

Estimation of $P(X > Y)$ for Weibull distribution based on hybrid censored samples

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Abstract

A Hybrid censoring scheme is mixture of Type-I and Type-II censoring schemes. Based on hybrid censored samples, this paper deals with the inference on $R = P(X > Y)$, when X and Y are two independent Weibull distributions with different scale parameters, but having the same shape parameter. The maximum likelihood estimator (MLE), and the approximate MLE (AMLE) of R are obtained. The asymptotic distribution of the maximum likelihood estimator of R is obtained. Based on the asymptotic distribution, the confidence interval of R is constructed. Two bootstrap confidence intervals are also proposed. We consider the Bayesian estimate of R , and propose the corresponding credible interval for R . Monte Carlo simulations are performed to compare the different proposed methods. Analysis of a real data set has also been presented for illustrative purposes.

Keywords: Approximate maximum likelihood estimator, Hybrid censoring, Maximum likelihood estimator, Stress-strength model.

1 Introduction

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X and is subjected to random stress Y . The component fails if the system stress exceeds its strength. Thus, $R = P(X > Y)$ is a measure of component reliability. Many applications of the stress-strength model are related to engineering problems. However, there are also some applications in medicine or psychology, which involve the comparison of two random variables. For example, if X is the response for a control group, and Y refers to a treatment group, R is a measure of the effect of the treatment. For more application of R , see

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Kotz *et al.* (2003). The estimation of the stress-strength parameter R has received considerable attention in the statistical literature and has been discussed by many authors. See, for example Awad *et al.* (1981), Kundu and Gupta (2005, 2006), Adimari and Chiogna (2006), Raqab *et al.* (2008), Rezaei *et al.* (2010), Al-Mutairi *et al.* (2013), Asgharzadeh *et al.* (2013) and Nadar *et al.* (2014) for complete samples. Recently, some authors have also investigated the estimation of R for some lifetime distributions based on record and censored data. See for example, the work of Baklizi (2008), Asgharzadeh *et al.* (2011), Saracoglu *et al.* (2012) and Valiollahi *et al.* (2013).

A mixture of Type-I and Type-II censoring schemes is known as the hybrid censoring scheme and it can be described as follows. Suppose n identical units are put on a life test. The lifetimes of the sample units are independent and identically distributed (i.i.d) random variables. The test is terminated when a pre-specified number r out of n units have failed or a pre-determined time T , has been reached. It is also assumed that the failed items are not replaced. Therefore, in hybrid censoring scheme, the experimental time and the number of failures will not exceed T and r , respectively. It is clear that Type-I and Type-II censoring schemes can be obtained as special cases of hybrid censoring scheme by taking $r = n$ and $T \rightarrow \infty$ respectively. Now we describe the data available under the hybrid censoring scheme. Note that, under the hybrid censoring scheme, it is assumed that r and T are known in advance. Therefore, under this censoring scheme we have one of the two following types of observations:

$$\begin{aligned} \text{Case I : } & \{y_{1:n} < y_{2:n} < \dots < y_{r:n}\} \text{ if } y_{r:n} < T \\ \text{Case II : } & \{y_{1:n} < y_{2:n} < \dots < y_{d:n}\} \text{ if } d < r, y_{d:n} < T < y_{d+1:n} \end{aligned}$$

where $y_{1:n} < y_{2:n} < \dots$ denote the observed ordered failure times of the experimental units. For further details on hybrid censoring and relevant references, see Epstein (1954), Fairbanks *et al.* (1982), Ebrahimi (1990), Gupta and Kundu (1998), Childs *et al.* (2003), and Kundu (2007).

The two-parameter Weibull distribution denoted by $W(\alpha, \theta)$ has the probability density function (pdf)

$$f(x, \alpha, \theta) = \frac{\alpha}{\theta} x^{\alpha-1} e^{-\frac{x^\alpha}{\theta}}, \quad x > 0, \alpha, \theta > 0, \quad (1)$$

and the cumulative distribution function (cdf)

$$F(x, \alpha, \theta) = 1 - e^{-\frac{x^\alpha}{\theta}}, \quad x > 0, \alpha, \theta > 0, \quad (2)$$

Here α is the shape parameter and θ is the scale parameter.

Let $X \sim W(\alpha, \theta_1)$ and $Y \sim W(\alpha, \theta_2)$ are two independent Weibull distributions with different scale parameters, but having the same shape parameter. While the estimation of $R = P(X > Y)$ were discussed in the literature for complete and progressively Type-II censored samples, this problem has not been studied so far in the literature based on hybrid censored samples. This paper considers the estimation R based on hybrid censored data on both variables.

The paper is organized as follows: In Section 2, we derive the maximum likelihood estimator (MLE) of R . It is observed that the MLE can not be obtained in a closed form. We propose an approximate MLE (AMLE) of R in Section 3, which can be obtained explicitly. Different confidence intervals are presented in Section 4. Bayesian solutions are presented in Section 5. Analysis of a real data set as well a Monte Carlo simulation based comparison of the proposed methods are performed in Section 6. Finally in Section 7, we conclude the paper.

2 Maximum Likelihood Estimator of R

Let $X \sim W(\alpha, \theta_1)$ and $Y \sim W(\alpha, \theta_2)$ be independent random variables. Then it can be easily seen that

$$R = P(X > Y) = \frac{\theta_1}{\theta_1 + \theta_2}. \quad (3)$$

Our interest is in estimating R based on hybrid censored data on both variables. To derive the MLE of R , first we obtain the MLE's of α , θ_1 and θ_2 . Suppose $\mathbf{X} = (X_{1:n}, X_{2:n}, \dots, X_{r_1:n})$ is a hybrid censored sample from $W(\alpha, \theta_1)$ with censored scheme (r_1, T_1) and $\mathbf{Y} = (Y_{1:m}, Y_{2:m}, \dots, Y_{r_2:m})$ is a hybrid censored sample from $W(\alpha, \theta_2)$ with censored scheme (r_2, T_2) . For notation simplicity, we will write $(X_1, X_2, \dots, X_{r_1})$ for $(X_{1:n}, X_{2:n}, \dots, X_{r_1:n})$ and $(Y_1, Y_2, \dots, Y_{r_2})$ for $(Y_{1:m}, Y_{2:m}, \dots, Y_{r_2:m})$. Therefore, the likelihood function of α , θ_1 and θ_2 is given by

$$L(\alpha, \theta_1, \theta_2) = \left[c_1 \prod_{i=1}^{r_1} f(x_i) [1 - F(u_1)]^{n-r_1} \right] \times \left[c_2 \prod_{j=1}^{r_2} f(y_j) [1 - F(u_2)]^{m-r_2} \right], \quad (4)$$

where

$$\begin{aligned} c_1 &= n(n-1)(n-2) \cdots (n-r_1+1), & U_1 &= \min(X_{R_1}, T_1), \\ c_2 &= m(m-1)(m-2) \cdots (m-r_2+1), & U_2 &= \min(Y_{R_2}, T_2). \end{aligned}$$

and

$$r_1 = \sum_{i=1}^{r_1} I \{x_i \leq u_1\}, \quad r_2 = \sum_{j=1}^{r_2} I \{y_j \leq u_2\}.$$

Upon using (1) and (2), we immediately have the likelihood function of the observed data as follows:

$$L(\text{data}|\alpha, \theta_1, \theta_2) = c_1 c_2 \alpha^{r_1+r_2} \theta_1^{-r_1} \theta_2^{-r_2} \prod_{i=1}^{r_1} x_i^{\alpha-1} \prod_{j=1}^{r_2} y_j^{\alpha-1} \\ \times \exp \left\{ -\frac{1}{\theta_1} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n-r_1)u_1^\alpha \right] - \frac{1}{\theta_2} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m-r_2)u_2^\alpha \right] \right\}. \quad (5)$$

From (5), the log-likelihood function for the hybrid censored data without the multiplicative constant is

$$l(\alpha, \theta_1, \theta_2) = (r_1 + r_2) \ln(\alpha) - r_1 \ln(\theta_1) - r_2 \ln(\theta_2) + (\alpha - 1) \times \left[\sum_{i=1}^{r_1} \ln(x_i) + \sum_{j=1}^{r_2} \ln(y_j) \right] \\ - \frac{1}{\theta_1} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n-r_1)u_1^\alpha \right] - \frac{1}{\theta_2} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m-r_2)u_2^\alpha \right]. \quad (6)$$

The MLEs of α , θ_1 and θ_2 , say $\hat{\alpha}$, $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively, can be obtained as the solution of

$$\frac{\partial l}{\partial \alpha} = \frac{r_1 + r_2}{\alpha} + \left[\sum_{i=1}^{r_1} \ln(x_i) + \sum_{j=1}^{r_2} \ln(y_j) \right] - \frac{1}{\theta_1} \left[\sum_{i=1}^{r_1} x_i^\alpha \ln(x_i) + (n-r_1)u_1^\alpha \ln(u_1) \right] \\ - \frac{1}{\theta_2} \left[\sum_{j=1}^{r_2} y_j^\alpha \ln(y_j) + (m-r_2)u_2^\alpha \ln(u_2) \right] = 0, \quad (7)$$

$$\frac{\partial l}{\partial \theta_1} = -\frac{r_1}{\theta_1} + \frac{1}{\theta_1^2} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n-r_1)u_1^\alpha \right] = 0, \quad (8)$$

$$\frac{\partial l}{\partial \theta_2} = -\frac{r_2}{\theta_2} + \frac{1}{\theta_2^2} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m-r_2)u_2^\alpha \right] = 0. \quad (9)$$

From (8) and (9), we obtain

$$\hat{\theta}_1(\alpha) = \frac{1}{r_1} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n-r_1)u_1^\alpha \right], \quad \text{and} \quad \hat{\theta}_2(\alpha) = \frac{1}{r_2} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m-r_2)u_2^\alpha \right]. \quad (10)$$

Substituting the expressions of $\hat{\theta}_1(\alpha)$ and $\hat{\theta}_2(\alpha)$ into (7), $\hat{\alpha}$ can be obtained as a fixed point solution of the following equation:

$$k(\alpha) = \alpha. \quad (11)$$

Here

$$k(\alpha) = \frac{r_1 + r_2}{U(\alpha) + V(\alpha) - W},$$

where

$$U(\alpha) = \frac{r_1 [\sum_{i=1}^{r_1} x_i^\alpha \ln(x_i) + (n - r_1)u_1^\alpha \ln(u_1)]}{\sum_{i=1}^{r_1} x_i^\alpha + (n - r_1)u_1^\alpha},$$

$$V(\alpha) = \frac{r_2 [\sum_{j=1}^{r_2} y_j^\alpha \ln(y_j) + (m - r_2)u_2^\alpha \ln(u_2)]}{\sum_{j=1}^{r_2} y_j^\alpha + (m - r_2)u_2^\alpha}$$

and

$$W = \sum_{i=1}^{r_1} \ln(x_i) + \sum_{j=1}^{r_2} \ln(y_j).$$

A simple iterative procedure $k(\alpha^{(j)}) = \alpha^{(j+1)}$, where $\alpha^{(j)}$ is the j -th iterate, can be used to find the solution of (11). The iterative procedure should be stopped when the absolute difference between $\alpha^{(j)}$ and $\alpha^{(j+1)}$ is sufficiently small. Once we obtain $\hat{\alpha}_{ML}$, the MLE of θ_1 and θ_2 , can be deduced from (10) as $\hat{\theta}_{1ML} = \hat{\theta}_1(\hat{\alpha}_{ML})$ and $\hat{\theta}_{2ML} = \hat{\theta}_2(\hat{\alpha}_{ML})$. Therefore, we compute the MLE of R as

$$\hat{R}_{ML} = \frac{\frac{1}{r_1} \sum_{i=1}^{r_1} x_i^{\hat{\alpha}_{ML}} + (n - r_1)u_1^{\hat{\alpha}_{ML}}}{\frac{1}{r_1} \sum_{i=1}^{r_1} x_i^{\hat{\alpha}_{ML}} + (n - r_1)u_1^{\hat{\alpha}_{ML}} + \frac{1}{r_2} \sum_{j=1}^{r_2} y_j^{\hat{\alpha}_{ML}} + (m - r_2)u_2^{\hat{\alpha}_{ML}}}. \quad (12)$$

Here the maximum likelihood approach does not give an explicit estimator for α and hence for R , based on a hybrid censored sample. In the next section, we propose the approximate maximum likelihood estimates which have explicit forms.

3 Approximate Maximum Likelihood Estimator of R

In this section, the approximate maximum likelihood method is used to estimate the stress-strength parameter R . It is based on the fact that if the random variable X has $W(\alpha, \theta)$, then $V = \ln(X)$, has the extreme value distribution with pdf as

$$f(v; \mu, \sigma) = \frac{1}{\sigma} e^{\frac{v-\mu}{\sigma} - e^{\frac{v-\mu}{\sigma}}}, \quad -\infty < v < \infty, \quad (13)$$

where $\mu = \frac{1}{\alpha} \ln \theta$ and $\sigma = \frac{1}{\alpha}$. The density function (13) is known as the density function of an extreme value distribution, with location, and scale parameters as μ and σ respectively. The standard extreme value distribution has the pdf and cdf as

$$g(v) = e^{v-e^v}, \quad G(v) = 1 - e^{-e^v}.$$

Suppose $X_1 < X_2 < \dots < X_{r_1}$ is a hybrid censored sample from $W(\alpha, \theta_1)$ with censored scheme (r_1, T_1) and $Y_1 < Y_2 < \dots < Y_{r_2}$ is a hybrid censored sample from $W(\alpha, \theta_2)$ with censored scheme (r_2, T_2) . Let us use the following notations: $T_i = \ln(X_i)$, $Z_i = \frac{T_i - \mu_1}{\sigma}$, $i = 1, \dots, r_1$ and $S_j = \ln(Y_j)$, $W_j = \frac{S_j - \mu_2}{\sigma}$, $j = 1, \dots, r_2$, where $\mu_1 = \frac{1}{\alpha} \ln \theta_1$, $\mu_2 = \frac{1}{\alpha} \ln \theta_2$ and $\sigma = \frac{1}{\alpha}$. The log-likelihood function of the observed data T_1, \dots, T_{r_1} and S_1, \dots, S_{r_2} is

$$l^*(\mu_1, \mu_2, \sigma) \propto -(r_1 + r_2) \ln \sigma + \sum_{i=1}^{r_1} \ln(g(z_i)) + (n - r_1) \ln(1 - G(u_1^*)) + \sum_{j=1}^{r_2} \ln(g(w_j)) + (m - r_2) \ln(1 - G(u_2^*)), \quad (14)$$

where $u_1^* = \frac{\ln u_1 - \mu_1}{\sigma}$, $u_2^* = \frac{\ln u_2 - \mu_2}{\sigma}$. Differentiating (14) with respect to μ_1 , μ_2 and σ , we obtain the likelihood equations as

$$\frac{\partial l^*}{\partial \mu_1} = -\frac{1}{\sigma} \sum_{i=1}^{r_1} \frac{g'(z_i)}{g(z_i)} + \frac{1}{\sigma} (n - r_1) \frac{g(u_1^*)}{1 - G(u_1^*)} = 0, \quad (15)$$

$$\frac{\partial l^*}{\partial \mu_2} = -\frac{1}{\sigma} \sum_{j=1}^{r_2} \frac{g'(w_j)}{g(w_j)} + \frac{1}{\sigma} (m - r_2) \frac{g(u_2^*)}{1 - G(u_2^*)} = 0, \quad (16)$$

$$\begin{aligned} \frac{\partial l^*}{\partial \sigma} &= -\frac{r_1 + r_2}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^{r_1} z_i \frac{g'(z_i)}{g(z_i)} + \frac{1}{\sigma} (n - r_1) u_1^* \frac{g(u_1^*)}{1 - G(u_1^*)} \\ &- \frac{1}{\sigma} \sum_{j=1}^{r_2} w_j \frac{g'(w_j)}{g(w_j)} + \frac{1}{\sigma} (m - r_2) u_2^* \frac{g(u_2^*)}{1 - G(u_2^*)} = 0. \end{aligned} \quad (17)$$

It is observed that the likelihood equations are not linear and do not admit explicit solutions. We approximate the terms $p(z_i) = \frac{g'(z_i)}{g(z_i)}$ and $q(u_1^*) = \frac{g(u_1^*)}{1 - G(u_1^*)}$ by expanding in Taylor series as follows. Suppose $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$ and $p_{r_i}^* = \frac{p_{r_i} + p_{r_i+1}}{2}$, $q_{r_i}^* = 1 - p_{r_i}^*$. We expand the function $p(z_i)$ around $G^{-1}(p_i) = \ln(-\ln(q_i)) = \mu_i$. Further, we also expand the term $q(u_1^*)$ around $G^{-1}(p_{R_1}) = \mu_{R_1}$ if $u_1 = x_{R_1}$. If $u_1 = T_1$, expand $q(u_1^*)$ around the point $G^{-1}(p_{r_1}^*) = \mu_{r_1}^*$.

Similarly, We approximate the function $\bar{p}(w_j) = \frac{g'(w_j)}{g(w_j)}$ around $G^{-1}(p_j) = \mu_j$. we also expand the term $\bar{q}(u_2^*) = \frac{g(u_2^*)}{1 - G(u_2^*)}$ around $G^{-1}(p_{R_2}) = \mu_{R_2}$ if $u_2 = x_{R_2}$ and If $u_2 = T_2$, we expand $\bar{q}(u_2^*)$ around the point $G^{-1}(p_{r_2}^*) = \mu_{r_2}^*$. Now, let $u_1 = x_{R_1}$ and $u_2 = x_{R_2}$. Then $r_1 = R_1$, $r_2 = R_2$ and $u_1^* = z_{R_1}$, $u_2^* = w_{R_2}$.

Considering only the first order derivatives and neglecting the higher order derivatives we get

$$\begin{aligned} p(z_i) &\approx \alpha_i + \beta_i z_i, & \bar{p}(w_j) &\approx \alpha_j + \beta_j w_j, \\ q(z_{R_1}) &\approx 1 - \alpha_{R_1} - \beta_{R_1} z_{R_1}, & \bar{q}(w_{R_2}) &\approx 1 - \alpha_{R_2} - \beta_{R_2} w_{R_2}, \end{aligned}$$

where

$$\alpha_i = 1 + \ln q_i(1 - \ln(-\ln q_i)), \quad \beta_i = \ln(q_i).$$

Therefore, (15), (16), and (17) can be approximated respectively as

$$\frac{\partial l^*}{\partial \mu_1} = -\frac{1}{\sigma} \left[\sum_{i=1}^{R_1} (\alpha_i + \beta_i z_i) - (n - R_1)(1 - \alpha_{R_1} - \beta_{R_1} z_{R_1}) \right] = 0, \quad (18)$$

$$\frac{\partial l^*}{\partial \mu_2} = -\frac{1}{\sigma} \left[\sum_{j=1}^{R_2} (\alpha_j + \beta_j w_j) - (m - R_2)(1 - \alpha_{R_2} - \beta_{R_2} w_{R_2}) \right] = 0, \quad (19)$$

$$\begin{aligned} \frac{\partial l^*}{\partial \sigma} &= -\frac{2(R_1 + R_2)}{\sigma} - \frac{1}{\sigma} \left[\sum_{i=1}^{R_1} z_i(\alpha_i + \beta_i z_i) - (n - R_1)(1 - \alpha_{R_1} - \beta_{R_1} z_{R_1}) \right] \\ &\quad - \frac{1}{\sigma} \left[\sum_{j=1}^{R_2} w_j(\alpha_j + \beta_j w_j) - (m - R_2)(1 - \alpha_{R_2} - \beta_{R_2} w_{R_2}) \right] = 0. \quad (20) \end{aligned}$$

If we denote $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\sigma}_1$ as the solutions of (18), (19) and (20) respectively, then observe that

$$\tilde{\mu}_1 = A_1 + B_1 \tilde{\sigma}, \quad \tilde{\mu}_2 = A_2 + B_2 \tilde{\sigma}, \quad \text{and} \quad \tilde{\sigma} = \frac{-D + \sqrt{D^2 - 4(R_1 + R_2)E}}{2(R_1 + R_2)},$$

where

$$A_1 = \frac{\sum_{i=1}^{R_1} \beta_i t_i + (n - R_1)\beta_{R_1} t_{R_1}}{\sum_{i=1}^{R_1} \beta_i + (n - R_1)\beta_{R_1}}, \quad B_1 = \frac{\sum_{i=1}^{R_1} \alpha_i - (n - R_1)(1 - \alpha_{R_1})}{\sum_{i=1}^{R_1} \beta_i + (n - R_1)\beta_{R_1}},$$

$$A_2 = \frac{\sum_{j=1}^{R_2} \beta_j s_j + (m - R_2)\beta_{R_2} s_{R_2}}{\sum_{j=1}^{R_2} \beta_j + (m - R_2)\beta_{R_2}}, \quad B_2 = \frac{\sum_{j=1}^{R_2} \alpha_j - (m - R_2)(1 - \alpha_{R_2})}{\sum_{j=1}^{R_2} \beta_j + (m - R_2)\beta_{R_2}},$$

$$\begin{aligned}
D &= \sum_{i=1}^{R_1} \alpha_i (t_i - 3A_1) - (n - R_1)(1 - \alpha_{R_1})(t_{R_1} - 3A_1) + 2A_1B_1 \sum_{i=1}^{R_1} \beta_i + 2A_1B_1(n - R_1)\beta_{R_1} \\
&+ \sum_{j=1}^{R_2} \alpha_j (s_j - 3A_2) - (m - R_2)(1 - \alpha_{R_2})(s_{R_2} - 3A_2) + 2A_2B_2 \sum_{j=1}^{R_2} \beta_j + 2A_2B_2(m - R_2)\beta_{R_2}, \\
E &= \sum_{i=1}^{R_1} \beta_i t_i (t_i - A_1) + (n - R_1)\beta_{R_1} t_{R_1}^2 - (n - R_1)A_1\beta_{R_1} t_{R_1} + \sum_{j=1}^{R_2} \beta_j s_j (s_j - A_2) \\
&+ (m - R_2)\beta_{R_2} s_{R_2}^2 - (m - R_2)A_2\beta_{R_2} s_{R_2}.
\end{aligned}$$

Once $\tilde{\sigma}$ is obtained, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ can be obtained immediately. Therefore, the approximate MLE of R is given by

$$\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1 + \tilde{\theta}_2},$$

where

$$\tilde{\theta}_1 = \exp\left(\frac{1}{\tilde{\sigma}}(A_1 + B_1\tilde{\sigma})\right), \quad \text{and} \quad \tilde{\theta}_2 = \exp\left(\frac{1}{\tilde{\sigma}}(A_2 + B_2\tilde{\sigma})\right).$$

Similar procedure as above can be used to obtain AMLE of R for other cases (See the Appendix).

4 Different Confidence Intervals of R

Based on asymptotic behavior of R , we present an asymptotic confidence interval of R . We further, propose two confidence intervals based on the non-parametric bootstrap method.

4.1 Asymptotic Confidence Intervals of R

In this subsection, we obtain the asymptotic variances and covariances of the MLEs, $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ by entries of the inverse of the observed Fisher information matrix $\mathbf{I}(\theta)$ where $\theta = (\alpha, \theta_1, \theta_2)$. From the log-likelihood function in (6), we obtain the observed Fisher information matrix of θ as

$$I(\theta) = - \begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \theta_1} & \frac{\partial^2 l}{\partial \alpha \partial \theta_2} \\ \frac{\partial^2 l}{\partial \theta_1 \partial \alpha} & \frac{\partial^2 l}{\partial \theta_1^2} & \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 l}{\partial \theta_2 \partial \alpha} & \frac{\partial^2 l}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 l}{\partial \theta_2^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix},$$

where

$$\begin{aligned}
-I_{11} &= -\frac{r_1 + r_2}{\alpha^2} - \frac{1}{\theta_1} \left[\sum_{i=1}^{r_1} x_i^\alpha [\ln(x_i)]^2 + (n - r_1) u_1^\alpha [\ln(u_1)]^2 \right] \\
&\quad - \frac{1}{\theta_2} \left[\sum_{j=1}^{r_2} y_j^\alpha [\ln(y_j)]^2 + (m - r_2) u_2^\alpha [\ln(u_2)]^2 \right], \\
-I_{22} &= \frac{r_1}{\theta_1^2} - \frac{2}{\theta_1^3} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n - r_1) u_1^\alpha \right], \\
-I_{33} &= \frac{r_2}{\theta_2^2} - \frac{2}{\theta_2^3} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m - r_2) u_2^\alpha \right], \\
-I_{12} &= \frac{1}{\theta_1^2} \left[\sum_{i=1}^{r_1} x_i^\alpha \ln(x_i) + (n - r_1) u_1^\alpha \ln(u_1) \right] = -I_{21}, \\
-I_{13} &= \frac{1}{\theta_2^2} \left[\sum_{j=1}^{r_2} y_j^\alpha \ln(y_j) + (m - r_2) u_2^\alpha \ln(u_2) \right] = -I_{31}, \\
I_{23} &= I_{32} = 0,
\end{aligned}$$

Now, the asymptotic variance-covariance matrix, $A = [a_{ij}]$, for the MLE's is obtained by inverting the Fisher information matrix, i.e. $A = I(\theta)^{-1} = [I_{ij}]^{-1}$. Therefore

$$\mathbf{A} = \frac{1}{u} \begin{pmatrix} I_{22}I_{33} & -I_{12}I_{33} & -I_{22}I_{13} \\ -I_{21}I_{33} & I_{11}I_{33} - I_{13}I_{31} & I_{21}I_{13} \\ -I_{22}I_{31} & I_{12}I_{31} & I_{11}I_{22} - I_{12}I_{21} \end{pmatrix},$$

where

$$u = I_{11}I_{22}I_{33} - I_{12}I_{21}I_{33} - I_{13}I_{31}I_{22}.$$

Now, the variance of \widehat{R} (say B) can be obtained also using delta method as follows. We have $\widehat{R} = g(\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2)$, where

$$g(\alpha, \theta_1, \theta_2) = \frac{\theta_1}{\theta_1 + \theta_2}.$$

Therefore, $B = \mathbf{b}^t \mathbf{A} \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} \frac{\partial g}{\partial \alpha} \\ \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \end{pmatrix} = \frac{1}{(\theta_1 + \theta_2)^2} \begin{pmatrix} 0 \\ \theta_2 \\ -\theta_1 \end{pmatrix},$$

It is easy to show that

$$B = \mathbf{b}'\mathbf{A}\mathbf{b} = \frac{1}{u(\theta_1 + \theta_2)^4} [(I_{11}I_{22} - I_{12}^2)\theta_1^2 - 2I_{12}I_{13}\theta_1\theta_2 + (I_{11}I_{33} - I_{13}^2)\theta_2^2].$$

To compute the confidence interval of R , the variance B needs to be estimated. We recommend using the MLE estimates of α , θ_1 and θ_2 , to estimate B , which is very convenient. As a consequence of that, a $100(1 - \gamma)\%$ asymptotic confidence intervals for R can be given by

$$(\widehat{R} - z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}, \widehat{R} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{B}}),$$

where z_γ is 100γ -th percentile of $N(0, 1)$. The confidence interval of R by using the asymptotic distribution of the AMLE of R can be obtained similarly by replacing α , θ_1 and θ_2 in B by their respective AMLE's.

It is of interest to observe that when the shape parameter α is known, the MLE of R can be obtained explicitly as

$$\widehat{R}_{ML} = \frac{S_1(\mathbf{x})}{S_1(\mathbf{x}) + \frac{r_1}{r_2}S_2(\mathbf{y})}, \quad (21)$$

where $S_1(\mathbf{x}) = \sum_{i=1}^{r_1} x_i^\alpha + (n - r_1)u_1^\alpha$ and $S_2(\mathbf{y}) = \sum_{j=1}^{r_2} y_j^\alpha + (m - r_2)u_2^\alpha$.

4.2 Bootstrap Confidence Intervals

In this section, we propose two confidence intervals based on the bootstrap methods: (i) percentile bootstrap method (we call it Boot-p) based on the idea of Efron (1982), and (ii) bootstrap-t method (we refer to it as Boot-t) based on the idea of Hall (1988). We illustrate briefly how to estimate confidence intervals of R using both methods.

(i) Boot-p method

1. Generate a bootstrap sample of size r_1 , $\{x_1^*, \dots, x_{r_1}^*\}$ from $\{x_1, \dots, x_{r_1}\}$, and generate a bootstrap sample of size r_2 , $\{y_1^*, \dots, y_{r_2}^*\}$ from $\{y_1, \dots, y_{r_2}\}$. Based on $\{x_1^*, \dots, x_{r_1}^*\}$ and $\{y_1^*, \dots, y_{r_2}^*\}$, compute the bootstrap estimate of R say \widehat{R}^* using (12).
2. Repeat 1 NBOOT times.
3. Let $H_1(x) = P(\widehat{R}^* \leq x)$ be the cumulative distribution function of \widehat{R}^* . Define $\widehat{R}_{Bp}(x) = H_1^{-1}(x)$ for a given x . The approximate $100(1 - \gamma)\%$ confidence

interval of R is given by

$$\left(\widehat{R}_{Bp}\left(\frac{\gamma}{2}\right), \widehat{R}_{Bp}\left(1 - \frac{\gamma}{2}\right) \right)$$

(ii) Boot-t method

1. From the sample $\{x_1, \dots, x_{r_1}\}$ and $\{y_1, \dots, y_{r_2}\}$, compute \widehat{R} .
2. Same as in Boot-p method, first generate bootstrap sample $\{x_1^*, \dots, x_{r_1}^*\}$, $\{y_1^*, \dots, y_{r_2}^*\}$ and then compute \widehat{R}^* , the bootstrap estimate of R . Also, compute the statistic

$$T^* = \frac{\widehat{R}^* - \widehat{R}}{\sqrt{\text{Var}(\widehat{R}^*)}},$$

and its $\text{Var}(\widehat{R}^*)$ as described in Section 4.1.

3. Repeat 1 and 2 NBOOT times.
4. Let $H_2(x) = P(T^* \leq x)$ be the cumulative distribution function of T^* . For a given x define $\widehat{R}_{Bt}(x) = \widehat{R} + H_2^{-1}(x) \sqrt{\text{Var}(\widehat{R})}$. The approximate $100(1 - \gamma)\%$ confidence intervals of R is given by

$$\left(\widehat{R}_{Bt}\left(\frac{\gamma}{2}\right), \widehat{R}_{Bt}\left(1 - \frac{\gamma}{2}\right) \right).$$

5 Bayes Estimation of R

In this section, we obtain the Bayes estimation of R under assumption that the shape parameter α and scale parameters θ_1 and θ_2 are independent random variables. It is also assumed that the prior density of θ_j is the inverse gamma $IG(a_j, b_j)$, $j = 1, 2$ with density function

$$\pi_j(\theta_j) = \pi(\theta_j | a_j, b_j) = e^{-\frac{b_j}{\theta_j}} \frac{\theta_j^{-(a_j+1)} b_j^{a_j}}{\Gamma(a_j)},$$

and α has the gamma $G(a_3, b_3)$ with density function

$$\pi_3(\alpha) = \pi(\alpha | a_3, b_3) = e^{-b_3 \alpha} \frac{\alpha^{a_3-1} b_3^{a_3}}{\Gamma(a_3)}.$$

The likelihood function of the observed data is

$$\begin{aligned} L(\text{data} | \alpha, \theta_1, \theta_2) &\propto \alpha^{r_1+r_2} \theta_1^{-r_1} \theta_2^{-r_2} \prod_{i=1}^{r_1} x_i^{\alpha-1} \prod_{j=1}^{r_2} y_j^{\alpha-1} \\ &\times \exp \left\{ -\frac{1}{\theta_1} \left[\sum_{i=1}^{r_1} x_i^\alpha + (n - r_1) u_1^\alpha \right] - \frac{1}{\theta_2} \left[\sum_{j=1}^{r_2} y_j^\alpha + (m - r_2) u_2^\alpha \right] \right\}. \end{aligned} \quad (22)$$

The joint density of the $\alpha, \theta_1, \theta_2$ and data can be obtained as

$$L(\alpha, \theta_1, \theta_2, data) = L(data|\alpha, \theta_1, \theta_2) \times \pi_1(\theta_1) \times \pi_2(\theta_2) \times \pi_3(\alpha).$$

Therefore the joint posterior density of α, θ_1 and θ_2 given the data is

$$L(\alpha, \theta_1, \theta_2|data) = \frac{L(data|\alpha, \theta_1, \theta_2) \pi_1(\theta_1) \pi_2(\theta_2) \pi_3(\alpha)}{\int_0^\infty \int_0^\infty \int_0^\infty L(data|\alpha, \theta_1, \theta_2) \pi_1(\theta_1) \pi_2(\theta_2) \pi_3(\alpha) d\theta_1 d\theta_2 d\alpha}. \quad (23)$$

From (23), it is observed that the Bayes estimate can not be obtained in a closed form. Therefore, we opt for stochastic simulation procedures, namely, the Gibbs and Metropolis samplers (Gilks et al., 1995) to generate samples from the posterior distributions and then compute the Bayes estimate of R and the corresponding credible interval of R . The posterior pdfs of θ_1 and θ_2 are as follows:

$$\theta_1|\alpha, \theta_2, data \sim IG \left(r_1 + a_1, b_1 + \sum_{i=1}^{r_1} x_i^\alpha + (n - r_1)u_1^\alpha \right),$$

$$\theta_2|\alpha, \theta_1, data \sim IG \left(r_2 + a_2, b_2 + \sum_{j=1}^{r_2} y_j^\alpha + (m - r_2)u_2^\alpha \right),$$

and

$$f(\alpha|\theta_1, \theta_2, data) \propto \alpha^{r_1+r_2+a_3-1} \prod_{i=1}^{r_1} x_i^{\alpha-1} \prod_{j=1}^{r_2} y_j^{\alpha-1} \\ \times \exp \left\{ -b_3\alpha - \frac{1}{\theta_1} \left(\sum_{i=1}^{r_1} x_i^\alpha + (n - r_1)u_1^\alpha \right) - \frac{1}{\theta_2} \left(\sum_{j=1}^{r_2} y_j^\alpha + (m - r_2)u_2^\alpha \right) \right\}.$$

The posterior pdf of α is not known, but the plot of its show that it is symmetric and close to normal distribution. So to generate random numbers from this distributions, we use the Metropolis-Hastings method with normal proposal distribution. Note that the Metropolis algorithm considers only symmetric proposal distributions. Therefore the algorithm of Gibbs sampling is as follows:

1. Start with an initial guess $(\alpha^{(0)}, \theta_1^{(0)}, \theta_2^{(0)})$.
2. Set $t = 1$.
3. Using Metropolis-Hastings, generate $\alpha^{(t)}$ from $f_\alpha = f(\alpha|\theta_1^{(t-1)}, \theta_2^{(t-1)}, data)$ with the $N(\alpha^{(t-1)}, 1)$ proposal distribution.
4. Generate $\theta_1^{(t)}$ from $IG(r_1 + a_1, b_1 + \sum_{i=1}^{r_1} x_i^{\alpha^{(t-1)}} + (n - r_1)u_1^{\alpha^{(t-1)}})$.
5. Generate $\theta_2^{(t)}$ from $IG(r_2 + a_2, b_2 + \sum_{j=1}^{r_2} y_j^{\alpha^{(t-1)}} + (m - r_2)u_2^{\alpha^{(t-1)}})$.

6. Compute $R^{(t)}$ from (3).
7. Set $t = t + 1$.
8. Repeat steps 3-7, M times.

Note that in step 3, we use the Metropolis-Hastings algorithm with $q \sim N(\alpha^{(t-1)}, \sigma^2)$ proposal distribution as follows:

- a. Let $x = \alpha^{(t-1)}$.
- b. Generate y from the proposal distribution q .
- c. Let $p(x, y) = \min\left\{1, \frac{f_\alpha(y) q(x)}{f_\alpha(x) q(y)}\right\}$.
- d. Accept y with probability $p(x, y)$ or accept x with probability $1 - p(x, y)$.

Now the approximate posterior mean, and posterior variance of R become

$$\begin{aligned}\widehat{E}(R|data) &= \frac{1}{M} \sum_{t=1}^M R^{(t)}, \\ \widehat{Var}(R|data) &= \frac{1}{M} \sum_{t=1}^M \left(R^{(t)} - \widehat{E}(R|data)\right)^2.\end{aligned}$$

Based on M and R values, using the method proposed by Chen and Shao (1999), a $100(1-\gamma)\%$ HPD credible interval can be constructed as $\left(R_{[\frac{\gamma}{2}M]}, R_{[(1-\frac{\gamma}{2})M]}\right)$, where $R_{[\frac{\gamma}{2}M]}$ and $R_{[(1-\frac{\gamma}{2})M]}$ are the $[\frac{\gamma}{2}M]$ -th smallest integer and the $[(1-\frac{\gamma}{2})M]$ -th smallest integer of $\{R_t, t = 1, 2, \dots, M\}$, respectively.

6 Numerical Computations

In this section, analysis of a real data set and a Monte Carlo simulation are presented to illustrate all the estimation methods described in the preceding sections.

6.1 Simulation Study

In this subsection, We compare the performances of the MLE, AMLE and the Bayes estimates with respect to the squared error loss function in terms of biases, and mean squares errors (MSE). We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLE, and two different bootstrap confidence intervals and the HPD credible intervals in terms of the average confidence lengths. For computing the Bayes estimates and

HPD credible intervals, we assume 2 priors as follows:

$$\text{Prior 1: } a_j = 0, \quad b_j = 0, \quad j = 1, 2, 3,$$

$$\text{Prior 2: } a_j = 1, \quad b_j = 2, \quad j = 1, 2, 3.$$

Prior 1 is the non-informative gamma prior for both the shape and scale parameters. Prior 2 is informative gamma prior.

For different censoring schemes and different priors, we report the average estimates, and MSE of the MLE, AMLE and Bayes estimates of R over 5000 replications. The results are reported in Table 1. In our simulation experiments for both the bootstrap methods, we have computed the confidence intervals based on 250 re-sampling.

Table 1. The average estimates(A.E) and MSE of the MLE, AMLE and Bayes estimates of R when $m = n = 30$ and $(\alpha, \theta_1, \theta_2) = (1.5, 1, 1)$.

(r_1, T_1)	(r_2, T_2)		MLE	AMLE	BS	
					prior 1	prior 2
(20, 1)	(20, 1)	A.E	0.4929	0.4921	0.5074	0.5067
		MSE	0.0081	0.0093	0.0047	0.0031
	(25, 1)	A.E	0.4941	0.5040	0.5068	0.5037
		MSE	0.0072	0.0084	0.0035	0.0032
	(20, 2)	A.E	0.5049	0.4951	0.5043	0.5039
		MSE	0.0074	0.0090	0.0045	0.0039
	(25, 2)	A.E	0.4957	0.4953	0.5049	0.5036
		MSE	0.0068	0.0071	0.0049	0.0041
	(30, 2)	A.E	0.4986	0.4979	0.5033	0.5027
		MSE	0.0065	0.0074	0.0053	0.0037
(25, 1)	(20, 1)	A.E	0.5058	0.4947	0.5051	0.5047
		MSE	0.0071	0.0078	0.0066	0.0061
	(25, 1)	A.E	0.5045	0.5052	0.5038	0.5031
		MSE	0.0066	0.0067	0.0054	0.0056
	(20, 2)	A.E	0.5041	0.5046	0.5039	0.5022
		MSE	0.0067	0.0069	0.0061	0.0057
	(25, 2)	A.E	0.5033	0.5039	0.5032	0.5023
		MSE	0.0058	0.0061	0.0055	0.0046
	(30, 2)	A.E	0.4979	0.4983	0.5024	0.5022
		MSE	0.0058	0.0049	0.0049	0.0038
(20, 2)	(20, 1)	A.E	0.5035	0.5036	0.5033	0.5027
		MSE	0.0067	0.0074	0.0057	0.0043
	(25, 1)	A.E	0.4971	0.4956	0.5029	0.5024
		MSE	0.0054	0.0069	0.0046	0.0033
	(20, 2)	A.E	0.5029	0.5031	0.5026	0.5025
		MSE	0.0066	0.0069	0.0048	0.0037
	(25, 2)	A.E	0.4979	0.4968	0.5022	0.5020
		MSE	0.0065	0.0059	0.0043	0.0031
	(30, 2)	A.E	0.4986	0.4977	0.5018	0.5016
		MSE	0.0056	0.0061	0.0039	0.0023
(25, 2)	(20, 1)	A.E	0.4974	0.4959	0.5027	0.5019
		MSE	0.0062	0.0071	0.0045	0.0037
	(25, 1)	A.E	0.4979	0.4964	0.5020	0.5018
		MSE	0.0051	0.0061	0.0043	0.0023
	(20, 2)	A.E	0.5023	0.4968	0.5021	0.5020
		MSE	0.0057	0.0070	0.0049	0.0027
	(25, 2)	A.E	0.5019	0.4982	0.5019	0.5017
		MSE	0.0050	0.0054	0.0036	0.0021
	(30, 2)	A.E	0.4988	0.4983	0.5014	0.5010
		MSE	0.0044	0.0041	0.0041	0.0013
(30, 2)	(20, 1)	A.E	0.5047	0.4952	0.5042	0.5029
		MSE	0.0066	0.0076	0.0059	0.0043
	(25, 1)	A.E	0.4951	0.4959	0.5031	0.5016
		MSE	0.0059	0.0064	0.0048	0.0032
	(20, 2)	A.E	0.5068	0.4953	0.5043	0.5021
		MSE	0.0064	0.0068	0.0065	0.0029
	(25, 2)	A.E	0.5034	0.5029	0.5016	0.5009
		MSE	0.0050	0.0057	0.0047	0.0014
	(30, 2)	A.E	0.5005	0.5013	0.5006	0.5003
		MSE	0.0043	0.0049	0.0051	0.0008

Table 2. Average confidence/credible length for estimators of R .

(r_1, T_1)	(r_2, T_2)	MLE	Boot-t	Boot-p	BS	
					prior 1	prior 2
(20, 1)	(20, 1)	0.3854	0.4071	0.3966	0.3717	0.3416
	(25, 1)	0.3779	0.3988	0.3824	0.3714	0.3197
	(20, 2)	0.3815	0.3874	0.3902	0.3759	0.2989
	(25, 2)	0.3426	0.3747	0.3592	0.3217	0.2996
	(30, 2)	0.3140	0.3706	0.3269	0.2975	0.2661
(25, 1)	(20, 1)	0.3785	0.3884	0.3877	0.3419	0.3055
	(25, 1)	0.3766	0.3899	0.3813	0.3124	0.2892
	(20, 2)	0.3711	0.3914	0.3732	0.3053	0.2971
	(25, 2)	0.3463	0.3618	0.3557	0.2902	0.2566
	(30, 2)	0.2939	0.3314	0.3273	0.2613	0.2387
(20, 2)	(20, 1)	0.3461	0.3597	0.3444	0.3143	0.2819
	(25, 1)	0.3211	0.3424	0.3229	0.3078	0.2413
	(20, 2)	0.3357	0.3527	0.3289	0.3053	0.2673
	(25, 2)	0.3103	0.3315	0.3071	0.2642	0.2374
	(30, 2)	0.2846	0.3363	0.2978	0.2560	0.1956
(25, 2)	(20, 1)	0.3484	0.3677	0.3565	0.2835	0.2634
	(25, 1)	0.3192	0.3275	0.2905	0.2764	0.2312
	(20, 2)	0.3248	0.3350	0.3132	0.2714	0.2276
	(25, 2)	0.2925	0.2850	0.3013	0.2558	0.2170
	(30, 2)	0.2614	0.2831	0.2738	0.2203	0.2111
(30, 2)	(20, 1)	0.3417	0.3605	0.3452	0.2304	0.2079
	(25, 1)	0.3311	0.3225	0.3308	0.2244	0.1716
	(20, 2)	0.3255	0.3600	0.3396	0.2372	0.1948
	(25, 2)	0.2779	0.2813	0.2800	0.2052	0.1620
	(30, 2)	0.2462	0.2804	0.2748	0.2034	0.0913

6.2 Data Analysis

Here we present a data analysis of the strength data reported by Badar and Priest (1982). This data, represent the strength measured in GPA for single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20mm (Data Set 1) and 10mm (Data Set 2). These data have been used previously by Raqab and Kundu (2005), Kundu and Gupta (2006), Kundu and Raqab (2009) and Asgharzadeh et al (2011). The data are presented in Tables 3 and 4.

Table 3. Data Set 1 (gauge lengths of 20 mm).

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 4. Data Set 2 (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Kundu and Gupta (2006) analyzed these data sets using two-parameter Weibull distribution after subtracting 0.75 from both these data sets. After subtracting 0.75 from all the points of these data sets, Kundu and Gupta (2006) observed that the Weibull distributions with equal shape parameters fit to both these data sets. Based on the complete data, we plot the histogram of the samples of α generated by MCMC method along with their exact posterior density function in Figure 1. From the Figure 1 it is clear that the exact posterior density function match quite well with the simulated samples obtained using MCMC method.

For illustrative purposes, we have generated two different hybrid censored samples from above data sets after subtracting 0.75:

Scheme 1: $r_1 = 45, T_1 = 2.5, r_2 = 40, T_2 = 2.5$.

The MLE and AMLE of R become 0.3958 and 0.3872; and the corresponding 95% confidence intervals become (0.2723, 0.5193) and (0.2617, 0.4891), respectively. We also obtain the 95% Boot-p and Boot-t confidence intervals as (0.3045, 0.5523) and (0.3106, 0.5526), respectively.

For the Bayesian estimate of R , we use non-proper priors on θ_1, θ_2 and α , i.e., $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$. Based on the above priors, we obtain 0.3933 as the Bayes estimate of R , under the squared error loss function. We also compute the 95% highest posterior density (HPD) credible interval of R as (0.2937, 0.4829).

Scheme 2: $r_1 = 35, T_1 = 1.7, r_2 = 25, T_2 = 2.2$.

In this case, the MLE, AMLE and Bayes estimations of R become 0.4024, 0.4092 and 0.4238; and the corresponding 95% confidence intervals become (0.2781, 0.5267), (0.2960, 0.5531) and (0.2870, 0.5371) respectively. We also obtain the 95% Boot-p and Boot-t confidence intervals as (0.3359, 0.5711) and (0.3484, 0.5964) respectively.

7 Conclusions

In this paper, we have addressed the inferential issues on $R = P(X > Y)$, when X and Y are independent and the have Weibull distribution with the same shape

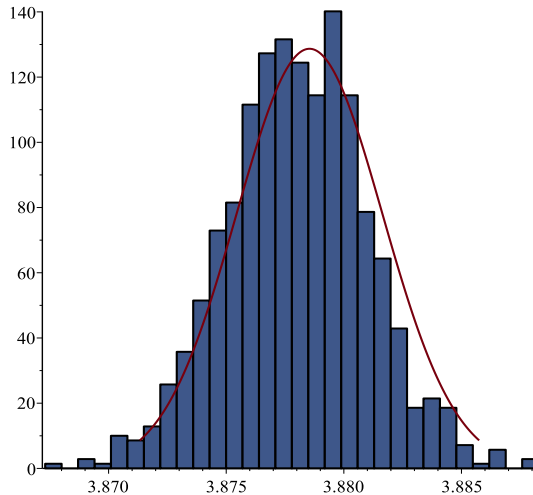


Figure 1: Posterior density function of α .

but different scale parameters. It is assumed that the data are hybrid censored on both X and Y . We have provided the MLE of R , and it is observed that the MLE of R cannot be obtained in explicit form. But it can be obtained by solving a one dimensional non-linear equation. We have also provided the AMLE of R , and it can be obtained explicitly. Extensive Monte Carlo simulations indicate that the performances of MLE and AMLE are very similar, hence for all practical purposes AMLE can be used in practice. We have further considered the Bayesian inference on R based on fairly general inverted gamma priors on the scale parameters, and an independent gamma prior on the common shape parameter. The Bayes estimator cannot be obtained in explicit form, we have Gibbs sampling technique to compute the Bayes estimate and also to compute the credible interval. It is observed that the performances of the Bayes estimators are very satisfactory, and if some prior informations are available on the unknown parameters, Bayes estimator should be preferred compared to MLE or AMLE, as expected.

Appendix

AMLE of R for other cases:

When $u_1 = T_1$ and $u_2 = T_2$, we expand $p(z_i)$, and $q(u_1^*)$ in Taylor series around the points μ_i and $\mu_{r_1}^*$, respectively. Further, we also expand the terms $\bar{p}(w_j)$, and $\bar{q}(u_2^*)$ around the points μ_j and $\mu_{r_2}^*$, respectively. Similarly, considering only the first order derivatives and neglecting the higher order derivatives, the AMLEs of α ,

θ_1 and θ_2 , can be obtained as

$$\tilde{\mu}_1 = A_1 + B_1\tilde{\sigma}, \quad \tilde{\mu}_2 = A_2 + B_2\tilde{\sigma}, \quad \text{and} \quad \tilde{\sigma} = \frac{-D + \sqrt{D^2 - 4(R_1 + R_2)E}}{2(R_1 + R_2)},$$

where

$$A_1 = \frac{\sum_{i=1}^{r_1} \beta_i t_i + (n - r_1)\beta_{r_1^*} \ln(T_1)}{\sum_{i=1}^{r_1} \beta_i + (n - r_1)\beta_{r_1^*}}, \quad B_1 = \frac{\sum_{i=1}^{r_1} \alpha_i - (n - r_1)(1 - \alpha_{r_1^*})}{\sum_{i=1}^{r_1} \beta_i + (n - r_1)\beta_{r_1^*}},$$

$$A_2 = \frac{\sum_{j=1}^{r_2} \beta_j s_j + (m - r_2)\beta_{r_2^*} s_{r_2}}{\sum_{j=1}^{r_2} \beta_j + (m - r_2)\beta_{r_2^*}}, \quad B_2 = \frac{\sum_{j=1}^{r_2} \alpha_j - (m - r_2)(1 - \alpha_{r_2^*})}{\sum_{j=1}^{r_2} \beta_j + (m - r_2)\beta_{r_2^*}},$$

$$\begin{aligned} D &= \sum_{i=1}^{r_1} \alpha_i (t_i - 3A_1) - (n - r_1)(1 - \alpha_{r_1^*})(\ln(T_1) - 3A_1) + 2A_1 B_1 \sum_{i=1}^{r_1} \beta_i + 2A_1 B_1 (n - r_1)\beta_{r_1^*} \\ &+ \sum_{j=1}^{r_2} \alpha_j (s_j - 3A_2) - (m - r_2)(1 - \alpha_{r_2^*})(\ln(T_2) - 3A_2) + 2A_2 B_2 \sum_{j=1}^{r_2} \beta_j + 2A_2 B_2 (m - r_2)\beta_{r_2^*}, \\ E &= \sum_{i=1}^{r_1} \beta_i t_i (t_i - A_1) + (n - r_1)\beta_{r_1^*} (\ln(T_1))^2 - (n - r_1)A_1 \beta_{r_1^*} \ln(T_1) + \sum_{j=1}^{r_2} \beta_j s_j (s_j - A_2) \\ &+ (m - r_2)\beta_{r_2^*} (\ln(T_2))^2 - (m - r_2)A_2 \beta_{r_2^*} \ln(T_2), \end{aligned}$$

here

$$\alpha_{r_i^*} = 1 + \ln(q_{r_i^*})(1 - \ln(-\ln(q_{r_i^*}))), \quad \beta_{r_i^*} = \ln(q_{r_i^*}), \quad i = 1, 2.$$

Therefore, the approximate MLE of R is given by

$$\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1 + \tilde{\theta}_2},$$

where

$$\tilde{\theta}_1 = \exp\left(\frac{1}{\tilde{\sigma}}(A_1 + B_1\tilde{\sigma})\right), \quad \text{and} \quad \tilde{\theta}_2 = \exp\left(\frac{1}{\tilde{\sigma}}(A_2 + B_2\tilde{\sigma})\right).$$

Similarly, AMLE of R for other cases (when $u_1 = T_1$, $u_2 = x_{R_2}$ or $u_1 = x_{R_1}$, $u_2 = T_2$) can be derived.

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