Authors' Rejoinder

N. Balakrishnan and Debasis Kundu

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First of all, we would like to express our sincere gratitude to Professor Fabrizio Ruggeri, the Editor-in-Chief, for his kind invitation for us to prepare this discussion paper and then making painstaking efforts for its review as well as to have numerous discussions on it done by various experts active on different aspects of Birnbaum-Saunders model. Next, we would like to express our great thanks and deep appreciation to all the discussants for their many incisive comments and suggestions on our expository article on Birnbaum-Saunders and related distributions. Below, we present our specific detailed responses to the comments made by each of the discussants.

Professor Anthony Desmond: It was an interesting read to see how Professor Desmond's interest in BS model originated from the work of Cramér and Leadbetter (1967) on properties of sample paths of stochastic processes including level crossings and maxima. He then went on to derive extensions of the famous Miner's rule in which the point process of local maxima played a key role. His approach in this direction led to an interesting finding that the damage process $D(t)$ could be modelled as a compound Poisson process. Of course, he was able to achieve this by exploiting the approximate Poisson nature of local maxima; then, without the Poisson approximation, by using only the mean and variance of the damage process $D(t)$, he was able to provide Birnbaum-Saunders distribution as an approximation for the fatigue lifetime distribution!

His idea of probability intervals for fatigue life is quite interesting and practically useful. His idea of allowing for some uncertainty in the material parameters, either in a Bayesian or frequentist approach, and then evaluating the sensitivity and robustness characteristics of these probability intervals will be very interesting, of course!

With regard to his comment about the potential use of estimating functions, many of the point estimators proposed in the literature (and also detailed in our article) are indeed solutions of some forms of estimating equations. Though the MLE score-based estimating equations are known to be optimal, use of estimating equations to come up with robust (to departures from the model assumptions) and efficient estimators will be an interesting and challenging problem to undertake!

In this regard, three different approaches have been taken in the literature, in our view, concerning robust estimation of model parameters of BS distribution. One is basing the estimation on censored samples, so that the effect of extreme order statistics are minimized in the estimation process. Second is the traditional influence function approach put forward by Dupuis and Mills (1998), and the third one is by basing the inference on generalized Birnbaum-Saunders distribution based on a heavy-tailed kernel (instead of the traditional normal one). It is true that, as correctly pointed out by Professor Desmond, that such a generalized model may lack a motivation from the underlying fatigue process itself, it does provide great flexibility while modelling observed data when the usual Birnbaum-Saunders distribution is found to be inadequate. The robustness of this approach has indeed been evaluated and verified by some authors through the insensitivity of estimates obtained from these models in the presence of outlying observations. Finally, we agree with Professor Desmond that mixtures of Birnbaum-Saunders distributions and associated inferential methods need to be studied further as they may be suitable models in situations wherein the crack size at each load has a positively skewed distribution!

Professor Augustine Wong: Professor Wong has detailed the third-order likelihood-based method for the estimation of the shape and scale parameters of the Birnbaum-Saunders distribution. As he aptly points out, this method seems to be very accurate even in case of very small sample sizes. But, the biggest problem in the use of this method of estimation is in obtaining $q(\psi)$, the standardized maximum likelihood estimate in the canonical parameter scale, especially in the case of censored data. Moreover, as Professor

Wong has pointed out, this method is not applicable for the case of tests of hypotheses involving the shape parameters.

His idea of using parametric bootstrap approach for determining an "empirical" Bartlett correction factor in the modified log-likelihood ratio statistic proposed by Bartlett (1937) is an interesting one and simple to implement! However, the efficiency of this method is not clear if the sample size n is indeed small, or when the sample involves censoring. We fully agree with Professor Wong that it will be very interesting to explore this direction further and examine the performance of the confidence intervals as well as tests of hypotheses for the model parameters, both in one- and multi-sample situations, under this approach!

Professors Artur Lemonte and Gauss Cordeiro: First, we acknowledge the two points made by Professors Lemonte and Cordeiro with regard to the typo present in the expression of $\mathbf{I}(\alpha, \beta)$ presented in Section 3.1 as well as its expression given by Lemonte et al. (2007).

While discussing inferential methods for the parameters of the classical Birnbaum-Saunders distribution, they have focused specifically on the case when the value of the shape parameter α is quite small, close to zero. They have pointed out that some numerical problems are encountered in using the Fisher information matrix due to the instability faced in the numerical evaluation of the function

$$h(\alpha) = \alpha \left(\frac{\pi}{2}\right)^{1/2} - \pi e^{2/\alpha^2} \left\{1 - \Phi\left(\frac{2}{\alpha}\right)\right\}$$

when α is close to zero. For this reason, they have then provided an approximation for the function $h(\alpha)$ for small values of α, and have demonstrated its usefulness through numerical computation as well as an illustrative example.

If $\phi(z)$ and $\Phi(z)$ denote the pdf and cdf of a standard normal distribution, respectively, then the function

$$M(z) = \frac{1 - \Phi(z)}{\phi(z)} = \sqrt{2\pi} e^{z^2/2} \{1 - \Phi(z)\}$$

is the well-known Mills' ratio. Many accurate asymptotic (as $z \to \infty$) approximations are available in the literature for this function; see Johnson et al. (1994) for some pertinent details and also the recent article by Gasull and Utzet (2014). Upon setting $z = 2/\alpha$, it is readily seen that

$$h(\alpha) = \alpha \left(\frac{\pi}{2}\right)^{1/2} - \sqrt{\frac{\pi}{2}} M \left(\frac{2}{\alpha}\right),$$

and so the well-known asymptotic approximations of $M(z)$ (as $z \to \infty$, or equivalently, as $\alpha \to 0$) can be readily used for evaluating the quantity $h(\alpha)$ in a computationally stable way for small values of α. This, in turn, can be utilized in developing efficient inferential procedures for the case when α value is small!

Professor Biswabrata Pradhan: As described in the paper, EM algorithms (and some variations) have been developed for the determination of maximum likelihood estimates of the parameters of the Birnbaum-Saunders distribution under complete, Type-I censored, Type-II censored, hybrid censored and progressively censored samples. In his discussion, Professor Pradhan has considered a progressive Type-I interval censoring scheme, along the lines of Aggarwala (2001), when the underlying lifetime distribution is Birnbaum-Saunders, and then has proposed an EM algorithm for the determination of maximum likelihood estimates of the model parameters α and β by proceeding as done by Ng et al. (2002). For the E-step, the mixture representation of the Birnbaum-Saunders distribution in terms of inverse Gaussian and reciprocal inverse Gaussian distributions has been utilized.

Even though Professor Pradhan has not explicitly mentioned, a large sample size n has been implicitly assumed for the implementation of this EM algorithm. This is so because the progressive censoring numbers R_i's are assumed to be pre-fixed and assumed to be so in his pseudo log-likelihood function. This may be violated in small to moderate values of the sample size n, since in this case, the pre-fixed number of units, R_i, for removal may not be available for removal at all stages. In this case, the numbers to be censored also become "random" because of the inherent form of progressive censoring involved. This may affect the E-step of the algorithm, and its effect may have to be evaluated through a Monte Carlo simulation study. Another difficulty may be in the evaluation of all the integrals involved in the E-step. Unfortunately, no closed-form expressions seem possible for most of these integrals. Though this is the case, it may not, however, be a big drawback since a stochastic method may be readily adopted for the evaluation of all the integrals as they are expectations of some functions of inverse Gaussian random variables; of course, this will tantamount to having a stochastic EM-algorithm!

Finally, we agree with his comment that the Birnbaum-Saunders distribution is close to the lognormal distribution in terms of its shape, characteristics and hazard properties, and hence discrimination between these two distributions for a given data will be of great interest and will be a problem worth studying!

Professor Hon Keung Tony Ng: In his detailed discussion, Professor Ng has specifically addressed two issues. The first is with regard to the similarity between Birnbaum-Saunders distribution and three other popular lifetime distributions, namely, inverse Gaussian, lognormal and Weibull. It is true that in terms of shape and characteristics, all these distributions share similar properties. He has then gone to using moment-ratio diagrams and maximum likelihood method to explore further this similarity and the model selection (specifically, probabilities of selection of the true model and the wrong models, among these four) through an empirical study. Though the probabilities of correct selection are understandably small when the sample size is small, they do increase substantially when the sample size increases. He has then considered the data in Example 1 and has demonstrated that the Birnbaum-Saunders, inverse Gaussian and lognormal distributions all provide good fit (though the Birnbaum-Saunders yields the largest maximized log-likelihood value), while the Weibull distribution does not.

In our viewpoint, though the two methods, viz., the moment-ratio and maximum likelihood methods, perform very similarly in the situations considered by Professor Ng, the latter certainly has the advantage that it is easily extendable to multivariate cases, and also would facilitate carrying out tests for validity at a specified level of significance (likely using a parametric bootstrap approach). These issues certainly deserves further attention!

Professors Longxiang Fang and Xiaojun Zhu: As the article clearly shows, considerable amount of work has gone on with regard to various properties, inferential methods, extensions and generalizations of Birnbaum-Saunders distribution during the last five decades. However, no work had been done with regard to stochastic orderings of systems with its components having heterogeneous Birnbaum-Saunders lifetime distributions until recently. Fang et al. (2016, 2018) and Fang and Balakrishnan (2018) carried out several stochastic comparisons of lifetimes of series and parallel systems with heterogeneous Birnbaum-Saunders components. However, their results were only for the case when the components are independent.

Quite interestingly, in their discussion, Professors Fang and Zhu have considered stochastic comparisons of systems by allowing dependence between the components. Specifically, they have done so by using Archimedean copula to model the dependence structure between the components of the system. Under such a dependence structure, they have then discussed stochastic comparisons of the lifetimes of parallel as well as series systems with respect to the usual stochastic order based on vector majorization of scale and shape parameters of the Birnbaum-Saunders distributions of the components. The results established earlier for the independent case follow readily by choosing the Archimedean copula generator to be $\phi(t) = e^{-t}$.

This, in our view, is an important development and opens many problems for further consideration. For example, it is not immediate whether such results could be established or not for systems with dependent generalized Birnbaum-Saunders components and also for more general copula structures? Moreover, stochastic comparisons for systems other than series and parallel systems will also be of interest!

Professors Isha Dewan and Swagata Nandi: In many practical datasets commonly found in reliability, survival and financial applications, the underlying data are non-negative. If one were to use a nonparametric method for the approximation of density function using kernel-based method, rather than adopting the usual parametric method, then it would necessitate the use of a kernel with support on the positive real line instead of the usual normal kernel. For this purpose, distributions such as exponential and Birnbaum-Saunders have been used in the literature. In their discussion, Professors Dewan and Nandi have carried out a Monte Carlo simulation study of kernel density estimators with normal, exponential and Birnbaum-Saunders distributions as kernels, and have shown empirically that the normal kernel leads to positive estimates for negative part of the real line, and also that the density estimates based on both exponential and Birnbaum-Saunders kernels give almost identical results. They further verified the same findings with a security-wise price volume historic data set from the Indian Stock Market.

Their idea to consider nonparametric kernel estimation of the joint density function of a vector-valued random variable wherein each component is non-negative with the use of multivariate Birnbaum-Saunders distribution and multivariate exponential distribution is certainly interesting and will be a very fruitful project to consider!

Professors Ahad Jamalizadeh, Farzane Hashemi and Mehrdad Naderi: In their interesting discussion, Professors Jamalizadeh, Hashemi and Naderi have first constructed a unified skew-ellipitical Brinbaum-Saunders distribution by basing it on the transformation in Eq. (5) of the paper and then assuming the underlying variable Z to have a SUE(λ,$\mathbf{\Omega}$,$\mathbf{h^{(1+q)}}$). This generalization includes some recently introduced extensions of the Birnbaum-Saunders distribution such as the skew-normal-Birnbaum-Saunders distribution of Vilca et al. (2011) and the skew-t-Birnbaum-Saunders distribution of Khosravi et al. (2016).

Professors Jamalizadeh, Hashemi and Naderi then go on to discuss the distributions of two-component series and parallel systems whose components jointly have a generalized bivariate Birnbaum-Saunders distribution and specifically express them as mixtures of unified skew-elliptical Birmbaum-Saunders distributions by proceeding in a manner analogous to those of Jamalizadeh and Balakrishnan (2010). They have then presented generalizations of these results to $p - r + 1$-out-of-p systems.

As these authors have mentioned, the proposed generalization does offer a great deal of flexibility in addition to including some of the previously introduced generalizations of Birnbaum-Saunders distribution. However, as can be seen from the plots presented in their Figure 1, the densities as well as hazard functions of these reliability systems can have bimodality in their shape. Interpretation of such shape characteristics for a given data would be a challenging task. Despite this, it would be interesting to carry out a detailed theoretical study of the properties of reliability and hazard functions of these systems in its general construct. Moreover, it would also be interesting to examine whether some stochastic ordering results could be proved for these systems along the lines of Fang et al.

(2016, 2018) and Fang and Balakrishnan (2018)!

Professors Victor Leiva, Camilo Lillo, Ivette Gomes and Marta Ferreira: In their discussion, Professors Leiva, Lillo, Gomes and Ferreira provide a discussion on another generalized model based on Birnbaum-Saunders distribution, namely, the extreme value Birnbaum-Saunders distribution. This distribution, based on a generalized extreme value distribution being used as a kernel in place of the normal one, was introduced originally by Ferreira et al. (2012). As the authors mention, this model has considerable flexibility and is a good alternative to the classical Birnbaum-Saunders distribution due to its flexibility and suitability for modelling heavy-tailed phenomena that could arise naturally, for example, in financial and environmental data.

They have then discussed a method of estimation of the model parameters based on L-moments, a regression model based on the extreme value Birnbaum-Saunders distribution, and have illustrated finally the usefulness of the model and the inferential methods with a financial data.

There are two points that are worth mentioning about this generalized model. First, the usual Birnbaum-Saunders distribution does not belong to this family of models, as the authors have already mentioned in their discussion. Second, in the case when the extreme value index parameter $\xi \neq 0$, the support of the distribution has a lower bound depending on the model parameters. While analyzing real-life data, could a meaningful interpretation of this lower bound be provided? Another important related matter is regarding the infeasibility in the estimation of model parameters. Specifically, when the support of a distribution is bounded by a function of the parameters, such as the case in this model, one would ideally like to obtain point estimators such that the data used to obtain these estimators are also bounded by the same function of these estimators. Unfortunately, this does not always happen with methods based on L-moments and probability weighted moments, as observed by Hosking (1986) and Chen and Balakrishnan (1995). The latter authors carried out a detailed empirical study and noted that for the generalized extreme value distribution, the probabilities of obtaining infeasible parameter estimates decrease for increasing sample sizes, but this depends on the value of the parameter ξ itself. If it is away from 0, then this probability may not be small even for large sample sizes. It should be noted that this problem will not arise if one were to use maximum likelihood method as the bound will be used in obtaining the estimates based on a constrained maximization algorithm (note, however, that this constrained optimization may pose some numerical challenge).

It, therefore, will be of great interest to examine this issue for the L-moment estimation of model parameters for the extreme value Birnbaum-Saunders distribution that the authors have proposed and discussed. If the mentioned issue is persistent in this generalized model, then it would also be of interest to propose a suitable modification to the method of estimation to avert this infeasibility issue!

Professors Michelli Barros and Gilberto Paula: While fitting the Birnbaum-Saunders model (as well as other related models) to real datasets, especially in a regression setup, it is important to assess the validity of the fitted model which is often done with the use of a residual analysis; see, for example, Leiva et al. (2007). In the case of Birnbaum-Saunders regression, this is done through quantile residuals based on the transformation in Eq. (4) of the paper. Then, using the asymptotic normality of these quantile residuals under the assumed model, one could readily assess the validity of the assumed Birnbaum-Saunders model. This is a simple and efficient way to validate the model, though a formal goodness-of-fit test can also be thought of for this specific purpose!

In their discussion, Professors Barros and Paula suggest such a validation method for bivariate and multivariate Birnbaum-Saunders models. For this specific purpose, they recommend in the bivariate case, for example, the usage of quantile residuals based on the marginal distribution of T_1 and the conditional distribution of T_2, given $T_1 = t_1$. As they have shown, one could establish joint asymptotic normality of these quantile residuals under the assumed model, and then use it to readily assess the validity of the assumed bivariate Birnbaum-Saunders model. As they have demonstrated its applicability using Example 5 in the paper, it is a simple and efficient way to validate the use of the bivariate model. It will, of course, be of great interest to develop some formal goodness-of-fit tests from this process and evaluate their efficiency and power properties!

Professor Filidor Vilca: We appreciate the review of several issues that Professor Vilca has provided, including two new and novel applications of Birnbaum-Saunders modelling in the analysis of neodymium and zinc measurements by Reyes et al. (2017) and in the analysis of dependent georeferenced data by Garcia-Papani et al. (2018). His comment about the use of heavy-tailed distributions as a kernel while developing generalized Birnbaum-Saunders distributions for the parameter estimation which is resistant to the presence of outliers is precisely what has been mentioned above in our response to Professor Desmond in this specific regard. In the case of multivariate data on dominant radius (Dr), radius (r), dominant ulna (Du) and ulna (u) of 25 newborn babies reported in Johnson and Wichern (1999), Professor Vilca is correct in that there are a few outlying observations, clearly visible from his plot. For this reason, some authors have used multivariate generalized Birnbaum-Saunders distribution for fitting these data in order to lessen the effect of these outliers. However, this does suggest that "robust estimation" of model parameters of a multivariate Birnbaum-Saunders distribution, *a la* Dupuis and Mills (1998), is an interesting problem that remains open! His other suggestion of using a test of fit for elliptical distribution as developed by Batsidis and Zografos (2013) for the validation of a multivariate generalized Birnbaum-Saunders distribution is quite interesting and its power and relative performance as compared to other goodness-of-fit methods and graphical tools will certainly be worth studying as well!

Professors José Díaz-Garcia and Francisco Caro-Lopera: Even though considerable amount of work has gone into various extensions, generalizations and multivariate forms, hardly any work had been done on matrix-variate forms of Birnbaum-Saunders distributions, as aptly mentioned by Dáz-Garcia and Caro-Lopera in their discussion. As they have mentioned, four most pertinent references in this direction are Caro-Lopera et al. (2012), Sánchez et al. (2015), Caro-Lopera and Díaz-Garcia (2016), and Díaz-Garcia and Caro-Lopera (2018). In their discussion, Professors Díaz-Garcia and Caro-Lopera have provided a discussion on the construction of matrix-variate Birnbaum-Saunders distribution through the transformation $\mathbf{T} = \mathbf{V}'\mathbf{V}$. Then, by using known properties of elliptical distributions and the results of Díaz-Garcia and Caro-Lopera (2018) and Díaz-Garcia and Domńguez Molina (2007), they have introduced a matrix-variate Birnbaum-Saunders distribution and have discussed its properties. They have specifically provided density and joint density functions, and then have demonstrated their use while fitting a data set. The results are interesting and also suggest many further problems to consider in the future in this direction.

Though the results are presented in their discussion for general elliptical family, they have specifically used the Kotz kernel $g(t)$ for their illustrative example. In this connection, one issue that will be of interest will be to consider model discrimination and model validation methods for a given data between different members of the elliptical family; see also the comments made by Professor Filidor Vilca in this regard and our response above to his comments. Another problem that deserves good attention is in developing efficient inferential methods for these matrix-variate models!

Professor Ramesh Gupta: Professor Gupta has highlighted the close connection between the Birnbaum-Saunders distribution and the inverse Gaussian distribution. Based on this, he has then discussed the role of Birnbaum-Saunders model as a frailty distribution in the analysis of univariate frailty data as well as in the analysis of bivariate shared frailty model. This has, in fact, been studied extensively in the works of Leão et al. (2017, 2018a,b), Balakrishnan and Liu (2018), and Liu and Balakrishnan (2018) for both frailty data and for frailty data with cure fraction. His comment that it is important to study some theoretical properties of the Birnbaum-Saunders frailty model is a valid one!

Professors Victor Leiva, Robert Aykroyd and Carolina Marchant: First, we want to express our appreciation to Professors Leiva, Aykroyd and Marchant for mentioning some of the most recent papers dealing with theory and applications of multivariate Birnbaum-Saunders distributions, in addition the ones that we have listed in our article. This certainly makes the bibliography more complete and up-to-date as well.

Then, based on the transformation in Eq. (5) of the paper for each of the variables $T_1, \cdots, T_m$ by assuming the underlying variables $Z_, \cdots, Z_m$ to have a m-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance (also correlation) matrix $\mathbf{\Gamma}$, they have considered the multivariate Birnbaum-Saunders distribution in a multivariate regression context, and specifically discussed the estimation of model parameters and model diagnostics. They have then demonstrated these methods on an economic data.

For the purpose of validation of the considered model for the dataset, they have used a Mahalanobis distance after transforming the data to normality based on Wilson-Hilferty approximation as well as Kolmogorov-Smirnov statistic for testing normality following the transformation. Though these are simple and convenient methods for validation, there is one potential problem, in our view, in that the statistic used will, in addition to capturing the effects due to model departure, also get adversely affected by the transformation itself (as it involves estimates of all the model parameters) as well as by the use of the approximation. This may be especially so if the sample size is not large enough, as is the case with the illustrative example they have considered. This clearly suggests the need for the development of more (direct?) model validation and diagnostic techniques for the multivariate Birnbaum-Saunders distribution as well as the regression model based on it!

Professors Helton Saulo, Jeremiaz Leão and Manoel Santos-Neto: In their discussion, Professors Saulo, Leão and Santos-Neto consider an extension of the reparametrized Birnbaum-Saunders regression model by allowing for a time-varying conditional mean in the underlying model. This is in contrast to the works on Birnbaum-Saunders time series models discussed recently in the literature by Bhatti (2010), Mayorov (2011), Saulo et al. (2017) and Rahul et al. (2018). For this purpose, they start with the reparametrized Birnbaum-Saunders distribution, which expresses the distribution in terms of mean and precision parameters, and then vary the mean parameter as a time-varying function of covariates. They have then discussed the estimation of model parameters and residual analysis, and have illustrated their model and inferential results with a simulated data.

An interesting problem in this direction would be to consider the differences between this type of modelling of time series data and other time series models based on Birnbaum-Saunders model such as conditional-median duration model and marginal Birnbaum-Saunders model. One could also examine the sensitivity to estimation and robustness properties of the fit achieved by these models. Furthermore, an extension of the model considered by Professors Saulo, Leão and Santos-Neto to the multivariate case will also worth exploring! In this regard, it is important to mention here the recent work of Tan (2017) in which a bivariate conditional median duration model for matched trade high frequency data has been discussed and associated inferential methods have been developed in detail.

Professors Shuangzhe Liu and Tiefeng Ma: We are thankful to Professors Liu and Ma for providing a concise review of various inferential issues for Birnbaum-Saunders distribution and its generalized forms, Birnbaum-Saunders regression and associated estimation and model diagnostics, and finally different time series models based on Birnbaum-Saunders distribution. Their discussion is along the lines of Professors Leiva, Aykroyd and Marchant and also those of Professors Saulo, Leão and Santos-Neto. So, our responses to their comments above hold true here as well. Their comment about the identification of change points of the Birnbaum-Saunders distribution with the use of Niu and Zhang's (2012) screening and ranking algorithm or by Fryzlewicz's (2014) wild binary segmentation algorithm is indeed interesting. Work along these lines for various generalized forms as well as multivariate forms of Birnbaum-Saunders distribution will be very challenging as well as useful, in our view!

Concluding Remarks: We wish to conclude this response by expressing our sincere thanks to all the discussants for their incisive comments as well as useful further details they have provided in the form of results and additional references. It is our sincere hope that our overview article, along with all these discussions and our response, reveal quite clearly to the readers that this is still a fertile area of research with several open problems and paths for further exploration!

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