Finite Volume Methods for Hyperbolic Problems[1]

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[1]LeVeque, R. (2002). Finite Volume Methods for Hyperbolic Problems (Cambridge Texts in Applied Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511791253

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Introduction

- Hyperbolic systems of partial differential equations can be used to model a wide variety of phenomena that involve wave motion or the advective transport of substances.
- An example of homogeneous first order PDE is

$$\boldsymbol{q}_t + \boldsymbol{A}(x)\boldsymbol{q}_x = 0, \tag{1}$$

  where $\boldsymbol{q}$ is a vector and $\boldsymbol{A}$ is a matrix.
- The Finite Volume Method (FVM) is a numerical technique that transforms the partial differential equations representing conservation laws over differential volumes into discrete algebraic equations over finite volumes (or elements or cells).

Advection

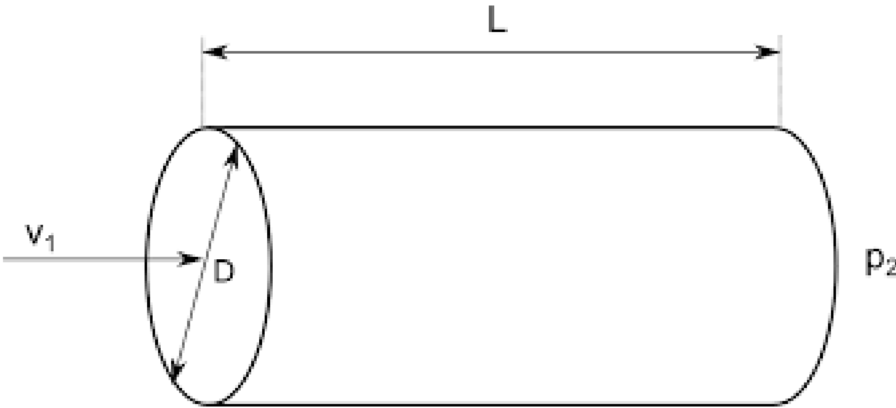


Figure: Flow in a pipe

Advection Equation

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = F_1(t) - F_2(t)$$

Where x_1, x_2 are boundary and $F_1(t)$ and $F_2(t)$ are fluxes at the boundary. $q(x,t)$ is the density of chemical tracer.
This the basic integral form of **conservation law**.
Flux=$f(q,x,t) = u(x,t)q$
If the velocity $\bar{u}$ is independent of x and t, then flux can be written Flux=f(q)=$\bar{u}q$
In this case the flux at any point and time can be determined directly from the value of the conserved quantity at that point, and does not depend at all on the location of the point in spacetime.
This type of equation is called **Autonomous equation**.

Advection equation

For a general autonomous flux

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t)) - f(q(x_2,t))$$

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -f(q(x,t))\Big|_{x_1}^{x_2}$$

If we assume that q and f are smooth functions, then this equation can be rewritten as

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x,t))dx$$

With further modification,as

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} f(q(x,t))\right] dx = 0$$

Advection Equation

$$\frac{\partial}{\partial t}q(x,t) + \frac{\partial}{\partial x}f(q(x,t)) = 0$$

$$q_t(x,t) + f(q(x,t))_x = 0$$

For flux=$f(q,x,t) = \bar{u}q$,

$$q_t + \bar{u}q_x = 0 \qquad \text{(Advection Equation)}$$

Advection equation is a scalar, linear, constant-coefficient PDE of hyperbolic type. Any smooth function of the form

$$q(x,t) = \tilde{q}(x - \bar{u}t)$$

satisfies the differential equation.

Characteristics of Equation

- Along the ray $X(t) = x_0 + \bar{u}t$ the value of $q(X(t), t)$ is equal to $\tilde{q}(x_0)$.
- Values of q simply advect (i.e., translate) with constant velocity $\bar{u}$, as we would expect physically, since the fluid in the pipe (and hence the density of tracer moving with the fluid) is simply advecting with constant speed.
- These rays X(t) are called the *characteristics* of the equation. More generally, characteristic curves for a PDE are curves along which the equation simplifies in some particular manner.

Characteristics

Along X(t) equation reduces to $\frac{d}{dt}q(X(t),t) = 0$

$$\begin{aligned}\frac{d}{dt}q(X(t),t) &= q_t(X(t),t) + X'(t)q_x(X(t),t)\\ &= q_t + \bar{u}q_x\\ &= 0.\end{aligned}$$

Solution

We need initial conditions and boundary conditions for solving advection equation

For infinitely long pipe and with initial conditon

$$q(x, t_0) = \mathring{q}(x)$$

q is given as

$$q(x, t) = \mathring{q}(x - \bar{u}(t - t_0))$$

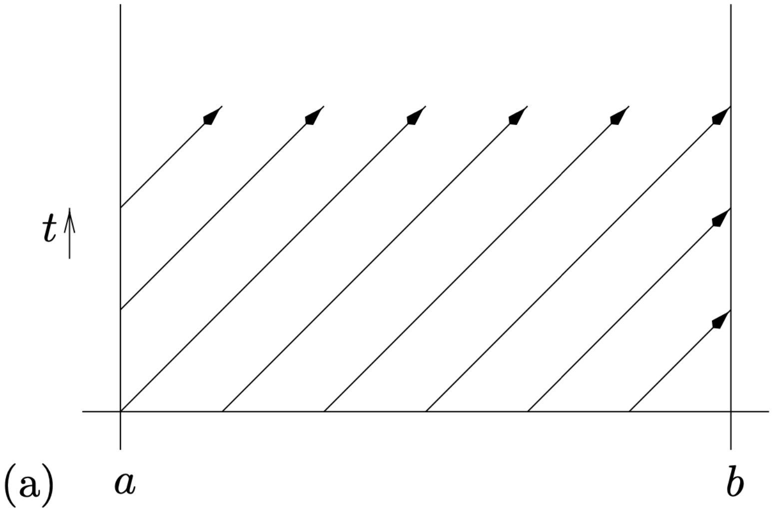
If pipe has a finite length $a < x < b$ and $\bar{u} > 0$, then we must specify value at x=a

$$q(a, t) = g_0(t)$$

The solution is then given by

$$q(x, t) = \begin{cases} g_0(t - (x - a)/\bar{u}) & \text{if } a \leq x \leq a + \bar{u}(t - t_0), \\ \mathring{q}(x - \bar{u}(t - t_0)) & \text{if } a + \bar{u}(t - t_0) \leq n \leq b \end{cases}$$

Solution using characteristics



Variable Coefficients

If the fluid velocity u varies with x, then the conservation law becomes

$$q_t + (u(x)q)_x = 0$$

In this case the characteristic curves X(t) are solutions to the ordinary differential equations

$$X'(t) = u(X(t))$$

This simplifies the above equation as

$$\frac{d}{dt} q(X(t), t) = -u'(X(t))q(X(t), t)$$

Note that when u is not constant, the curves are no longer straight lines and the solution q is no longer constant along the curves, but still the original partial differential equation has been reduced to solving sets of ordinary differential equations

Advection-Diffusion Equation

Density q also changes due to molecular diffusion if density is not same everywhere. Ficks law of diffusion states that the net flux is proportional to the gradient of q.
Flux=$f(q, q_x) = \bar{u}q - \beta q_{xx}$
Where β is the diffusion coefficient. Putting this flux in conservation law

$$q_t(x,t) + f(q(x,t))_x = 0$$

We get **Advection-diffusion equation**

$$q_t + \bar{u}q_x = \beta q_x$$

These are second order parabolic PDEs

Source Term

Conservation law with source term $\psi(q(x,t),x,t)$

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x,t))dx + \int_{x_1}^{x_2} \psi(q(x,t),x,t)dx$$

this leads to the PDE

$$q_t(x,t) + f(q(x,t))_x = \psi(q(x,t),x,t)$$

Non-linear Equation in Fluid Dynamics

$$q_t(x,t) + f(q(x,t))_x = 0$$

writing density as ρ and flux as ρu, we get

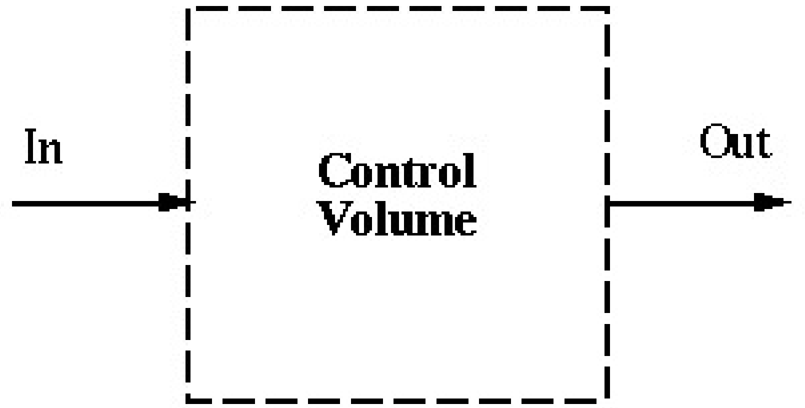
$$\rho_t + (\rho u)_x = 0 \qquad \text{(Continuity Equation)}$$

In addition to this equation we now need a second equation for the velocity. The velocity itself is not a conserved quantity, but the momentum is. The product ρ(x, t)u(x, t) gives the density of momentum.

Non-linear Equations in Fluid Dynamics

- The momentum flux past any point x consists of two parts.
- First there is momentum carried past this point along with the moving fluid. For any density function q this flux has the form qu.
- For the momentum $q = \rho u$ this contribution to the flux is $(\rho u)u = \rho u^2$
- This is essentially an advective flux, although in the case where the quantity being advected is the velocity or momentum of the fluid itself, the phenomenon is often referred to as **convection** rather than advection.

- In addition to this macroscopic convective flux, there is also a microscopic momentum flux due to the pressure of the fluid.

- 

Therefore momentum flux becomes
Momentum flux$=\rho u^2 + p$
This gives **Conservation of Momentum**

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

- We have a new variable here, pressure. To solve these equation we need a new conservation law. Since, pressure is not a conserved quantity we introduce a fourth variable, the energy, and an additional equation for the **conservation of energy**. The density of energy will be denoted by E(x, t).
- This still does not determine the pressure, and to close the system we must add an **equation of state**, an algebraic equation that determines the pressure at any point in terms of the mass, momentum, and energy at the point.

Isentropic flow

In Isentropic flow we have a very simple equation of state that detemines p from ρ alone. So, our equations for this case becomes:

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = 0$$

This is a coupled system of two nonlinear conservation laws, which we can write in the form

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0$$

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}, \mathbf{f}(\mathbf{q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p(\rho) \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)/q^1 + p(q^1) \end{bmatrix}$$

Quasi linear Form

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0$$

$$\mathbf{q}_t + \mathbf{f}'(\mathbf{q})\mathbf{q}_x = 0$$

This is called **Quasilinear form** of the equation because it resembles the linear system

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0$$

In the linear case this matrix does not depend on q, while in the quasilinear equation it does.

Linear Acoustics

- An acoustic wave is a very small pressure disturbance that propagates through the compressible gas, causing infinitesimal changes in the density and pressure of the gas via small motions of the gas with infinitesimal values of the velocity u.
- The magnitudes of disturbances from the background state are so small that products or powers of the perturbation amplitude can be ignored.

Linearization:

$$\mathbf{q}(x, t) = \mathbf{q_0} + \tilde{\mathbf{q}}(x, t)$$

Where $\mathbf{q_0} = (\rho_0, \rho_0 u_0)$ is the background state and $\tilde{\mathbf{q}}$ is the perturbation.

Linearized equations:

$$\tilde{\mathbf{q}}_t + \mathbf{f}'(\mathbf{q_0})\tilde{\mathbf{q}}_x = 0$$

where $\mathbf{A} = \mathbf{f}'(\mathbf{q_0}) = \begin{bmatrix} 0 & 1 \\ -u_0^2 + p'(\rho_0) & 2u \end{bmatrix}$

Physically it is often more natural to model perturbations in velocity and pressure, since these can often be measured directly. To obtain such equations, pressure perturbations can be related to density perturbations through the equation of state.

Final Acoustics equations are

$$\begin{bmatrix} p \\ u \end{bmatrix}_t + \begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}_x = 0$$

Here and from now on we will generally drop the tilde on p and u. Where $K_0 = \rho_0 p'(\rho_0)$ is the bulk modulus.The bulk modulus of a substance is a measure of how resistant to compressibility that substance is. It is defined as the ratio of the infinitesimal pressure increase to the resulting relative decrease of the volume.

Sound Waves

If we solve the equations just obtained for linear acoustics in a stationary gas, we expect the solution to consist of sound waves propagating to the left and right.
When we put the Ansatz $\mathbf{q}(x,t) = \bar{\mathbf{q}}(x - st)$, for some speed s in linear acoustics equation we get,

$$\mathbf{A}\bar{\mathbf{q}}'(x - st) = s\bar{\mathbf{q}}'(x - st)$$

Since s is a scalar while **A** is a matrix, this is only possible if s is an eigenvalue of the matrix **A**, and $\bar{\mathbf{q}}'$ must also be a corresponding eigenvector of **A**
Eigen values of **A** are $\lambda^1 = -c_0$ and $\lambda^2 = c_0$, where $c_0 = \sqrt{K_0/\rho_0}$ is the speed of sound in air.

The eigenvectors of **A** are

$$r^1 = \begin{bmatrix} -Z_0 \\ 1 \end{bmatrix}, r^2 = \begin{bmatrix} Z_0 \\ 1 \end{bmatrix}$$

Where $Z_0 = \rho_0 c_0$, is called the impedance of the medium.
Now, sound wave propagating to the left with velocity c_0 is given by

$$\mathbf{q}(x,t) = \bar{w}^1(x + c_0 t)\mathbf{r^1}$$

and right going wave as

$$\mathbf{q}(x,t) = \bar{w}^2(x - c_0 t)\mathbf{r^2}$$

The general solution is given by

$$\mathbf{q}(x,t) = \bar{w}^1(x + c_0 t)\mathbf{r^1} + \bar{w}^2(x - c_0 t)\mathbf{r^2}$$

Let initial conditions be

$$\mathbf{q}(x,0) = \mathring{\mathbf{q}}(x) = \begin{bmatrix} \mathring{p}(x) \\ \mathring{u}(x) \end{bmatrix}$$

By equating with initial condition we can find $\bar{w}^1$ and $\bar{w}^2$

$$p(x,t) = \frac{1}{2}[\mathring{p}(x+c_0t) + \mathring{p}(x-c_0t)] - \frac{Z_0}{2}[\mathring{u}(x+c_0t) - \mathring{u}(x-c_0t)],$$

$$u(x,t) = -\frac{1}{2Z_0}[\mathring{p}(x+c_0t) - \mathring{p}(x-c_0t)] + \frac{1}{2}[\mathring{u}(x+c_0t) + \mathring{u}(x-c_0t)].$$

Hyperbolicity of linear equations

Definition 2.1. *A linear system of the form*

$$q_t + Aq_x = 0$$

is called hyperbolic *if the $m \times m$ matrix A is diagonalizable with real eigenvalues.*

- The matrix is diagonalizable if there is a complete set of eigenvectors. In this case the matrix
  $\mathbf{R} = [\mathbf{r}^1|\mathbf{r}^2|....|\mathbf{r}^m]$
  formed by collecting the vectors $r^1, r^2, ..r^m$ together is nonsingular and has an inverse R^{-1}. we then have
  $\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathbf{\Lambda}$ and $\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$
  where

$$\Lambda = \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & .. & \\ & & & \lambda^m \end{bmatrix}$$

The importance of this from the standpoint of the PDE is that we can then rewrite the linear system as

$$\mathbf{R}^{-1}\mathbf{q_t} + \mathbf{R}^{-1}\mathbf{A}\mathbf{R}\mathbf{R}^{-1}\mathbf{q_x} = 0$$

If we define $\mathbf{w}(x,t) = \mathbf{R}^{-1}\mathbf{q}(x,t)$, then this takes the form

$$\mathbf{w}_t + \mathbf{\Lambda}\mathbf{w}_x = 0$$

- $\lambda^1, \lambda^2 ..., \lambda^m$ are called **Characteristic Speeds**
- functions $\mathbf{w}(x,t)$ are called **Characterisitcs variables**
- $X(t) = x_0 + \lambda^p(t)$ are called **characteristic curves**

- If A is a symmetric matrix , then A is always diagonalizable with real eigenvalues and the system is said to be **symmetric** hyperbolic.
- If A has distinct real eigenvalues $\lambda^1 < \lambda^2 < ..... < \lambda^m$ then the eigenvectors must be linearly independent and the system is called strictly hyperbolic.
- If A has real eigenvalues but is not diagonalizable, then the system is called **weakly** hyperbolic

References

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