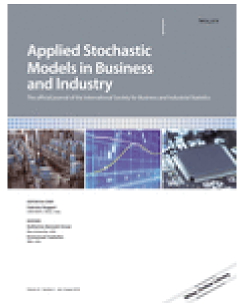

Bayesian Optimal Life–Testing Plan Under the Balanced Two Sample Type–II Progressive Censoring Scheme

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Bayesian optimal life-testing plan under the balanced two sample type-II progressive censoring scheme

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Abstract

Joint progressive censoring schemes are quite useful to conduct comparative life-testing experiment of different competing products. Recently, Mondal and Kundu ("A New Two Sample Type-II Progressive Censoring Scheme", Communications in Statistics-Theory and Methods, 2018) introduced a joint progressive censoring scheme on two samples known as the balanced joint progressive censoring (BJPC) scheme. Optimal planning of such progressive censoring scheme is an important issue to the experimenter. This article considers optimal life-testing plan under the BJPC scheme using the Bayesian precision and D-optimality criteria, assuming that the lifetimes follow Weibull distribution. In order to obtain the optimal BJPC life-testing plans, one needs to carry out an exhaustive search within the set of all admissible plans under the BJPC scheme. However, for large sample size, determination of the optimal life-testing plan is difficult by exhaustive search technique. A meta-heuristic algorithm based on the variable neighborhood search method is employed for computation of the optimal life-testing plan. Optimal plans are provided under different scenarios. The optimal plans depend upon the values of the hyper-parameters of the prior distribution. The effect of different prior information on optimal scheme is studied.

Key Words and Phrases: Joint censoring scheme, Optimal life-testing plan, Precision criterion, Bayesian D-optimality criterion, Variable neighborhood search.

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1 Introduction

Joint censoring schemes are quite useful to conduct comparative life-testing experiments of different competing products. Recently, Mondal and Kundu [13] introduced a joint progressive censoring scheme on two samples, called the balanced joint progressive censoring (BJPC) scheme. Due to its analytical flexibility and some other advantages, the BJPC scheme can be used quite conveniently in practice. In accelerated life-testing problem, the BJPC scheme can be applied where the experimental units are subject to different stress levels. It can be used in acceptance sampling plan to make decision on acceptance or rejection of lots of products coming from different sources of production in a single experiment.

The BJPC scheme can be described as follows. Suppose there are two different kind of products from line A and line B and it is required to study the relative merits of these two kind of products. A sample of size m is drawn from the line A and is called sam-A. Another sample of size n is drawn from line B and is called sam-B. Let k be the total number of failures to be observed from the life-testing experiment and $R_1,\dots,R_{k-1}$ be pre-specified non-negative integers such that $\sum_{i=1}^{k-1}(R_i+1) < \min(m,n)$. The units of these two samples are simultaneously put on the test. Suppose the first failure comes from sam-A and the failure time is denoted by W_1. Then at W_1, R_1 units are removed randomly from the remaining $m-1$ surviving units of sam-A and R_1+1 units are chosen randomly from the remaining n surviving units of sam-B and they are removed from the experiment. Next, if the second failure comes from sam-B at time point W_2, R_2+1 units are withdrawn from the remaining $m-R_1-1$ units of sam-A and R_2 units are withdrawn from the remaining $n-R_1-2$ units of sam-B randomly at W_2. The testing is continued until the k-th failure takes place and at the kth failure time point W_k, the testing gets terminated with the removal of all the remaining surviving units from both the samples. A schematic diagram of the BJPC scheme is provided in Figure 1. For more details on the BJPC scheme, see Mondal and Kundu [13].

To conduct a life-testing experiment under the BJPC scheme, the choice of the design parameters m, n, k and $\mathscr{R} = (R_1, \ldots, R_{k-1})$ is an important task to the experimenters. Then the question arises how to select the design parameters. In this work, we consider the optimal design of the BJPC scheme based on some optimality criteria. Optimal design of the life-testing plan under different censoring schemes have received considerable attention in the literature. See for example, Pradhan and Kundu [18], Pareek et al.[16], Kundu [10], Kundu and Pradhan [11], Kundu and Pradhan [12], Bhattacharya et al. [4], Budhiraja and Pradhan [6]. The choice of suitable optimality criteria is important for optimal design of life-testing experiment. In classical approach, optimality criteria depend on the parameter values of lifetime distribution. In practice, it is sometimes difficult to get the exact values of the parameters of lifetime distribution. Instead, information on the parameters may be available in terms of probability distributions. Therefore, Bayesian optimization is a natural choice. Also most of the conventional classical criteria are based on large sample approximation. For small sample sizes, those result may not be valid. Then, optimality criteria are developed by Bayesian method. Bayesian design of life-testing under different censoring schemes has also received considerable amount of attention in the literature. Kundu [10], Kundu and Pradhan [11], Kundu and Pradhan [12] and Pareek and Kundu [16] provided the optimal life-testing plan under the progressive censoring scheme based on the posterior variance of any p-th quantile (for $p \in (0, 1)$) of lifetime distributions. Optimal Bayes design of the accelerated life-testing is considered by Zhang and Meeker [24], Xu and Tang [21, 22], Nasir and Pan [15]. Bhattacharya and Pradhan [3] provided the optimal hybrid censoring scheme using a precision criterion based on the posterior variance of any p-th quantile using the idea of Zhang and Meeker [24]. Roy and Pradhan [19] provided the Bayesian D-optimal criterion based on the information matrix for the progressive Type-I interval censoring scheme. Chaloner and Verdinelli [7] considered Bayesian C-optimality criterion based on the p-th quantile of the lifetime distribution. Recently, Roy and Pradhan [20] discussed the Bayesian C-optimal design under the progressive Type-I interval censoring scheme.

In this work, we consider Bayes design of optimal life-testing plan under the BJPC scheme. It may be noted that the existing design critaria are appropriate to single sample problem. In case of BJPC scheme, we need to deal with two or more than two samples. We modify the existing criteria for single sample problem to two samples case. In general, this can be extended to more than two samples. Mondal and Kundu [14] considered a precision criterion based on the expected volume of the credible region and obtained the optimal life-testing plan under the BJPC scheme for $n = m$. Based on the existing literature, a natural choice of a precision criterion can be formulated by adding the posterior variance of any p-th quantile (for $p \in (0, 1)$) from both the populations. But this criterion is computationally intensive and time consuming. We apply the large sample approximation techniques proposed by Zhang and Meeker [24] and reduce to a simpler version of this criterion which is easier to compute. We also consider the Bayesian D-optimality criteria based on the information matrix under the BJPC scheme. It is assumed that lifetimes of two populations follow Weibull distributions with same shape parameter and different scale parameters. We consider beta-gamma prior for the scale parameters and independent gamma prior for the common shape parameter as suggested in Mondal and Kundu [14]. This article considers determination of the optimal $R_1, R_2, \ldots, R_{k-1}$ for given m, n and k under the proposed optimality criteria. For given m, n and k, the total number of choices of $\mathscr{R} = (R_1, \ldots, R_{k-1})$ such that $\sum_{i=1}^{k-1}(R_i + 1) < \min(m, n)$ is $\binom{\min(m,n)-1}{k-1}$ which is large even for small values of m, n, k. Therefore, a complete search is quite time consuming and challenging in practice. To avoid this complexity, we rely on the variable neighborhood search algorithm as suggested in Bhattacharya et al. [5] to determine the optimal $R_1, \ldots, R_{k-1}$. Further, a sensitivity analysis is performed to study the influence of prior hyper-parameters on the optimal life-testing plans.

Rest of the paper is arranged as follows. In Section 2, we define some notations and provide the model and the likelihood function. The Bayesian prior assumptions and the Bayesian design criteria are presented in Section 3. In Section 4, we present a computational algorithm based on the variable neighborhood search approach to find out the optimal plan. In Section 5, the optimal life-testing plans are obtained under different set-ups. A sensitivity analysis based on the prior hyper-parameters is also under-taken in this section. Finally we make few concluding remarks in Section 6.

2 Notation, model description and likelihood function

2.1 Notation

i.i.d. : Independent and identically distributed.

CDF : Cumulative distribution function.

MLE : Maximum likelihood estimator.

PDF : Probability density function.

$Beta(a, b)$: Beta distribution with PDF:
$$f_{Beta}(x, a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 < x < 1, a, b > 0.$$

$Exp(\theta)$: Exponential distribution with PDF:
$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}; x > 0, \theta > 0.$

$GA(a, b)$: Gamma distribution with PDF:
$$f_{GA}(x, a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx},\ x > 0, a, b > 0.$$

$WE(\alpha, \lambda)$: Weibull distribution with PDF:
$f_{WE}(x, \alpha, \lambda) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}}\ x > 0, \alpha, \lambda > 0$
and the CDF, $F_{WE}(x, \alpha, \lambda) = 1 - e^{-\lambda x^{\alpha}}$.

2.2 Model and likelihood function

Suppose $X_1, \dots, X_m$ denote the lifetimes of m units from line-A, and it is assumed that they are i.i.d. $WE(\alpha, \lambda_1)$. Similarly, it is assumed that $Y_1, \dots, Y_n$ denote the lifetimes of n units from line-B, and they are i.i.d. $WE(\alpha, \lambda_2)$ random variables. Define a new set of random variables $Z_1,\dots,Z_k$ where $Z_i = 1$ or 0 if ith failure comes from sam-A or sam-B, respectively, for $i = 1, \dots, k$.. Under the BJPC scheme, the censored sample can be obtained as $(\mathbf{W}, \mathbf{Z})$ where $\mathbf{W} = (W_1, \dots W_k)$ and $\mathbf{Z} = (Z_1, \dots, Z_k)$. Here, $K_1 = \sum_{i=1}^{k} Z_i$ and $K_2 = \sum_{i=1}^{k}(1 - Z_i)$ denote the total number of failures from sam-A and sam-B, respectively, in the experiment. Here, we denote $\boldsymbol{\theta} = (\alpha, \lambda_1, \lambda_2)$. As studied in Mondal and Kundu [13], based on the observed data $(\mathbf{w}, \mathbf{z})$, the likelihood function can be obtained as

$$L(\boldsymbol{\theta}|\mathbf{w}, \mathbf{z}) \propto \alpha^k \lambda_1^{k_1} \lambda_2^{k_2} \prod_{i=1}^{k} w_i^{\alpha-1} e^{-\lambda_1 A_1(\alpha,\mathbf{w}) - \lambda_2 A_2(\alpha,\mathbf{w})},$$

where

$$\begin{aligned}
k_1 &= \sum_{i=1}^{k} z_i, \;\; k_2 = \sum_{i=1}^{k}(1 - z_i), \\
A_1(\alpha, \mathbf{w}) &= \sum_{i=1}^{k-1}(R_i + 1)w_i^{\alpha} + \Big(m - \sum_{i=1}^{k-1}(R_i + 1)\Big) w_k^{\alpha}, \\
\text{and } A_2(\alpha, \mathbf{w}) &= \sum_{i=1}^{k-1}(R_i + 1)w_i^{\alpha} + \Big(n - \sum_{i=1}^{k-1}(R_i + 1)\Big) w_k^{\alpha}.
\end{aligned}$$

The Fisher information matrix is given by

$$\begin{aligned}
I(\boldsymbol{\theta}) &= -E\Big[\frac{\partial^2 \ln L(\boldsymbol{\theta}|\mathbf{W}, \mathbf{Z})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}\Big] \\
&= \begin{bmatrix} \frac{k}{\alpha^2} + \lambda_1 E\Big(A_1''(\alpha, \mathbf{W})\Big) + \lambda_2 E\Big(A_2''(\alpha, \mathbf{W})\Big) & E\Big(A_1'(\alpha, \mathbf{W})\Big) & E\Big(A_2'(\alpha, \mathbf{W})\Big) \\ E\Big(A_1'(\alpha, \mathbf{W})\Big) & \frac{E(K_1)}{\lambda_1^2} & 0 \\ E\Big(A_2'(\alpha, \mathbf{W})\Big) & 0 & \frac{E(K_2)}{\lambda_2^2} \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
A_1^{'}(\alpha,\mathbf{W}) &= \frac{\partial A_1(\alpha,\mathbf{W})}{\partial\alpha} = \sum_{i=1}^{k-1}(R_i+1)(\ln W_i)W_i^{\alpha} + \Big(m-\sum_{i=1}^{k-1}(R_i+1)\Big)(\ln W_k)W_k^{\alpha},\\
A_1^{''}(\alpha,\mathbf{W}) &= \frac{\partial^2 A_1(\alpha,\mathbf{W})}{\partial\alpha^2} = \sum_{i=1}^{k-1}(R_i+1)(\ln W_i)^2W_i^{\alpha} + \Big(m-\sum_{i=1}^{k-1}(R_i+1)\Big)(\ln W_k)^2W_k^{\alpha},\\
A_2^{'}(\alpha,\mathbf{W}) &= \frac{\partial A_2(\alpha,\mathbf{W})}{\partial\alpha} = \sum_{i=1}^{k-1}(R_i+1)(\ln W_i)W_i^{\alpha} + \Big(n-\sum_{i=1}^{k-1}(R_i+1)\Big)(\ln W_k)W_k^{\alpha},\\
\text{and } A_2^{''}(\alpha,\mathbf{W}) &= \frac{\partial^2 A_2(\alpha,\mathbf{W})}{\partial\alpha^2} = \sum_{i=1}^{k-1}(R_i+1)(\ln W_i)^2W_i^{\alpha} + \Big(n-\sum_{i=1}^{k-1}(R_i+1)\Big)(\ln W_k)^2W_k^{\alpha}.
\end{aligned}$$

Here, all the expectations are computed based on the results of the distributional properties of W_i's and Z_i's from Mondal and Kundu [13]. Note that the expected values of K_1 and K_2 can be obtained as

$$\begin{aligned}
E(K_1) &= \sum_{i=1}^{k}E(Z_i) = \sum_{i=1}^{k}\frac{\lambda_1\Big(m-\sum_{j=1}^{i-1}(R_j+1)\Big)}{\lambda_1\Big(m-\sum_{j=1}^{i-1}(R_j+1)\Big)+\lambda_2\Big(n-\sum_{j=1}^{i-1}(R_j+1)\Big)},\\
\text{and } E(K_2) &= k-E(K_1).
\end{aligned}$$

3 Optimal design

Among all possible life-testing plans under a certain censoring scheme, optimal life-testing plan provides maximum *information* about the unknown model parameters based on some specific design criterion. Hence, to determine the optimal life-testing plan, we need to propose a proper design criterion and then compare all possible life-testing plans based on that criterion. In Bayesian framework, the prior knowledge of the model parameters usually influences the design criterion. In this section, first we discuss the choice of prior distributions of the model parameters followed by the corresponding posterior density. Next, we propose two Bayesian design criteria under this framework. We thoroughly discuss the computational procedure of the optimal life-testing plan based on the proposed criteria in the subsequent sections.

3.1 Prior distribution and the joint posterior density function

Following the idea in Pena and Gupta [17] and Kundu and Pradhan [12], it is assumed that

$$\frac{\lambda_1}{(\lambda_1+\lambda_2)} \sim Beta(a_1, a_2)$$
$$\text{and } \lambda_1+\lambda_2 \sim GA(a_0, b_0),$$

and they are independently distributed. Therefore, the joint PDF of λ_1 and λ_2 can be obtained as follows

$$\pi_1(\lambda_1,\lambda_2) = \begin{cases} C^*(\lambda_1+\lambda_2)^{a_0-a_1-a_2} & \\ \times\lambda_1^{a_1-1}\lambda_2^{a_2-1} \times\ e^{-b_0}(\lambda_1+\lambda_2), & \text{if } 0<\lambda_1,\ \lambda_2<\infty, \\ 0 & \text{if } -\infty<\lambda_1,\ \lambda_2\le 0, \end{cases}$$

where the normalizing constant $C^* = \dfrac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} b_0^{a_0}$. It is known as the beta-gamma PDF and it will be denoted by $BG(a_0, b_0, a_1, a_2)$. The following result will be useful for further development and they can be established very easily.

Result 1: If $(\lambda_1,\lambda_2) \sim BG(a_0, b_0, a_1, a_2)$, then for $i = 1, 2$,

$$\begin{aligned} E(\lambda_i) &= \frac{a_0}{b_0} \times \frac{a_i}{(a_1+a_2)}, \\ Var(\lambda_i) &= \frac{a_0}{b_0^2} \times \frac{a_i}{(a_1+a_2)} \left\{ \frac{(a_i+1)(a_0+1)}{a_1+a_2+1} - \frac{a_0 a_i}{a_1+a_2} \right\}, \\ \text{and } cov(\lambda_1,\lambda_2) &= \frac{a_0}{b_0^2} \times \frac{a_1 a_2 (a_1+a_2-a_0)}{(a_1+a_2)^2 (a_1+a_2+1)}. \end{aligned}$$

Proof: See Pena and Gupta [17]. ∎

Next we consider prior on α. Following the idea of Berger and Sun [2], it is assumed that $\pi_2(\alpha)$, the prior on α, has a support on the positive real line, and it has a log-concave PDF. Moreover, α and (λ_1,λ_2) are independently distributed. For specific implementation, we assume α follows a gamma prior and

$$\pi_2(\alpha) = \begin{cases} \frac{d^c}{\Gamma(c)}\alpha^{c-1}e^{-d\alpha}, & \text{if } x>0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the joint prior density of $\boldsymbol{\theta} = (\alpha, \lambda_1, \lambda_2)$ can be obtained as

$$\pi(\boldsymbol{\theta}) = \pi_1(\lambda_1, \lambda_2) \times \pi_2(\alpha).$$

From the above prior assumptions, we can derive the covariance matrix of α, λ_1 and λ_2, which is required for some further development. Let $S(\boldsymbol{\theta})$ denote the prior covariance matrix, then

$$S(\boldsymbol{\theta}) = \begin{bmatrix} Var(\alpha) & 0 & 0 \\ 0 & Var(\lambda_1) & cov(\lambda_1, \lambda_2) \\ 0 & cov(\lambda_1, \lambda_2) & Var(\lambda_2) \end{bmatrix}$$

where $Var(\alpha) = \frac{c}{d^2}$ and $Var(\lambda_i)$ for $i = 1, 2$ and $cov(\lambda_1, \lambda_2)$ are obtained in Result 1. Therefore, the prior precision matrix, i.e. $S^{-1}(\boldsymbol{\theta})$ can be obtained as

$$S^{-1}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{d^2}{c} & 0 & 0 \\ 0 & \frac{Var(\lambda_2)}{Var(\lambda_1)Var(\lambda_2)-\{cov(\lambda_1,\lambda_2)\}^2} & -\frac{cov(\lambda_1,\lambda_2)}{Var(\lambda_1)Var(\lambda_2)-\{cov(\lambda_1,\lambda_2)\}^2} \\ 0 & -\frac{cov(\lambda_1,\lambda_2)}{Var(\lambda_1)Var(\lambda_2)-\{cov(\lambda_1,\lambda_2)\}^2} & \frac{Var(\lambda_1)}{Var(\lambda_1)Var(\lambda_2)-\{cov(\lambda_1,\lambda_2)\}^2} \end{bmatrix}.$$

Based on the likelihood function and the prior distribution, the posterior density can be obtained as

$$\begin{aligned} \pi(\boldsymbol{\theta}|data) &\propto \lambda_1^{a_1+k_1-1}\lambda_2^{a_2+k_2-1}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}e^{-(\lambda_1+\lambda_2)(b_0+U(\alpha))} \\ &\quad \times e^{-\lambda_1(A_1(\alpha)-U(\alpha))}e^{-\lambda_2(A_2(\alpha)-U(\alpha))}\alpha^{c+k-1}e^{-\alpha(d-\sum_{i=1}^{k}\ln w_i)}, \end{aligned}$$

where $U(\alpha)$ =min$(A_1(\alpha), A_2(\alpha))$. Hence, the joint posterior density of $\boldsymbol{\theta}$ can be decomposed as

$$\pi(\boldsymbol{\theta}|data) \propto \pi_1^*(\lambda_1, \lambda_2|\alpha, data) \times \pi_2^*(\alpha|data) \times g(\alpha, \lambda_1, \lambda_2),$$

where

$$\pi_1^*(\lambda_1, \lambda_2|\alpha, data) \sim BG(a_0+k, b_0+U(\alpha), a_1+k_1, a_2+k_2),$$
$$\pi_2^*(\alpha|data) \propto \frac{\alpha^{c+k-1}e^{-\alpha(d-\sum_{i=1}^{k}\ln w_i)}}{(b_0+U(\alpha))^{a_0+k}},$$
and $g(\alpha, \lambda_1, \lambda_2) = e^{-\lambda_1(A_1(\alpha)-U(\alpha))}e^{-\lambda_2(A_2(\alpha)-U(\alpha))}$.

Note that, the posterior density of α will be a proper density only if $d - \sum_{i=1}^{k}\ln w_i$ is positive. For a fixed d, $w_1, \ldots, w_k$ might be such that $d - \sum_{i=1}^{k}\ln w_i < 0$. In such cases, we cannot decompose the posterior density function as described above. In this case, we can apply methods like Metropolis-Hastings algorithm based on $\pi(\boldsymbol{\theta}|data)$ or we can divide the data by a suitable constant say $\eta(>1)$. If $X \sim WE(\alpha,\lambda)$, then $Y = \frac{X}{\eta} \sim WE(\alpha,\lambda^*)$ where $\lambda^* = \eta^\alpha * \lambda$. Therefore, after rescaling the data by the constant η, the model parameter will be $(\alpha, \lambda_1^* = \eta^\alpha\lambda_1, \lambda_2^* = \eta^\alpha\lambda_2)$. Therefore, the joint prior density of $(\alpha,\lambda_1^*,\lambda_2^*)$ can be obtained by

$$\begin{aligned}\pi(\alpha,\lambda_1^*,\lambda_2^*) &\propto \frac{1}{\eta^{a_0\alpha}}\alpha^{c-1}e^{-d\alpha}(\lambda_1^*+\lambda_2^*)^{a_0-a_1-a_2}e^{-\frac{b_0}{\eta^\alpha}(\lambda_1^*+\lambda_2^*)}\lambda^{*a_1-1}_1\lambda^{*a_2-1}_2\\ &= \alpha^{c-1}e^{-\alpha(d+a_0\ln(\eta))}(\lambda_1^*+\lambda_2^*)^{a_0-a_1-a_2}e^{-\frac{b_0}{\eta^\alpha}(\lambda_1^*+\lambda_2^*)}\lambda^{*a_1-1}_1\lambda^{*a_2-1}_2.\end{aligned}$$

Let $data^*$ be obtained from $data$ after rescaling and $w_i^* = \frac{w_i}{\eta}$. The posterior density will be

$$\pi(\alpha,\lambda_1^*,\lambda_2^*|data^*) \propto \pi_1^*(\lambda_1^*,\lambda_2^*|\alpha,data^*) \times \pi_2^*(\alpha|data^*) \times g(\alpha,\lambda_1^*,\lambda_2^*)$$

where,

$$\begin{aligned}\pi_1^*(\lambda_1^*,\lambda_2^*|\alpha,data^*) &= BG(a_0+k,\frac{b_0}{\eta^\alpha},a_1+k_1,a_2+k_2)\\ \pi_2^*(\alpha|data^*) &\propto \frac{\alpha^{c+k-1}e^{-\alpha(d+a_0\ln(\eta)-\sum_{i=1}^k \ln w_i^*)}}{(\frac{b_0}{\eta^\alpha}+U(\alpha,w^*))}\\ g(\alpha,\lambda_1^*,\lambda_2^*) &= e^{-\lambda_1^*(A_1(\alpha,w^*)-U(\alpha,w^*))} * e^{-\lambda_2^*(A_2(\alpha,w^*)-U(\alpha,w^*))}.\end{aligned}$$

Using importance sampling technique (see, Mondal and Kundu [14]), we can compute the Bayes estimators and associated credible intervals of α, λ_1^* and λ_2^*. Finally, we can compute the Bayes estimators of λ_1, λ_2 as function of α and λ_1^*, λ_2^*.

3.2 Bayesian design criteria

In the Bayesian framework, several works have been done in finding the optimal life testing plan. Kundu [10], Pareek and Kundu et al. [16], Kundu and Pradhan [16], Kundu and

Pradhan [12] provided the optimal life-testing plan based on the posterior variance of any p-th quantile (for $p \in (0,1)$) of the lifetime distributions under different censoring schemes. Bhattacharya and Pradhan [3] reduced the criterion in simplified form using the large sample approximation under the hybrid censoring scheme. Roy and Pradhan [19] suggested the Bayesian D-optimal criterion based on the information matrix under the progressive censoring scheme. Here, we consider the quantile based precision criterion and D-optimality criterion for determination of the optimal life-testing plans under the BJPC scheme.

Quantile based criterion: Zhang and Meeker [23] suggested a design criterion based on the posterior variance of the logarithm of p-th ($p \in (0,1)$) quantile of Weibull distribution. Kundu and Pradhan [12] extended the result for two competing causes and they proposed the criterion as the weighted sum of the expected posterior variances of the logarithm of p-th ($p \in (0,1)$) quantile of the two lifetime distributions due to two competing causes. Following the idea of Kundu and Pradhan [12], here we consider a design criterion as the weighted sum of the expected posterior variances of the logarithm of p-th ($p \in (0,1)$) quantile from two populations. The p-th quantiles $x_p(\boldsymbol{\theta})$ and $y_p(\boldsymbol{\theta})$ of $WE(\alpha, \lambda_1)$ and $WE(\alpha, \lambda_2)$, respectively, can be obtained as

$$x_p(\boldsymbol{\theta}) = [-\frac{1}{\lambda_1}\ln(1-p)]^{\frac{1}{\alpha}},$$
$$\text{and } y_p(\boldsymbol{\theta}) = [-\frac{1}{\lambda_2}\ln(1-p)]^{\frac{1}{\alpha}}.$$

We propose the precision criterion as

$$\varphi_1(\mathscr{R}) = \delta E_{data;\mathscr{R}}\Big[Var_{Posterior}\big(\ln x_p(\boldsymbol{\theta})|data\big)\Big] + (1-\delta)E_{data;\mathscr{R}}\Big[Var_{Posterior}\big(\ln y_p(\boldsymbol{\theta})|data\big)\Big], \quad (1)$$

where $\delta \in (0,1)$ and $Var_{Posterior}(\cdot|data)$ is the variance of the corresponding posterior distribution given the data and E_{data} is unconditional with respect to the data.

The design problem now can be considered as to find $\mathscr{R}$ for given m and n such that (1) is minimum. However, computationally it is tedious and time consuming. By applying delta method, $Var_{Posterior}\big(\ln x_p(\boldsymbol{\theta})|data\big)$ can be approximated as

$$Var_{Posterior}\big(\ln x_p(\boldsymbol{\theta})|data\big) = \gamma_1^T Var_{Posterior}\big(\boldsymbol{\theta}|data)\big)\gamma_1,$$

where $\gamma_1^T = (\frac{\partial \ln x_p(\boldsymbol{\theta})}{\partial \alpha}, \frac{\partial \ln x_p(\boldsymbol{\theta})}{\partial \lambda_1}, \frac{\partial \ln x_p(\boldsymbol{\theta})}{\partial \lambda_2})$. Similarly, $Var_{Posterior}\big(\ln y_p(\boldsymbol{\theta})|data\big)$ can be approximated as

$$Var_{Posterior}\big(\ln y_p(\boldsymbol{\theta})|data\big) = \gamma_2^T Var_{Posterior}\big(\boldsymbol{\theta}|data)\big)\gamma_2,$$

where $\gamma_2^T = (\frac{\partial \ln y_p(\boldsymbol{\theta})}{\partial \alpha}, \frac{\partial \ln y_p(\boldsymbol{\theta})}{\partial \lambda_1}, \frac{\partial \ln y_p(\boldsymbol{\theta})}{\partial \lambda_2})$.

Therefore, the criterion defined in (1) can be approximated as

$$\varphi_1(\mathscr{R}) \approx E_{data;\mathscr{R}}\Big[\delta\gamma_1^T Var_{Posterior}\big(\boldsymbol{\theta}|data)\big)\gamma_1 + (1-\delta)\gamma_2^T Var_{Posterior}\big(\boldsymbol{\theta}|data)\big)\gamma_2\Big]. \quad (2)$$

Using the result in Berger [1], which states that on reasonably large sample size, a multivariate normal distribution provides a good approximation for the posterior distribution, $Var_{Posterior}(\boldsymbol{\theta}|data)$ can be approximated by

$$Var_{Posterior}(\boldsymbol{\theta}|data) \approx \Big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\Big]^{-1}, \quad (3)$$

where $I(\widehat{\boldsymbol{\theta}})$ is the information matrix $I(\boldsymbol{\theta})$ evaluated at the MLE $\widehat{\boldsymbol{\theta}}$.

Therefore, applying the approximation in (3), the criterion in (2) can further be expressed as

$$\varphi_1(\mathscr{R}) \approx E_{data;\mathscr{R}}\Big[\delta\gamma_1^T\big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\big]^{-1}\gamma_1 + (1-\delta)\gamma_2^T\big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\big]^{-1}\gamma_2\Big]. \quad (4)$$

Note that, $\Big[\delta\gamma_1^T\Big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\Big]^{-1}\gamma_1 + (1-\delta)\gamma_2^T\Big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\Big]^{-1}\gamma_2\Big]$ depends on data only through the MLE $\widehat{\boldsymbol{\theta}}$. Following the result in Zhang and Meeker [24], we assume that $P(\widehat{\boldsymbol{\theta}})$ is the Bayesian posterior predictive distribution function of $\widehat{\boldsymbol{\theta}}$. Thus the pre-posterior expectation in (4) of the data can be derived by taking expectation over $P(\widehat{\boldsymbol{\theta}})$. Therefore, (4) can be expressed as

$$\varphi_1(\mathscr{R}) \approx \int \Big[\delta\gamma_1^T\Big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\Big]^{-1}\gamma_1 + (1-\delta)\gamma_2^T\Big[S^{-1}(\boldsymbol{\theta}) + I(\widehat{\boldsymbol{\theta}})\Big]^{-1}\gamma_2\Big] dP(\widehat{\boldsymbol{\theta}}). \quad (5)$$

Now, $P(\widehat{\boldsymbol{\theta}})$ can be obtained by integrating sampling distribution of $\widehat{\boldsymbol{\theta}}$ with respect to prior distribution of $\boldsymbol{\theta}$. For large sample size, we assume that $\widehat{\boldsymbol{\theta}}|\boldsymbol{\theta}$ converges to $\boldsymbol{\theta}$ in probability. Hence, $P(\widehat{\boldsymbol{\theta}})$ can be approximated by $\pi(\boldsymbol{\theta})$, the prior distribution of $\boldsymbol{\theta}$. Using this approximation, (5) can be approximated as

$$\varphi_1(\mathscr{R}) \approx \int_{\boldsymbol{\theta}} h(\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta},$$

where $h(\boldsymbol{\theta}) = \delta\gamma_1^T[S^{-1}(\boldsymbol{\theta}) + I(\boldsymbol{\theta})]^{-1}\gamma_1 + (1-\delta)\gamma_2^T[S^{-1}(\boldsymbol{\theta}) + I(\boldsymbol{\theta})]^{-1}\gamma_2$.

Therefore, using all those results of approximation on large sample approach, we consider the design criterion as

$$\varphi_1(\mathscr{R}) = \int_{\boldsymbol{\theta}} \Big[\delta\gamma_1^T[S^{-1}(\boldsymbol{\theta}) + I(\boldsymbol{\theta})]^{-1}\gamma_1 + (1-\delta)\gamma_2^T[S^{-1}(\boldsymbol{\theta}) + I(\boldsymbol{\theta})]^{-1}\gamma_2\Big]\pi(\boldsymbol{\theta})d\boldsymbol{\theta}. \tag{6}$$

From now onwards, $\varphi_1(\mathscr{R})$ indicates its final form as expressed in (6). Based on the criterion $\varphi_1(\mathscr{R})$, the life-testing plan $\mathscr{R}_1$ is said to be better than the life-testing plan $\mathscr{R}_2$ if $\mathscr{R}_1$ produces lesser value of $\varphi_1(\mathscr{R})$ than $\mathscr{R}_2$.

Bayesian D-optimality criterion : The Bayesian D-optimality criterion is a popular Bayesian design criterion. A detailed discussion on this criterion can be found in Chaloner and Verdinelli [7]. Here, we consider a design criterion based on the Bayesian D-optimality as follows

$$\varphi_2(\mathscr{R}) = \int \ln(det(I(\boldsymbol{\theta})))\pi(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Based on the criterion $\varphi_2(\mathscr{R})$, the life-testing plan $\mathscr{R}_1$ is said to be better than the life-testing plan $\mathscr{R}_2$ if $\mathscr{R}_1$ produces larger value of $\varphi_2(\mathscr{R})$ than $\mathscr{R}_2$.

4 Computation of optimal design

Here, we obtain the optimal censoring plan by using the proposed criteria for fixed sample sizes m, n and effective sample size k. Then the task is to obtain the optimal life-testing plan $\mathscr{R}$. To find out the optimal life-testing plan, we need to maximize or minimize a criterion $\varphi(\mathscr{R})$ with respect to the life-testing plan $\mathscr{R} = (R_1, \ldots, R_{k-1})$ such that $\sum_{i=1}^{k-1}(R_i + 1) < \min(m, n)$. This is a problem of discrete optimization where solution space consists of non-negative integers $R_1, \ldots, R_{k-1}$ with cardinality $\binom{\min(m,n)-1}{k-1}$. Note that, even for small m, n and k, the quantity $\binom{\min(m,n)-1}{k-1}$ is quite large. Therefore, exhaustive search requires prolonged computational time and it is a challenging task in practice.

We consider a meta-heuristic approach based on the variable neighborhood search (VNS) to find out the optimal life-testing plan as proposed in Bhattacharya et al. [5]. The VNS approach is illustrated here followed by the pseudo code.

4.1 Variable neighborhood search algorithm

Hansen and Mladenović [9] introduced the VNS algorithm which works on the strategy of neighborhood change in search of better local optimal. The concept is that neighborhoods are connected and local optimal for one neighborhood structure may not be necessarily identical with other neighborhood structure. A global optimal is a local optimal for all possible neighborhood structures. Therefore, a stopping rule can be set for fixed number of iterations.

Let us denote $\min(m, n) = m_0$ and hence, under the BJPC scheme, a life-testing plan $\mathscr{R} = (R_1, \ldots, R_{k-1})$ satisfies $\sum_{i=1}^{k-1}(R_i + 1) < m_0$. Therefore, we can write $\sum_{i=1}^{k-1} R_i \leq m_0 - k$. Let us denote $R_k = m_0 - k - \sum_{i=1}^{k-1} R_i$. Then, we can re-write $\mathscr{R}$ as $\mathscr{R} = (R_1, \ldots, R_{k-1}, R_k)$. Thus the construction of neighborhood problem under the BJPC scheme is reduced to the construction of neighborhood structure under the conventional progressive Type-II censoring scheme. Following the idea of Bhattacharya et al.[5], we can construct neighborhood for a given initial life-testing plan. Suppose for fixed m, n and k, we fix an initial life-testing plan

$\mathscr{R}_0 = (R_{10}, R_{20}, \ldots, R_{k0})$. For $\min(m,n) = m_0$, the conditions are given by $\sum_{i=1}^{k-1}(R_i + 1) < m_0$ and $R_{k0} = m_0 - k - \sum_{i=1}^{k-1} R_{i0}$. Let us denote $i_{max} = \max_{1\leq j\leq k} R_{0j}$, then for $i = 1, \ldots, i_{max}$, we construct neighborhood $N_i(\mathscr{R}_0)$ such that any life-testing plan $\mathscr{R}$ in $N_i(\mathscr{R}_0)$ can be obtained by shifting any component of $\mathscr{R}_0$ maximum of i units such that $||\mathscr{R} - \mathscr{R}_0|| < i + 1$, where, $||\cdot||$ is the Euclidean norm. Next, in the neighborhood $N_i(\mathscr{R}_0)$ we search for a local optimal discretely among all possible life-testing plans. The search continues until we reach the neighborhood $N_{i_{max}}(\mathscr{R}_0)$. The pseudo code of the algorithm is given below.

Initial guess : Set an initial guess $\mathscr{R}_0$ and set $i_{max} = \max_{1\leq j\leq k} R_{0j}$.

Iterations : Set $i = 1$, until $i = i_{max}$ continue the following steps.

Local search : Construct the neighborhood $N_i(\mathscr{R}_0)$ and find out the local optimal, say $\mathscr{R}_{\text{opt}_i}$.

Decision to move : If $\mathscr{R}_0 = \mathscr{R}_{\text{opt}_i}$, set $i = i + 1$ and go back to the Local search.
If $\mathscr{R}_0 \neq \mathscr{R}_{\text{opt}_i}$, set the initial guess as $\mathscr{R}_{\text{opt}_i}$ and repeat all steps.

Finally, the global optimal is achieved by comparing all local optimal obtained through intermediate steps.

5 Numerical illustration and sensitivity analysis

5.1 Numerical illustration

In this section, we compute the optimal life-testing plans based on the proposed criteria in Section 3.2 under different input settings. For different values of sample sizes m, n and effective sample size k, we obtain the optimal plan using the VNS algorithm. For computation of the optimal scheme, we need to set the prior hyper parameter values. For illustration, we set the prior hyper-parameters as in Prior-1.

Prior -1: $(a_0, b_0) = (9, 2), (a_1, a_2) = (2, 5)$ and $(a, b) = (9, 2)$

In $\varphi_1(\mathscr{R})$, we set $\delta = 0.5, p = 0.5$. We start the algorithm with different initial guesses based on Prior-1 and the final optimal plans are reported in the Table 1 for (m, n) =$(25, 22)$, $(25, 25)$, $(40, 25)$, $(40, 45)$ and $(60, 80)$. Note that we use the notation $0_{(h)}$ which represents 0 is repeated h times. For instance, $\mathscr{R} = (5, 0_{(13)}, 2)$ implies $R_1 = 5, R_2 = \cdots = R_{14} = 0, R_{15} = 2$.

In Table 1, we notice that whatever the sample sizes (m, n) and effective sample size k are, the optimal life-testing plans are the one stage censoring plan based on the criteria $\varphi_1(\mathscr{R})$ and $\varphi_2(\mathscr{R})$. This result is not quite unexpected. Once $\min(m, n)$ is obtained, the optimization problem reduces to determination of the optimal life-testing plan under the progressive Type-II censoring scheme. From the classical results in the progressive Type-II censoring scheme, under Weibull model with A- or D-optimal criteria, the optimal life-testing plan reduces to one step censoring plan (see Dahmen et al.[8]). Intuitively, we obtain the one step censoring life-testing plan as the optimal solution in the Bayesian set-up also. Although it is difficult to prove analytically, nevertheless, the numerical study supports our claim in the Bayesian paradigm. It is also evident that both the design criteria give same optimal life-testing plans. In Tables 3 and 4, we notice that even for different initial guesses, the VNS algorithm reaches the same result in every cases investigated here. Hence, we can conclude that in this context the search algorithm is quite robust with respect to the initial guess.

5.2 Sensitivity analysis

The proposed method of determining the optimal life-testing plan depends on the values of the hyper-parameters of the prior distribution. It is, therefore, important to study the effect of the hyper-parameter values on the optimal life-testing plan. For illustrations, along with the Prior-1, we consider another two sets of prior hyper-parameter values as follows,

$$Prior-2: \quad (a_0, b_0) = (9,4), (a_1, a_2) = (4,5), \text{ and } (a,b) = (9,4),$$

$$Prior-3: \quad (a_0, b_0) = (9,2), (a_1, a_2) = (2,5) \text{ and } (a,b) = (4,2).$$

In order to check the effect of the population quantile p and the weight function δ on the optimal plan, we carry out a comparative study by computing optimal plans for different values of p and δ. Criterion $\varphi_1(\mathscr{R})$ is used for illustration and the computed optimal plans based on prior-1, 2 and 3 are reported in Table 3. From Table 3, it is observed that the prior hyper-parameters affect the optimal life-testing plans for fixed m, n, k and p, δ. Also, for a specific prior, the optimal life-testing plans depend on the values of p and δ. For criterion $\varphi_2(\mathscr{R})$, we also determine the optimal life-testing plans based on prior-1, 2 and 3 for fixed values of m, n and k in Table 4. It is observed that the prior hyper-parameters affect the optimal life-testing plans obtained through criterion 2. These results indicate that the optimal life-testing plans are sensitive due to the perturbation of the hyper-parameter values for both the criteria.

6 Conclusion

In this article, we have considered optimal design of the two sample BJPC scheme under the Bayesian set-up. We have proposed some precision criteria under this set-up. Under flexible prior assumption of the model parameters, different design criteria are applied to obtain the optimal plan using the variable neighborhood search algorithm. It is observed that optimal life-testing plans are one step censoring plan. The sensitivity analysis indicates that the optimal solutions are affected due to the perturbation of the hyper-parameter values. Apart from the proposed criteria, Bayesian C-optimality criterion (see in Chaloner and Verdinelli [7]) could be considered to attain precision in estimation of a particular lifetime quantile. For the BJPC scheme, the Bayesian C-optimality criterion can be obtained as minimize $\varphi_3(\mathscr{R})$, where

$$\varphi_3(\mathscr{R}) = \int [\gamma_1^T I^{-1}(\boldsymbol{\theta})\gamma_1 + \gamma_2^T I^{-1}(\boldsymbol{\theta})\gamma_2]\pi(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Here γ_1 and γ_2 are same as discussed in $\varphi_1(\mathscr{R})$. The optimal plan will minimize the criterion $\varphi_3(\mathscr{R})$. Here, we only restrict our work on two sample problem. In practice, the proposed methodology can be extended to more than two samples. An extension of the present work is going on in that direction and we hope to report our findings in future articles.

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Table 1: Optimal life-testing plans obtained through VNS algorithm based on Prior-1 for different m, n and k

Sample Size (m,n)	**Effective Sample Size** (k)	**Initial Guess** $(\mathscr{R}_o)$	**Design Criterion** $\varphi_1(\mathscr{R})$		$\varphi_2(\mathscr{R})$	
			Optimal Plan	**Optimal Value**	**Optimal Plan**	**Optimal Value**
$(25,22)$	15	$(5,0_{(13)},2)$	$(7,0_{(14)})$	0.015	$(7,0_{(14)})$	1.377
$(25,25)$	20	$(2,0_{(18)},3)$	$(5,0_{(19)})$	0.014	$(5,0_{(19)})$	2.079
$(25,25)$	15	$(5,0_{(6)},5,0_{(7)})$	$(10,0_{(9)})$	0.017	$(10,0_{(9)})$	1.216
$(40,25)$	20	$(2,0_{(6)},2,0_{(6)},1,0_{(5)})$	$(5,0_{(19)})$	0.010	$(5,0_{(19)})$	2.333
$(40,25)$	10	$(5,0_{(4)},5,0_{(3)},5)$	$(15,0_{(9)})$	0.017	$(15,0_{(9)})$	0.229
$(40,45)$	35	$(1,0_{(13)},1,0_{(7)},1,0_{(3)},1,0_{(7)},1)$	$(0_{(34)},5)$	0.009	$(0_{(34)},5)$	3.643
$(40,45)$	30	$(5,0_{(28)},5)$	$(0_{(29)},10)$	0.010	$(0_{(29)},10)$	3.200
$(40,45)$	25	$(5_{(3)},0_{(22)})$	$(0_{(24)},15)$	0.012	$(0_{(24)},15)$	2.665
$(40,45)$	15	$(10,0_{(6)},10,0_{(6)},5)$	$(0_{(14)},25)$	0.018	$(0_{(14)},25)$	1.148
$(60,80)$	25	$(5,0_{(10)},15,0_{(12)},15)$	$(0_{(24)},35)$	0.013	$(0_{(24)},35)$	2.572
$(60,80)$	50	$(5,0_{(30)},5,0_{(18)})$	$(0_{(49)},10)$	0.008	$(0_{(49)},10)$	4.570

Table 2: Optimal life-testing plans obtained through VNS algorithm based on Prior-1 with $(m, n) = (25, 25), (40, 45)$ and $k = 20, 25$ for different initial guesses

Initial Guess	Design Criteria	
$\mathscr{R}_o$	$\varphi_1(\mathscr{R})$	$\varphi_2(\mathscr{R})$
$(2, 0_{(18)}, 3)$	$(5, 0_{(19)})$	$(5, 0_{(19)})$
$(2, 0_{(6)}, 2, 0_{(6)}, 1, 0_{(5)})$	$(5, 0_{(19)})$	$(5, 0_{(19)})$
$(0_{(19)}, 5)$	$(5, 0_{(19)})$	$(5, 0_{(19)})$
$(1, 0_{(3)}, 1, 0_{(4)}, 1, 0_{(5)}, 1, 0_{(3)}, 1)$	$(5, 0_{(19)})$	$(5, 0_{(19)})$
$(5_{(3)}, 0_{(22)})$	$(0_{(24)}, 15)$	$(0_{(24)}, 15)$
$(2, 0_{(4)}, 2, 0_{(3)}, 2, 0_{(2)}, 2, 0_{(3)}, 2, 0_{(4)}, 2, 0_{(2)}, 3)$	$(0_{(24)}, 15)$	$(0_{(24)}, 15)$
$(0_{(22)}, 5_{(3)})$	$(0_{(24)}, 15)$	$(0_{(24)}, 15)$
$(7, 0_{(23)}, 8)$	$(0_{(24)}, 15)$	$(0_{(24)}, 15)$

Table 3: Optimal life-testing plans based on criterion $\varphi_1(\mathscr{R})$ for different values of p and δ obtained through VNS algorithm based on Prior-1, 2 and 3 with $(m, n) = (25, 22)$ and $k = 15$ for different initial guesses

	(p, δ)	**Initial guess**	$\varphi_1(\mathscr{R})$	
			Optimal Plan	Optimal Value
Prior-1	$(0.1, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	0.043
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	
		$(0_{(14)}, 7)$	$(0_{(12)}, 7, 0_{(2)})$	
	$(0.1, 0.5)$	$(5, 0_{(13)}, 2)$	$(7, 0_{(14)})$	0.039
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(7, 0_{(14)})$	
		$(0_{(14)}, 7)$	$(7, 0_{(14)})$	
	$(0.9, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	0.007
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	
		$(0_{(14)}, 7)$	$(0_{(12)}, 7, 0_{(2)})$	
	$(0.9, 0.5)$	$(5, 0_{(13)}, 2)$	$(7, 0_{(14)})$	0.012
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(7, 0_{(14)})$	
		$(0_{(14)}, 7)$	$(7, 0_{(14)})$	
Prior-2	$(0.1, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(14)}, 7)$	0.107
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(14)}, 7)$	
		$(0_{(14)}, 7)$	$(0_{(14)}, 7)$	
	$(0.1, 0.5)$	$(5, 0_{(13)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	0.102
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	
		$(0_{(14)}, 7)$	$(0_{(8)}, 7, 0_{(6)})$	
	$(0.9, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(14)}, 7)$	0.079
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(14)}, 7)$	
		$(0_{(14)}, 7)$	$(0_{(14)}, 7)$	
	$(0.9, 0.5)$	$(5, 0_{(13)}, 2)$	$(0_{(4)}, 7, 0_{(10)})$	0.041
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(4)}, 7, 0_{(10)})$	
		$(0_{(14)}, 7)$	$(0_{(4)}, 7, 0_{(10)})$	
Prior-3	$(0.1, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	0.523
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	
		$(0_{(14)}, 7)$	$(0_{(8)}, 7, 0_{(6)})$	
	$(0.1, 0.5)$	$(5, 0_{(13)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	0.475
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	
		$(0_{(14)}, 7)$	$(0_{(8)}, 7, 0_{(6)})$	
	$(0.9, 0.1)$	$(5, 0_{(13)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	0.038
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(8)}, 7, 0_{(6)})$	
		$(0_{(14)}, 7)$	$(0_{(8)}, 7, 0_{(6)})$	
	$(0.9, 0.5)$	$(5, 0_{(13)}, 2)$	$(7, 0_{(14)})$	0.158
		$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(7, 0_{(14)})$	
		$(0_{(14)}, 7)$	$(7, 0_{(14)})$	

Table 4: Optimal life-testing plans based on criterion $\varphi_2(\mathscr{R})$ obtained through VNS algorithm based on Prior-1, 2 and 3 for $(m, n) = (25, 22)$ and $k = 15$

	Initial guess	$\varphi_2(\mathscr{R})$ Optimal Plan	Optimal Value
Prior-1	$(5, 0_{(13)}, 2)$	$(7, 0_{(14)})$	1.377
	$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(14)}, 7)$	
	$(0_{(14)}, 7)$	$(0_{(14)}, 7)$	
Prior-2	$(5, 0_{(13)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	5.097
	$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(0_{(12)}, 7, 0_{(2)})$	
	$(0_{(14)}, 7)$	$(0_{(12)}, 7, 0_{(2)})$	
Prior-3	$(5, 0_{(13)}, 2)$	$(7, 0_{(14)})$	3.161
	$(3, 0_{(6)}, 2, 0_{(6)}, 2)$	$(7, 0_{(14)})$	
	$(0_{(14)}, 7)$	$(7, 0_{(14)}, 7)$	

Figure 1: Illustration of BJPC scheme on two samples

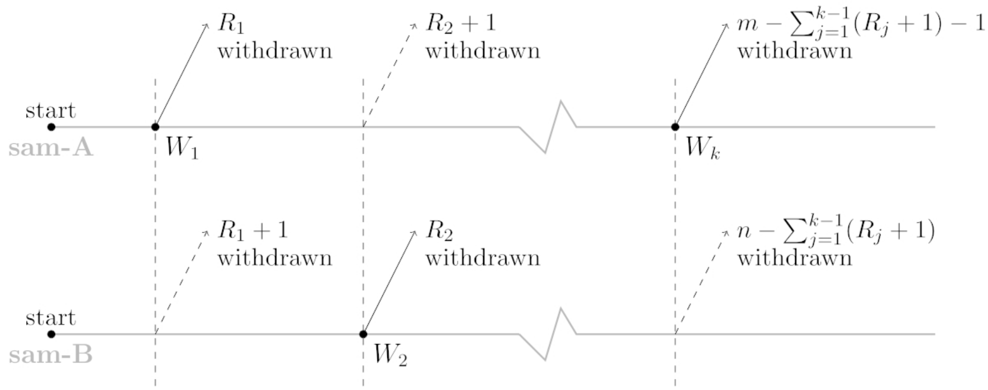

Figure 1: Illustration of BJPC Scheme On Two Samples

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