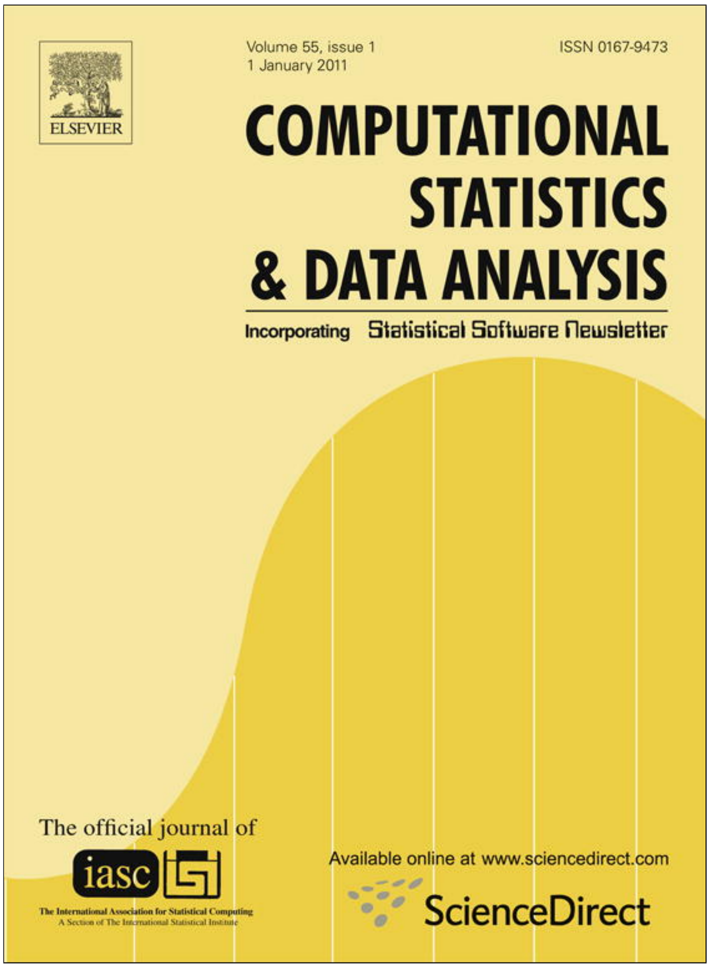
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Inferences on Weibull parameters with conventional type-I censoring

Avijit Joarder [a,1], Hare Krishna [b], Debasis Kundu [c,*]

[a] Reserve Bank of India (RBI), Main Branch, Mumbai Pin 400001, India

[b] Department of Statistics, C.C.S. University, Meerut, Pin 200005, India

[c] Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur, Pin 208016, India

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ABSTRACT

In this article we consider the statistical inferences of the unknown parameters of a Weibull distribution when the data are Type-I censored. It is well known that the maximum likelihood estimators do not always exist, and even when they exist, they do not have explicit expressions. We propose a simple fixed point type algorithm to compute the maximum likelihood estimators, when they exist. We also propose approximate maximum likelihood estimators of the unknown parameters, which have explicit forms. We construct the confidence intervals of the unknown parameters using asymptotic distribution and also by using the bootstrapping technique. Bayes estimates and the corresponding highest posterior density credible intervals of the unknown parameters are also obtained under fairly general priors on the unknown parameters. The Bayes estimates cannot be obtained explicitly. We propose to use the Gibbs sampling technique to compute the Bayes estimates and also to construct the highest posterior density credible intervals. Different methods have been compared by Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

1. Introduction

Type-I and Type-II censoring schemes are the two most popular censoring schemes used in the reliability and life testing experiments. In this article we consider the conventional Type-I censored lifetime data, when the lifetime of the experimental unit follows a two-parameter Weibull distribution. A Type-I censoring sampling scheme can be described as follows. Suppose n units, denoted by $Y_1, \ldots, Y_n$, are placed on a life testing experiment. The lifetimes of the sample units are independent and identically distributed (*i.i.d.*) random variables. The test is terminated when a pre-specified time point, T, on test has been reached. It is also assumed that the failed items are not replaced.

The Weibull distribution is one of the most popular distributions in analyzing skewed data. Because of its various shapes of the probability density functions and due to the monotonicity property of the hazard function it has been used quite extensively in place of the gamma distribution. Since it has a closed form cumulative distribution function, it can be used very effectively for analyzing censored data. The Weibull distribution was originally proposed by Waloddi Weibull, a Swedish mechanical engineer, way back in 1937, and it became widely known in 1951. Extensive work has been done on the Weibull distribution since then. Almost a book length treatment on Weibull distribution can be found in Chapter 21 of Johnson et al. (1995).

It is well known that the maximum likelihood estimators (MLEs) of the unknown parameters of a Weibull distribution cannot be obtained in closed form. In this paper it is observed that the MLEs can be obtained by solving a fixed point

* Corresponding author. Tel.: +91 512 2597141; fax: +91 512 2597500.
E-mail addresses: ajoarder@rbi.org.in (A. Joarder), hkrishnastats@yahoo.com (H. Krishna), kundu@iitk.ac.in (D. Kundu).

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type equation. We propose a simple iterative scheme to compute the MLEs, and the proposed method works quite well. Since the MLEs cannot be obtained in explicit forms, we propose approximate maximum likelihood estimators (AMLEs), which can be obtained by expanding the normal equations using Taylor series. AMLEs have explicit forms, and therefore they can be computed very easily. In the case of a two-parameter Weibull distribution, it is not possible to compute the exact distributions of the MLEs. We have used the asymptotic distribution of the MLEs to construct approximate confidence intervals of the unknown parameters, based on MLEs. Since we are not able to obtain the exact and asymptotic distribution of the AMLEs, we use the asymptotic distribution of the MLEs, and by replacing the MLEs with the AMLEs, we construct the approximate confidence intervals based on AMLEs. We construct confidence intervals of the unknown parameters, based on the bootstrapping method also.

We further consider the Bayesian inference of the unknown parameters of a two-parameter Weibull distribution. For the Bayesian inference, we need to assume some prior distributions of the unknown parameters. If the shape parameter of the Weibull distribution is known, the most natural choice of the prior of the scale parameter is the conjugate gamma prior. If the shape parameter is also unknown, the continuous conjugate priors do not exist, see for example Kaminsky and Krivtsov (2005), although there exists a continuous-discrete prior distribution, see Soland (1969). The continuous component of this distribution is related to the scale parameter, and the discrete one is related to the shape parameter. This method has been widely criticized in the literature because of its difficulty in applications to real life problems, see Kaminsky and Krivtsov (2005). Another approach is to use the same conjugate prior on the scale parameter, and use an independent uniform prior on the shape parameter. Some authors use independent uniform priors on both the shape and scale parameters, see for example Smith and Naylor (1987) or Dellaportas and Wright (1991). Clearly they have their own limitations.

In this paper we have used the same gamma prior on the scale parameter, but we have used a fairly general prior on the shape parameter, and that will be explained in detail in Section 6. The Bayes estimates of the unknown parameters cannot be obtained in explicit forms as expected. We have used the Markov chain Monte Carlo (MCMC) technique to compute the Bayes estimates and also to construct the highest posterior density (HPD) credible intervals. Extensive simulations are performed to compare the performances of the MLEs, approximate MLEs and the Bayes estimators. One data set has been analyzed for illustrative purposes.

It should be mentioned that in the frequentist set-up the comparison of the different estimators of the Weibull parameters can be found in Hossain and Zimmer (2002). Jeng and Meeker (2000) compared the performances of the different confidence intervals. But none of them considered the AMLEs. Moreover, the comparison of the MLEs or AMLEs with the Bayesian estimators and the comparison between the confidence intervals with the corresponding credible intervals are not available in the literature. We believe that is the main contribution of this paper.

The rest of the paper is organized as follows. In Section 2, we describe the model and notations. The MLEs and AMLEs are provided in Sections 3 and 4 respectively. Bootstrap confidence intervals and Bayesian inferences are provided in Sections 5 and 6 respectively. Simulation results are presented in Section 7. One data set is analyzed and the results are presented in Section 8. Finally we conclude the paper in Section 9.

2. Model description and notations

Suppose the lifetime random variable Y has a Weibull distribution with shape and scale parameters α and λ respectively, i.e., the probability density function (PDF) of Y is;

$$f_Y(y;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-(\frac{y}{\lambda})^{\alpha}}; \quad y > 0, \tag{1}$$

where $\alpha > 0, \lambda > 0$. If the random variable Y has density function (1), then $X = \ln Y$ has the extreme value distribution with PDF;

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma} e^{\left(\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}\right)}; \quad -\infty < x < \infty, \tag{2}$$

where $\mu = \ln\lambda, \sigma = \frac{1}{\alpha}$. The density function as described by (2) is known as the density function of an extreme value distribution with location and scale parameters as μ and σ respectively.

Models (1) and (2) are equivalent models in the sense the procedure developed under one model can be easily used for the other model. Although they are equivalent models, sometimes it is easier to work with model (2) than model (1), because in model (2), the two parameters μ and σ appear as location and scale parameters. In fact it is observed that in deriving the AMLEs, it is easier to work with model (2) than model (1). For $\mu = 0$ and $\sigma = 1$, model (2) is known as the standard extreme value distribution and it has the following PDF;

$$f_Z(z;0,1) = e^{z-e^{z}}; \quad -\infty < z < \infty. \tag{3}$$

Now we describe the data available under Type-I censoring scheme. Note that under this scheme, it is assumed that n identical items are put on a test and the lifetime random variables of the n items are denoted by $Y_1, \ldots, Y_n$ and censoring time T is known in advance. We denote the ordered lifetimes of these life testing items by $Y_{1:n}, \ldots, Y_{n:n}$. Let $d\ (\leq n)$ be the number of units that fail up to and including the pre-fixed time point T. Therefore, under this conventional Type-I censoring scheme, the observations are

$$\{Y_{1:n}, \ldots, Y_{d:n}\} \quad \text{where } 0 \le d \le n \text{ and } Y_{d:n} < T < Y_{d+1:n}. \tag{4}$$

It may be mentioned that although we do not observe $Y_{d+1:n}$, but $Y_{d:n} < T < Y_{d+1:n}$ means that the d-th failure took place before T and no failure took place between $Y_{d:n}$ and T, i.e. $Y_{d+1:n}, \ldots, Y_{n:n}$ are not observed.

3. Maximum likelihood estimation

In this section we provide the MLEs of the unknown parameters. Based on the observed data, the likelihood function is

$$l(\alpha, \lambda) = \frac{n!}{(n-d)!}\left(\frac{\alpha}{\lambda}\right)^d \prod_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha-1} e^{-\left[\sum_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha}+(n-d)\left(\frac{T}{\lambda}\right)^{\alpha}\right]} \quad \text{if} \quad d > 0, \tag{5}$$

$$= e^{-n\left(\frac{T}{\lambda}\right)^{\alpha}} \quad \text{if } d = 0. \tag{6}$$

It is clear from (6) that if $d = 0$, the MLEs of α and λ do not exist. Therefore, from now on we assume that $d \neq 0$. The logarithm of (5) without the constant term can be written as

$$L(\alpha, \lambda) = d(\ln\alpha - \ln\lambda) + (\alpha - 1)\left[\sum_{i=1}^{d}\ln y_{i:n} - d\ln\lambda\right] - \sum_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha} - (n-d)\left(\frac{T}{\lambda}\right)^{\alpha}. \tag{7}$$

Taking derivatives with respect to α and λ of (7) and equating them to zero, we obtain

$$\frac{\partial L(\alpha, \lambda)}{\partial\lambda} = -\frac{\alpha d}{\lambda} + \frac{\alpha}{\lambda^{\alpha+1}}\left[\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha}\right] = 0 \tag{8}$$

$$\frac{\partial L(\alpha, \lambda)}{\partial\alpha} = \frac{d}{\alpha} + \sum_{i=1}^{d}\ln y_{i:n} - d\ln\lambda - \sum_{i=1}^{d}\left(\frac{y_{i:n}}{\lambda}\right)^{\alpha}(\ln y_{i:n} - \ln\lambda) - (n-d)\left(\frac{T}{\lambda}\right)^{\alpha}(\ln T - \ln\lambda) = 0. \tag{9}$$

From (8), we obtain

$$\lambda^{\alpha} = \frac{\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha}}{d} = u(\alpha) \quad \text{(say)}. \tag{10}$$

Using (10), (9) can be rewritten as

$$\frac{d}{\alpha} = \frac{d}{\alpha}\ln u(\alpha) - \sum_{i=1}^{d}\ln y_{i:n} + \sum_{i=1}^{d}\frac{y_{i:n}^{\alpha}}{u(\alpha)}\left(\ln y_{i:n} - \frac{1}{\alpha}\ln u(\alpha)\right) + (n-d)\frac{T^{\alpha}}{u(\alpha)}\left(\ln T - \frac{1}{\alpha}\ln u(\alpha)\right) \tag{11}$$

or

$$\frac{1}{\alpha}\left[d - d\ln u(\alpha) + \sum_{i=1}^{d}\frac{y_{i:n}^{\alpha}}{u(\alpha)} \times \ln u(\alpha) + (n-d)\frac{T^{\alpha}}{u(\alpha)} \times \ln u(\alpha)\right]$$

$$= -\sum_{i=1}^{d}\ln y_{i:n} + \sum_{i=1}^{d}\frac{y_{i:n}^{\alpha}}{u(\alpha)} \times \ln y_{i:n} + (n-d)\frac{T^{\alpha}}{u(\alpha)} \times \ln T. \tag{12}$$

Note that (12) can be written in the form:

$$\alpha = h(\alpha) \tag{13}$$

where

$$h(\alpha) = \frac{d(1 - \ln u(\alpha)) + \frac{\ln u(\alpha)}{u(\alpha)}\left(\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha}\right)}{-\sum_{i=1}^{d}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}\ln y_{i:n} + (n-d)T^{\alpha}\ln T\right]}$$

$$= \frac{d}{-\sum_{i=1}^{d}\ln y_{i:n} + \frac{1}{u(\alpha)}\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}\ln y_{i:n} + (n-d)T^{\alpha}\ln T\right]}.$$

We propose a simple iterative scheme to solve for α from (13). Start with an initial guess of α, say $\alpha^{(0)}$, obtain $\alpha^{(1)} = h(\alpha^{(0)})$ and, proceeding in this way, obtain $\alpha^{(n+1)} = h(\alpha^{(n)})$. Stop the iterative procedure, when $\left|\alpha^{(n+1)} - \alpha^{(n)}\right| < \epsilon$, some pre-assigned tolerance limit.

Since the MLEs, when they exist, are not in compact forms, we propose the following approximate maximum likelihood estimators (AMLEs) which have explicit expressions.

4. Approximate maximum likelihood estimators

Let us use the following notations; $x_{i:n} = \ln y_{i:n}$ and $S = \ln T$. Therefore, the likelihood equation of the observed data $x_{i:n}$ is

$$l(\mu,\sigma) = \frac{c}{\sigma^d}\prod_{i=1}^{d} g(z_{i:n})\left(\bar{G}(V)\right)^{n-d}, \tag{14}$$

where $z_{i:n} = \frac{(x_{i:n}-\mu)}{\sigma}$, $i = 1,\ldots,d$, $V = \frac{S-\mu}{\sigma}$, $g(x) = \mathrm{e}^{x-\mathrm{e}^x}$, $\bar{G}(x) = \mathrm{e}^{-\mathrm{e}^x}$, $\mu = \ln\lambda$, $\sigma = \frac{1}{\alpha}$ and $c =$ constant.

Ignoring the constant term, we obtain, using (14), the log-likelihood equation as,

$$L(\mu,\sigma) = \ln[l(\mu,\sigma)] = -d\ln\sigma + \sum_{i=1}^{d}\ln\left(g(z_{i:n})\right) + (n-d)\ln\left(\bar{G}(V)\right). \tag{15}$$

Taking derivatives with respect to μ and σ of $L(\mu,\sigma)$, and equating them to zero, gives

$$\frac{\partial L(\mu,\sigma)}{\partial\mu} = -\left(\frac{1}{\sigma}\right)\sum_{i=1}^{d}\frac{g'(z_{i:n})}{g(z_{i:n})} + (n-d)\times\left(\frac{1}{\sigma}\right)\times\frac{g(V)}{\bar{G}(V)} = 0 \tag{16}$$

$$\frac{\partial L(\mu,\sigma)}{\partial\sigma} = -\frac{d}{\sigma} - \sum_{i=1}^{d}\frac{g'(z_{i:n})}{g(z_{i:n})}\times\frac{z_{i:n}}{\sigma} + (n-d)\times\frac{g(V)}{\bar{G}(V)}\times\frac{V}{\sigma} = 0. \tag{17}$$

Note that the above two Eqs. (16) and (17) can be written equivalently as

$$-\sum_{i=1}^{d}\frac{g'(z_{i:n})}{g(z_{i:n})} + (n-d)\times\frac{g(V)}{\bar{G}(V)} = 0, \tag{18}$$

$$-d - \sum_{i=1}^{d}\frac{g'(z_{i:n})}{g(z_{i:n})}\times z_{i:n} + (n-d)\times\frac{Vg(V)}{\bar{G}(V)} = 0. \tag{19}$$

Clearly, (18) and (19) do not have explicit analytical solutions. We consider a first-order Taylor approximation to $g'(z_{i:n})/g(z_{i:n})$ and $g(V)/\bar{G}(V)$ by expanding around the actual mean μ_i, the means of standardized order statistic $Z_{i:n}$ and μ_d^* respectively, where $\mu_i = G^{-1}(p_i) = \ln(-\ln q_i)$, $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$ for $i = 1,\ldots,d$, and $\mu_d^* = G^{-1}(p_d^*) = \ln(-\ln q_d^*)$, $p_d^* = \frac{(p_d+p_{d+1})}{2}$, $q_d^* = 1 - p_d^*$ similar to Balakrishnan and Varadan (1991). Note that for $i = 1,\ldots,d$

$$\frac{g'(z_{i:n})}{g(z_{i:n})} \approx \alpha_i - \beta_i z_{i:n} \tag{20}$$

where

$$\alpha_i = \frac{g'(\mu_i)}{g(\mu_i)} - \mu_i\left[\frac{g''(\mu_i)}{g'(\mu_i)} - \left(\frac{g'(\mu_i)}{g(\mu_i)}\right)^2\right] = 1 + \ln q_i(1 - \ln(-\ln q_i)),$$

$$\beta_i = \left[-\frac{g''(\mu_i)}{g'(\mu_i)} + \left(\frac{g'(\mu_i)}{g(\mu_i)}\right)^2\right] = -\ln q_i$$

and

$$\frac{g(V)}{\bar{G}(V)} \approx 1 - \alpha_d^* + \beta_d^* V \tag{21}$$

where

$$\alpha_d^* = \frac{g'(\mu_d^*)}{g(\mu_d^*)} - \mu_d^*\left[\frac{g''(\mu_d^*)}{g'(\mu_d^*)} - \left(\frac{g'(\mu_d^*)}{g(\mu_d^*)}\right)^2\right] = 1 + \ln q_d^*(1 - \ln(-\ln q_d^*)),$$

$$\beta_d^* = \left[-\frac{g''(\mu_d^*)}{g'(\mu_d^*)} + \left(\frac{g'(\mu_d^*)}{g(\mu_d^*)}\right)^2\right] = -\ln q_d^*.$$

Using the approximations (20) and (21) in (18) and (19), we obtain

$$\left[\left(\sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}\right) - \left(\sum_{i=1}^{d} \mu_i e^{\mu_i} + D\mu_d^* e^{\mu_d^*}\right) - d\right]\sigma + \left[\sum_{i=1}^{d} X_{i:n} e^{\mu_i} + DSe^{\mu_d^*}\right] - \mu\left[\sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}\right] \approx 0 \qquad (22)$$

and

$$\left[d\left(\sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}\right)\right]\sigma^2 + \left[\left(\sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}\right)\left(\sum_{i=1}^{d} \mu_i X_{i:n} e^{\mu_i} + D\mu_d^* Se^{\mu_d^*} + d\bar{X}\right)\right]\sigma$$
$$- \left[\left(\sum_{i=1}^{d} X_{i:n} e^{\mu_i} + DSe^{\mu_d^*}\right)\left(\sum_{i=1}^{d} \mu_i e^{\mu_i} + D\mu_d^* e^{\mu_d^*} + d\right)\right]\sigma + \left[\sum_{i=1}^{d} X_{i:n} e^{\mu_i} + DSe^{\mu_d^*}\right]^2$$
$$- \left[\sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}\right]\left[\sum_{i=1}^{d} X_{i:n}^2 e^{\mu_i} + DS^2 e^{\mu_d^*}\right] \approx 0. \qquad (23)$$

The above two approximations (22) and (23) can be written as

$$(c_1 - c_2 - d)\sigma + d_1 - \mu c_1 \approx 0 \qquad (24)$$

$$A\sigma^2 + B\sigma + C \approx 0 \qquad (25)$$

where $c_1 = \sum_{i=1}^{d} e^{\mu_i} + De^{\mu_d^*}$, $c_2 = \sum_{i=1}^{d} \mu_i e^{\mu_i} + D\mu_d^* e^{\mu_d^*}$, $d_1 = \sum_{i=1}^{d} X_{i:n} e^{\mu_i} + DSe^{\mu_d^*}$, $d_2 = \sum_{i=1}^{d} X_{i:n}^2 e^{\mu_i} + DS^2 e^{\mu_d^*}$, $d_3 = \sum_{i=1}^{d} \mu_i X_{i:n} e^{\mu_i} + D\mu_d^* Se^{\mu_d^*}$, $A = dc_1$, $B = c_1(d_3 + d\bar{X}) - d_1(c_2 + d)$, $C = {d_1}^2 - c_1 d_2$ and $D = n - d$. The solution to the preceding equations yields the AMLEs

$$\widehat{\mu} = \frac{(c_1 - c_2 - d)\widehat{\sigma} + d_1}{c_1} \qquad (26)$$

$$\widehat{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \qquad (27)$$

Here we consider only positive root of σ. It is easily seen that these approximate estimators are equivalent but not unbiased. Unfortunately, it is not possible to compute the exact bias of $\widehat{\mu}$ and $\widehat{\sigma}$ theoretically because of intractability encountered in finding the expectation of $\sqrt{B^2 - 4AC}$.

5. Bootstrap confidence intervals

In this section we propose the confidence intervals based on the bootstrapping. The percentile bootstrap (Boot-p) method, proposed by Efron (1982), is widely used in practice. We have mainly used the parametric bootstrap method. To estimate the Boot-p confidence interval, we proceed as follows:

[1] Estimate $\widehat{\alpha}$ and $\widehat{\lambda}$ from the sample generated, using (13) and (10).

[2] Generate bootstrap sample $\{Y_{1:n}^*, \ldots, Y_{d^*:n}^*\}$, using $\widehat{\alpha}$, $\widehat{\lambda}$ and T. Obtain the bootstrap estimate of α and λ say, $\widehat{\alpha}^*$ and $\widehat{\lambda}^*$ respectively, using the bootstrap sample.

[3] Repeat Step [2] NBOOT times.

[4] Let $\widehat{CDF}(x) = P(\widehat{\alpha}^* \le x)$ and $\widehat{CDF}(y) = P(\widehat{\lambda}^* \le y)$, be the cumulative distribution functions of $\widehat{\alpha}^*$ and $\widehat{\lambda}^*$ respectively. Define $\widehat{\alpha}_{Boot\text{-}p}(x) = \widehat{CDF}^{-1}(x)$ for a given x. The approximate $100(1-\delta)\%$ Boot-p confidence interval for α is given by

$$\left(\hat{\alpha}_{Boot\text{-}p}\left(\frac{\delta}{2}\right), \hat{\alpha}_{Boot\text{-}p}\left(1 - \frac{\delta}{2}\right)\right).$$

Similarly, define $\widehat{\lambda}_{Boot\text{-}p}(y) = \widehat{CDF}^{-1}(y)$ for a given y. The approximate $100(1-\delta)\%$ confidence interval for λ is given by

$$\left(\hat{\lambda}_{Boot\text{-}p}\left(\frac{\delta}{2}\right), \hat{\lambda}_{Boot\text{-}p}\left(1 - \frac{\delta}{2}\right)\right).$$

6. Bayesian analysis

In this section we consider the Bayes estimation of the unknown parameters and also construct the HPD credible intervals. We re-parameterize the model as follows: $\theta = \frac{1}{\lambda^\alpha}$. Based on the new parameterization we consider the Bayes estimates of α and θ. In this case our likelihood function is,

$$l(\alpha,\theta) = k_1\alpha^d\theta^d\prod_{i=1}^{d} y_{i:n}^{\alpha-1}\mathrm{e}^{-\theta\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}+(n-d)T^{\alpha}\right]} \quad \text{if } d>0 \tag{28}$$

$$= k_2\mathrm{e}^{-n\theta T^{\alpha}} \quad \text{if } d=0 \tag{29}$$

where k_1 and k_2 are constants.

6.1. Prior and posterior distributions

Following the approach of Berger and Sun (1993), it is assumed that θ has a gamma prior, $Gamma(a, b)$, for $a, b > 0$*i.e.*

$$\pi_1(\theta) \propto \theta^{a-1}\mathrm{e}^{-b\theta}; \quad \theta > 0. \tag{30}$$

Here a and b are the known hyper-parameters. No specific form of prior $\pi_2(\alpha)$ on α is assumed here. It is only assumed that the support of $\pi_2(\alpha)$ on α is $(0,\infty)$, it is independent of θ, and $\pi_2(\alpha)$ is log-concave. It may be mentioned that many well known distributions, like normal, Weibull, gamma, log-normal etc. have log-concave density functions.

Based on the above prior assumptions, the joint density function of the *data*, α and θ becomes:

$$l(data,\alpha,\theta) \propto \alpha^d\theta^{a+d-1}\prod_{i=1}^{d} y_{i:n}^{\alpha-1}\mathrm{e}^{-\theta\left[\sum_{i=1}^{d} y_{i:n}^{\alpha}+(n-d)T^{\alpha}+b\right]}\pi_2(\alpha) \quad \text{if } d>0 \tag{31}$$

$$\propto \theta^{a-1}\mathrm{e}^{-\theta[nT^{\alpha}+b]}\pi_2(\alpha) \quad \text{if } d=0 \tag{32}$$

and we obtain the joint posterior density functions of α and θ given the *data* as

$$l(\alpha,\theta|data) = \frac{l(\alpha,\theta,data)}{\int_0^{\infty}\int_0^{\infty} l(\alpha,\theta,data)\mathrm{d}\alpha\mathrm{d}\theta}. \tag{33}$$

Suppose we compute the Bayes estimate of any function of α,θ, say $g(\alpha,\theta)=\gamma$, then the Bayes estimate of γ with respect to the squared error loss function becomes the posterior mean, i.e.,

$$\int_0^{\infty}\int_0^{\infty} l(\alpha,\theta/data)g(\alpha,\theta)\mathrm{d}\alpha\mathrm{d}\theta. \tag{34}$$

Note that even if we know $\pi_2(\alpha)$ explicitly, (34) cannot be computed analytically most of the time. We adopt the Gibbs sampling procedures to compute the Bayes estimates of α and λ.

It can be easily observed that the conditional density function of θ given α and *data* is

$$\pi_1(\theta|\alpha,data) = \begin{cases} Gamma\left(a+d, \sum_{i=1}^{d} y_{i:n}^{\alpha}+(n-d)T^{\alpha}+b\right) & \text{if} \quad d>0 \\ Gamma\left(a, nT^{\alpha}+b\right) & \text{if} \quad d=0. \end{cases} \tag{35}$$

We need the following results for further development.

Theorem 1. *The conditional density function of* α *given the data is log-concave*

Proof. See in the Appendix. □

It easily follows from the result of the Appendix that if the prior distribution of α is gamma (the shape parameter can be less than one or can be zero also) then the posterior density function of α is log-concave if $d \geq 1$. We use the method proposed by Devroye (1984) to generate a sample from a log-concave density function for Gibbs sampling purposes and they can be used to compute Bayes estimate and also to construct the HPD credible interval of γ. We use the following algorithm to compute the Bayes estimate of γ, say $\widehat{\gamma}_B$, and to construct its HPD credible interval.

Algorithm:

- Step 1: Generate α_1, from the log-concave density $l(.|data)$, as given in (37) or (38) depending on the value of d using the method proposed by Devroye (1984).
- Step 2: Generate θ_1 from $\pi_1(.|\alpha, data)$ as provided in (35).
- Step 3: Repeat Steps 1 and 2, M times and obtain α_i, θ_i and $\gamma_i = g(\alpha_i,\theta_i)$, for $i = 1,\ldots,M$.
- Step 4: $\widehat{\gamma}_B$ can be obtained as

$$\frac{1}{M}\sum_{i=1}^{M} g(\alpha_i,\theta_i).$$

- Step 5: Arrange γ_i for $i = 1,\ldots,M$, say $\gamma_{(1)} < \cdots < \gamma_{(M)}$.

- Step 6: From the ordered γ_i's a $100(1-2\beta)\%$ credible interval can be obtained as

$$(\gamma_{(i)}, \gamma_{([M(1-2\beta)]+i)}), \quad i = 1, \ldots, [M2\beta],$$

where $[x]$ denotes the largest integer less than or equal to x. Now the HPD credible interval can be obtained by choosing that interval which has the shortest length.

7. Numerical results and discussions

Since the performance of the different methods cannot be compared theoretically, we perform Monte Carlo simulations to compare the performances of the different estimators and also different confidence/ credible intervals for different sampling schemes. The term *different sampling schemes* means different n and T values. We mainly compare the performances of the MLEs, AMLEs and Bayes estimators of the unknown parameters, in terms of their biases and mean squared errors (MSEs). We also compare the average lengths of the asymptotic confidence/ credible intervals and their coverage percentages. All the computations are performed with a Pentium IV processor using FORTRAN-77 programs. In all cases we use the random deviate generator RAN2 proposed in Press et al. (1991).

Since λ is the scale parameter, we have taken in all cases $\lambda = 1$ without loss of generality. For simulation purposes, we present the results when T is of the form $(\lambda T)^{\frac{1}{\alpha}}$. The reason to choose T in that form is the following: if $\widehat{\alpha}$ represents the MLE or AMLE of α, then the distributions of $\frac{\widehat{\alpha}}{\alpha}$ and $\frac{\widehat{\lambda}}{\lambda}$ become independent of α and λ. Moreover, if we have two sets of parameters and T values, say $(\alpha_1, \lambda_1, T_1)$ and $(\alpha_2, \lambda_2, T_2)$, so that $(\lambda_1 T_1)^{\frac{1}{\alpha_1}} = (\lambda_2 T_2)^{\frac{1}{\alpha_2}}$, then the results associated with $\frac{\widehat{\alpha}_1}{\alpha_1}$ and $\frac{\widehat{\lambda}_1}{\alpha_1}$, will be the same as those of $\frac{\widehat{\alpha}_2}{\alpha_2}$, and $\frac{\widehat{\lambda}_2}{\lambda_2}$ respectively. For that reason without loss of generality, we report the result only for $\alpha = 1$ and $\lambda = 1$. But these results can be used for any other α and λ also.

For a given n and T, we generate $Y_1, \ldots, Y_n$, and consider only $Y_{1:n}, \ldots, Y_{d:n}$ such that $Y_{d:n} \le T$. We consider different n and T values. We have used three different choices of n and for each n we have taken four different T values. In each case we compute the MLEs and AMLEs of the unknown parameters. In computing the MLE, we use $\epsilon = 10^{-6}$, and we have used the true value of α as the initial guess value of α. It is also observed that if we use the AMLE of α as the initial guess value of α, then we also get the same results.

We compute the 95% confidence intervals based on the asymptotic distribution of the MLEs and also obtained by replacing the MLEs with AMLEs. We replicate the process 1000 times and report the average estimates, the MSEs, the average confidence lengths and coverage percentages. We also compute the 95% Boot-p confidence interval based on the MLEs and AMLEs. All the results are reported in Tables 2–5.

For computing the Bayes estimators, it is assumed that α and θ have $Gamma(a_1, b_1)$ and $Gamma(a_2, b_2)$ priors respectively. Moreover we use the non-informative priors of both α and θ, i.e. $a_1 = b_1 = a_2 = b_2 = 0$. In this case also we obtain the average estimates over 1000 replications and the associated MSEs. We also compute the HPD credible intervals in each replication and obtain the average length and the coverage percentages over 1000 replications. The results are reported in Table 6.

Now we compare different confidence intervals in terms of their average lengths and coverage percentages. In general it is observed that most of the methods work well unless n and T are very small. For most of the methods, it is observed that the average confidence lengths decrease as n increases for fixed T or the other way. Both MLEs and AMLEs, behave very similarly although the confidence intervals for AMLEs are slightly shorter than the confidence intervals for MLEs.

From Tables 2–6, the following general observations can be made. For all the methods, it is observed that (a) for fixed n when T increases from 0.75 to 2.00, the MSEs decrease and (b) for fixed T as n increases from 20 to 40 the MSEs decrease. The performances of the MLEs and AMLEs are very similar in all aspects. The MSEs of the Boot-p estimators and the Bayes estimators are marginally larger than those of the MLEs or AMLEs for small n but for large n they are the other way. The average lengths of the confidence interval based on the Boot-p approach are larger than the average lengths of other confidence/credible intervals, but the coverage percentages are usually larger than the other confidence intervals but smaller than the credible intervals. The average credible lengths are smaller than the average confidence lengths in all the cases considered, but the coverage percentages of the credible intervals are usually larger than the confidence intervals in most cases considered. Finally, it should be mentioned that Boot-p and Bayes estimates are most computationally expensive followed by MLEs and AMLEs.

8. Data analysis

In this section we present a data analysis for illustrative purposes. The data set is available in Bain and Engelhardt (1991). The data set represents the remission times of leukemia patients due to administering a new drug. Forty patients were administered the new drug which induces remission in leukamia and the experiment was terminated after 7 months (210 days). The following 22 remission times were observed: 47, 56, 58, 64, 77, 79, 89, 128, 131, 142, 144, 149, 163, 166, 175, 176, 184, 184, 188, 190, 191 and 204. Clearly, it is a Type-I censored sample with $n = 40$, and $D = 22$. Just for computational ease, we have divided all the data points by 100. It does not affect any inferential procedure.

Table 1
MLEs, AMLEs, 95% asymptotic and bootstrap confidence intervals of α and λ.

Methods	MLEs		AMLEs	
Parameters	α	θ	α	θ
Estimates	2.3539	0.1452	2.3343	0.1479
ACI	(1.4416, 3.2662)	(0.0412, 0.2493)	(1.4220, 3.2466)	(0.0439, 0.2519)
BCI	(1.4219, 3.2811)	(0.0387, 0.2518)	(1.4198, 3.2812)	(0.0379, 0.2610)

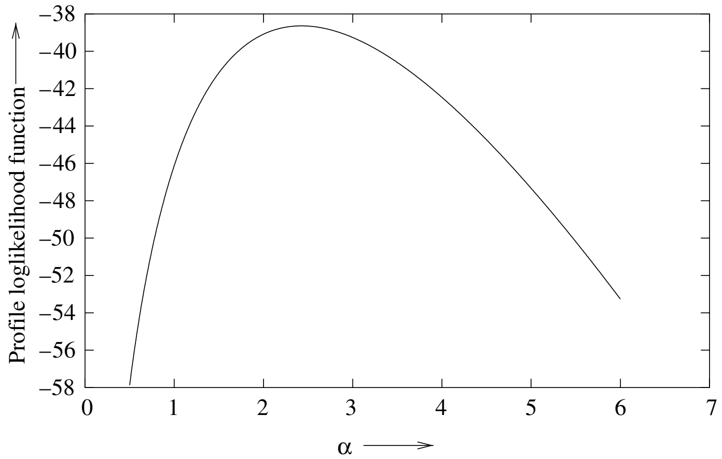

Fig. 1. Profile log-likelihood function of α for Type-I censored sample.

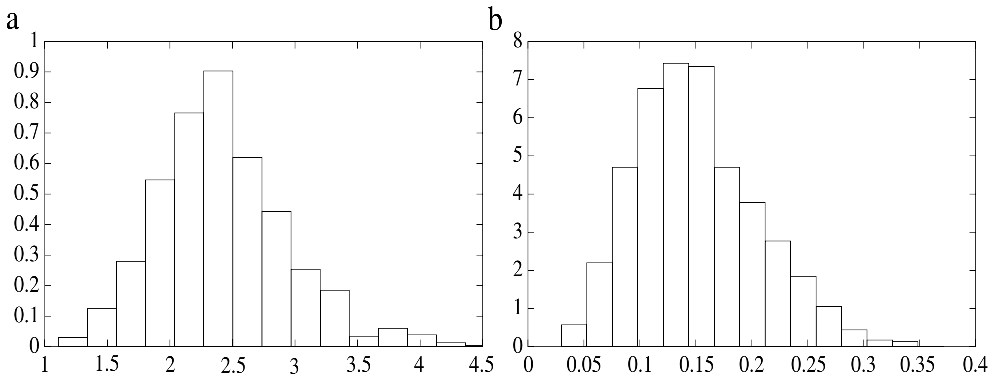

Fig. 2. Histogram plots: (a) MCMC samples of α (b) MCMC samples of λ.

Based on the observed sample we compute the AMLEs, MLEs, 95% asymptotic confidence intervals (ACI) and 95% bootstrap confidence intervals (BCI) of both the parameters. The results are reported in Table 1. To compute the MLE of α, we use the iterative process as mentioned in (13). We use the initial guess of α the same as the AMLE of α namely 2.3343, and $\epsilon = 10^{-6}$. The iteration stops after 7 steps and it produces the MLE of α as provided in Table 1. We have tried the iteration with an initial guess of α as 1, and in that case the iteration stops after 10 steps, and it produces the same result as before. We provide the profile log-likelihood function of α in Fig. 1, and it clearly indicates that the MLE of α indeed maximizes the profile log-likelihood function.

Finally we compute the Bayes estimates of the unknown parameters based on the gamma priors as mentioned in Section 6. Since we do not have any prior information of the unknown parameters, we take the non-informative priors, namely $a_1 = b_1 = a_2 = b_2 = 0.0$. We generate 1000 MCMC samples as has been suggested in Section 6. We provide the histogram plots of generated α and λ in Fig. 2. Based on the MCMC samples we obtain the Bayes estimates of α and λ as 2.4215 and 0.1508 respectively. The associated 95% credible intervals of α and λ are (1.5051, 3.6864) and (0.0597, 0.2734) respectively.

Now the natural question is whether Weibull provides a good fit or not. We compute the Kolmogorov–Smirnov distances between the empirical distribution function and the fitted distribution functions based on MLEs, AMLEs and Bayes estimates and they are 0.1629, 0.1631 and 0.1611, and the associated p values are 0.603, 0.598, and 0.628 respectively. Therefore, based on the p values we can say that the Weibull distribution fits quite well to the above data set.

Table 2
The average estimates, the mean squared errors and average confidence interval based on MLE.

n		$T = 0.75$	$T = 1.00$
20	α	1.1032(0.1111), 1.3574(92.7)	1.0843(0.0863), 1.0912(96.2)
	λ	1.1124(0.3603), 1.6554(87.3)	1.0767(0.1756), 1.2555(89.4)
30	α	1.0720(0.0709), 1.0251(95.8)	1.0573(0.0553), 0.8637(95.8)
	λ	1.0525(0.1438), 1.2377(89.7)	1.0391(0.0885), 0.9903(90.5)
40	α	1.0456(0.0465), 0.8643(95.6)	1.0370(0.0382), 0.7327(95.0)
	λ	1.0460(0.0857), 1.0419(91.0)	1.0354(0.0544), 0.8392(89.4)
n		$T = 1.50$	$T = 2.00$
20	α	1.0665(0.0600), 0.9142(96.2)	1.0613(0.0495), 0.8431(96.4)
	λ	1.0430(0.0779), 1.0768(92.2)	1.0327(0.0635), 1.0501(93.1)
30	α	1.0434(0.0401), 0.7340(94.1)	1.0422(0.0326), 0.6838(95.6)
	λ	1.0275(0.0516), 0.8589(92.9)	1.0185(0.0436), 0.8437(94.4)
40	α	1.0286(0.0278), 0.6271(94.9)	1.0293(0.0228), 0.5803(96.1)
	λ	1.0243(0.0374), 0.7310(92.5)	1.0161(0.0320), 0.7105(94.4)

Table 3
The average estimates, the mean squared errors and average confidence interval based on AMLE.

n		$T = 0.75$	$T = 1.00$
20	α	1.0915(0.1092), 1.3160(92.4)	1.0735(0.0850), 1.0808(95.9)
	λ	1.1110(0.3737), 1.6014(87.0)	1.0709(0.1783), 1.2423(89.3)
30	α	1.0641(0.0700), 1.0238(95.2)	1.0500(0.0548), 0.8581(95.4)
	λ	1.0508(0.1471), 1.2352(89.7)	1.0350(0.0892), 0.9831(90.6)
40	α	1.0398(0.0461), 0.8581(95.3)	1.0316(0.0380), 0.7291(94.8)
	λ	1.0445(0.0865), 1.0337(90.9)	1.0322(0.0547), 0.8346(89.5)
n		$T = 1.50$	$T = 2.00$
20	α	1.0569(0.0594), 0.9073(95.7)	1.0525(0.0492), 0.8541(96.0)
	λ	1.0317(0.0780), 1.0657(91.8)	1.0177(0.0632), 1.0635(93.4)
30	α	1.0368(0.0398), 0.7303(93.9)	1.0362(0.0325), 0.6798(95.4)
	λ	1.0200(0.0517), 0.8531(92.6)	1.0086(0.0435), 0.8357(94.3)
40	α	1.0236(0.0276), 0.6247(94.7)	1.0248(0.0227), 0.5784(96.1)
	λ	1.0186(0.0374), 0.7273(92.5)	1.0085(0.0320), 0.7064(94.3)

Table 4
The average confidence interval based on Boot-p estimate when the MLEs are used and the associated coverage percentages are presented.

n		$T = 0.75$	$T = 1.00$
20	α	1.4868(92.0)	1.2216(92.8)
	λ	1.4633(93.2)	1.2146(91.3)
30	α	1.0882(93.0)	0.9282(93.7)
	λ	2.0338(92.6)	1.2874(91.8)
40	α	0.8844(95.0)	0.7652(95.1)
	λ	1.3537(93.3)	0.9631(92.9)
n		$T = 1.50$	$T = 2.00$
20	α	0.9985(92.2)	0.9197(92.3)
	λ	1.2963(92.3)	1.0647(91.8)
30	α	0.7780(93.8)	0.7143(93.1)
	λ	0.9190(92.2)	0.8042(92.2)
40	α	0.6466(95.8)	0.5991(92.3)
	λ	0.7489(93.6)	0.6934(93.7)

9. Conclusions

In this paper we discuss the conventional Type-I censored data for the two-parameter Weibull distribution. It is observed that the MLE of the shape parameter can be obtained by using an iterative procedure. The proposed AMLEs of the shape and scale parameters can be obtained in explicit forms. Various approximate confidence/credible intervals can be constructed along with coverage percentages. Bayes estimates of the unknown parameters can be obtained using Gibbs sampling procedures and the performances of the Bayes estimators under the assumption of the non-informative priors are quite similar to the MLEs or the AMLEs. It is also observed that the Bayes estimates work quite well unless n and T are very small or n is very large. One important point should be mentioned here is that when $d = 0$ MLEs or approximate MLEs do not exist, but Bayes estimates or the corresponding credible intervals can be constructed. This is definitely one major advantage of the Bayes estimates. Considering all the points, we suggest that if we do not have any prior information of the unknown parameters, the AMLEs and the associated proposed confidence intervals can be used for all practical purposes. Finally,

Table 5
The average confidence interval based on Boot-p estimate when the AMLEs are used and the associated coverage percentages are presented.

n		$T = 0.75$	$T = 1.00$
20	α	1.4529(93.0)	1.2341(93.0)
	λ	1.4944(92.6)	1.1976(92.4)
30	α	1.1167(94.2)	0.9199(93.1)
	λ	2.1123(92.8)	1.2699(92.3)
40	α	0.8987(94.6)	0.7711(94.8)
	λ	1.3459(94.1)	0.9711(93.4)
n		$T = 1.50$	$T = 2.00$
20	α	0.9812(93.3)	0.9261(93.6)
	λ	1.2899(91.9)	1.1112(92.3)
30	α	0.7818(94.0)	0.7231(94.0)
	λ	0.8999(92.6)	0.8221(93.1)
40	α	0.6646(94.9)	0.6113(93.0)
	λ	0.7501(94.1)	0.7016(94.4)

Table 6
The average estimates, the mean squared errors and average credible interval based on the Bayes estimate.

n		$T = 0.75$	$T = 1.00$
20	α	1.0436(0.1134), 1.3123(96.2)	1.0322(0.0829), 1.1200(95.1)
	λ	1.0491(0.1552), 1.3988(94.3)	1.0120(0.0972), 1.1389(94.1)
30	α	1.0213(0.0670), 0.9907(94.9)	1.0104(0.0535), 0.8677(94.7)
	λ	1.0398(0.0862), 1.1519(95.9)	1.0063(0.0653), 0.9155(93.8)
40	α	1.0185(0.0459), 0.8477(95.5)	1.0165(0.0348), 0.7569(96.7)
	λ	1.0156(0.0616), 0.9601(94.5)	1.0093(0.0435), 0.8104(95.0)
n		$T = 1.50$	$T = 2.00$
20	α	1.0247(0.0565), 0.9412(97.0)	1.0167(0.0467), 0.8396(94.4)
	λ	1.0233(0.0839), 0.9941(93.6)	1.0090(0.0658), 0.9685(95.4)
30	α	1.0185(0.0397), 0.7454(95.1)	1.0196(0.0308), 0.6885(95.7)
	λ	1.0128(0.0494), 0.8168(94.5)	1.0038(0.0468), 0.7693(94.3)
40	α	1.0123(0.0272), 0.6393(95.6)	1.0174(0.0254), 0.5794(94.3)
	λ	1.0130(0.0365), 0.7027(94.0)	0.9993(0.0325), 0.6722(94.2)

although here we have assumed that the lifetime distributions are Weibull most of the methods can be extended for other distributions also, like Log-normal, Gamma, or GE distributions. Work is in progress, and they will be reported later.

One of the referees had asked a very genuine question: why we had chosen the gamma prior on the shape parameter, and not other log-concave priors like log-normal or Weibull etc. Yes, definitely, it is an important question. Although the gamma prior is a very flexible prior, theoretically any other log-concave priors can be chosen. Moreover, choosing a proper prior is an important issue, which is beyond the scope of this paper. Definitely, it needs separate attention.

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Appendix

To prove Theorem 1, we need the following lemma.

Lemma. *For $x_i \geq 0$ and $b \geq 0$, define $g(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} + b$. Then $\frac{d^2}{d\alpha^2} \ln g(\alpha) \geq 0$.*

Proof. Note that,

$$g'(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} \ln x_i \quad \text{and} \quad g''(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2.$$

Since

$$\left(\sum_{i=1}^{n} x_i^{\alpha} (\ln x_i)^2\right) \times \left(\sum_{i=1}^{n} x_i^{\alpha}\right) - \left(\sum_{i=1}^{n} x_i^{\alpha} \ln x_i\right)^2 = \sum_{1 \leq i \leq j \leq n} x_i^{\alpha} x_j^{\alpha} (\ln x_i - \ln x_j)^2 \geq 0,$$

therefore for $b \geq 0$,

$$g''(\alpha)g(\alpha) \geq (g'(\alpha))^2. \tag{36}$$

Proof of Theorem 1: Here, considering our case, we get the conditional density function of α given the *data* is

$$l(\alpha|data) \propto \frac{\alpha^d \pi_2(\alpha) \prod_{i=1}^{d} y_{i:n}^{\alpha-1}}{\left(\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha} + b\right)^{a+d}}, \quad \text{if } d > 0 \tag{37}$$

$$\propto \frac{\pi_2(\alpha)}{(nT^{\alpha} + b)^{a}}, \quad \text{if } d = 0. \quad \square \tag{38}$$

Therefore, ignoring the additive constant, the log-likelihood function of the posterior density function of α can be written as

$$\ln(l(\alpha|data)) = \ln \pi_2(\alpha) + d \ln \alpha + (\alpha - 1)\left(\sum_{i=1}^{d} \ln y_{i:n}\right) - (a+d)\ln\left(\sum_{i=1}^{d} y_{i:n}^{\alpha} + (n-d)T^{\alpha} + b\right) \quad \text{if } d > 0 \tag{39}$$

$$= \ln \pi_2(\alpha) - a \ln(nT^{\alpha} + b) \quad \text{if } d = 0. \tag{40}$$

Therefore, using the above defined lemma and the assumption on $\pi_2(\alpha)$, it easily follows that $\ln(l(\alpha|data))$ is log-concave.

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