5288, May 1989.

Figures

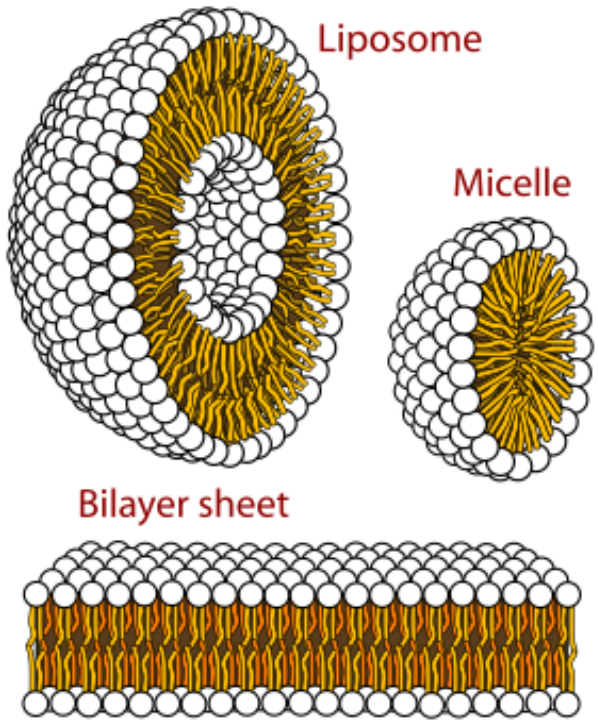

Figure 1: Bilayer membranes. Source: Google

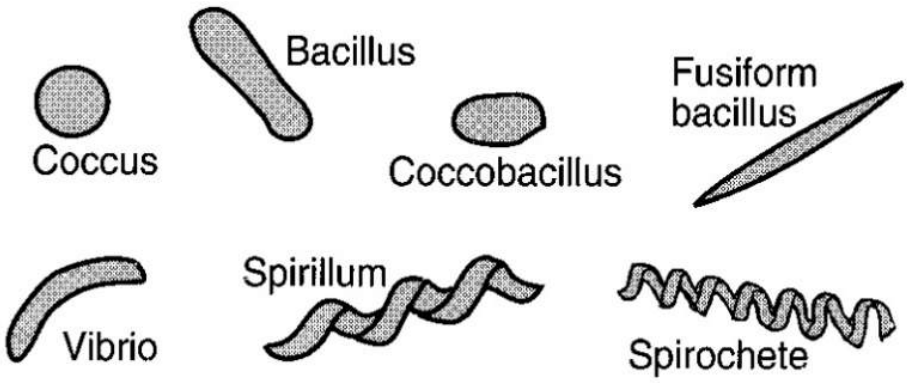

Figure 2: Different shapes of the bacteria. Source: Google

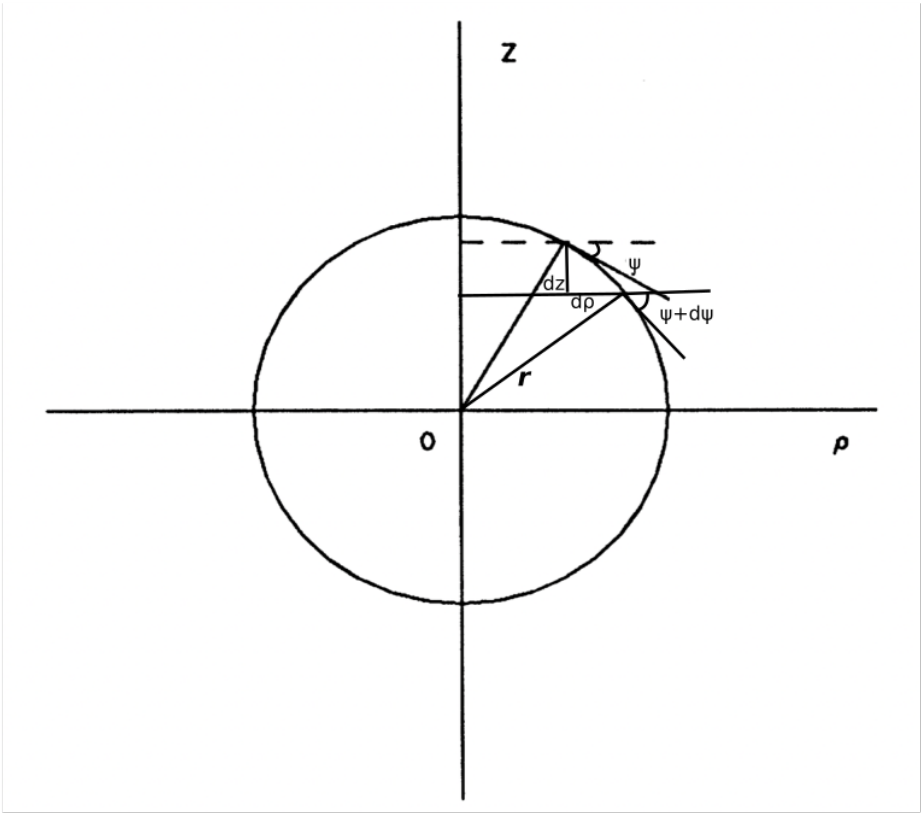

Figure 3: Schematic of axisymmetric vesicle

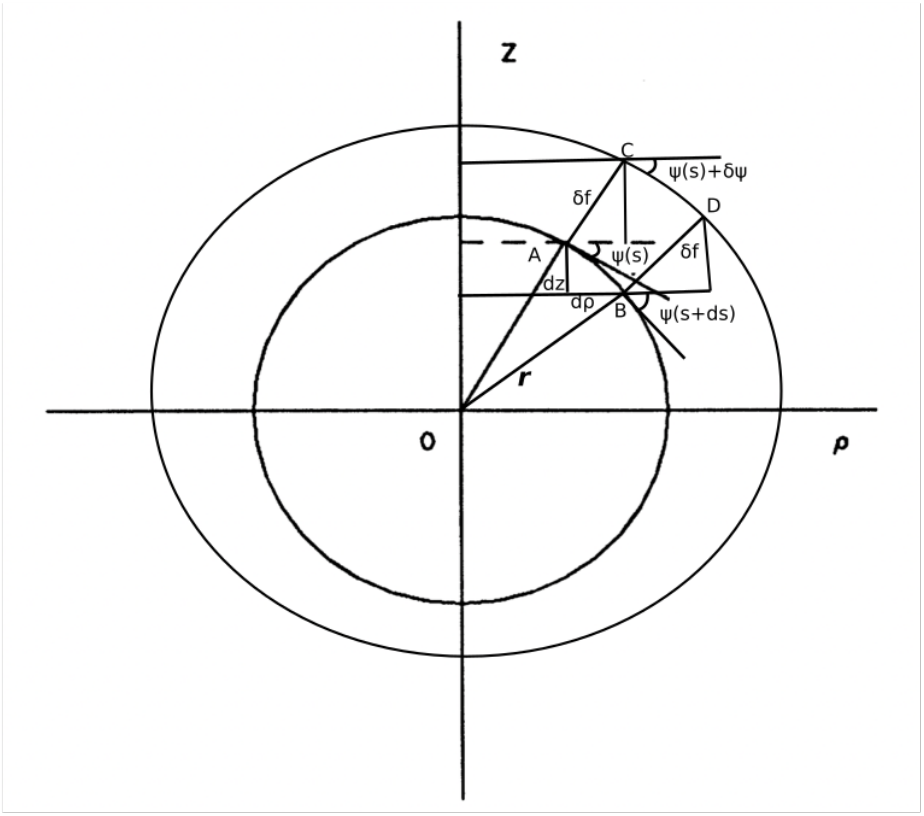

Figure 4: Schematic of axisymmetric vesicle with variation only in normal direction

Tables